Geometrical arguments concerning two-sided submanifolds, flat submanifolds and pinched bicollars *

by

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1. Expository discussion of two-sidedness and main results. A connected n-manifold M^n without boundary in the interior of an $(n+1)$-manifold N^{n+1} is *two-sided* if there is a connected open neighborhood U of M in N such that $U-M$ has exactly two components each of which is open in N and each of which has M as its frontier relative to U. A connected n-manifold M^n with boundary contained in the interior of an $(n+1)$-manifold N^{n+1} is *two-sided* if $\operatorname{Int} M^n$ is two-sided. It is easy to obtain submanifolds which are not two-sided. For instance, the center 1-sphere of a Möbius band is not two-sided in the Möbius band and a Möbius band in E^3 is not two-sided. An embedding f of a connected n-manifold M^n into the interior of an $(n+1)$-manifold N^{n+1} will be said to be *two-sided* if $f(M^n)$ is two-sided in N^{n+1}. Of course, it is possible to have one embedding of a manifold M^n into a manifold N^{n+1} which is two-sided and another which is not. (For instance, there are two such embeddings of S^1 into a Möbius band.) However, if the manifolds M and N possess certain properties, one naturally suspects that every embedding of M into N is two-sided. One classical result of this nature is the Jordan–Brouwer separation theorem which has as a corollary that every embedding of S^n into S^{n+1} is two-sided. Brouwer's techniques can be generalized to show that every embedding of a closed (i.e., compact and without boundary), orientable n-manifold into S^{n+1} is two-sided. Of course, the orientability condition in the previous sentence can be removed since a non-orientable, closed n-manifold cannot be embedded in S^{n+1} (c.f., p. 179 of [5]). However, it is false that every embedding of a nonclosed n-manifold into S^{n+1} is two-sided, since such embeddable manifolds do not have to be orientable. The duality techniques of algebraic topology used to establish the above results do not seem to suffice even to show that every

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embedding of a non-closed, orientable n-manifold into S^{n+1} is two-sided. In fact, about the most general result of this nature that one can obtain with those techniques seems to be that an orientable, closed n-manifold in an orientable, closed $(n+1)$-manifold with n-dimensional Betti number zero is two-sided (c.f., p. 294 of [10]) (¹). Many times, however, one is only interested in locally flat embeddings, as we shall see later in this paper. Our first theorem states that locally flat embeddings of simply-connected n-manifolds into arbitrary $(n+1)$-manifolds are two-sided. (Of course, simply connected manifolds are orientable (c.f., p. 116 of [5]).) The proof is completely geometrical and involves no algebraic topology. Thus, for the special case of locally flat embeddings of closed, simply-connected manifolds, the classical result mentioned above can be proved without using algebraic topology or the formal notion of orientability. In fact, for such embeddings, that result is generalized by dropping the requirement that the ambient manifold be orientable and have (n)-Betti number zero. It is quite important to notice that our first theorem is not true if the simply-connectivity hypothesis is replaced by an orientability hypothesis. To see an example, take a Möbius band with the boundary excluded and cross it with $(0,1)$. Then, the center 1-sphere crossed with $(0,1)$ is a locally flat, orientable (non-simply-connected) submanifold of the product which is not two-sided. By taking further products with $(0,1)$, this example generalizes to higher dimensions. Similar examples, where the manifolds are closed, are obtained by taking the natural inclusion of each odd dimensional projective space into the next higher even dimensional projective space.

Because of the innocent appearance of the first result, especially for the case the ambient manifold is E^{n+1}, it is natural for one to feel that he can find a rigorous half-page proof in a few minutes. Perhaps it would be wise to comment on what the author believes is the "obvious proof" of Theorem 1 that would quickly occur to almost any topologist.

"Well, we have this orientable n-manifold M^n sitting in E^{n+1}, so at every point x of M we have some local orientation. Now, take a flattening neighborhood of M at x and point your index finger in the direction of the orientation and call the part of the flattening neighborhood in the direction your thumb is pointing the plus part and the part in the other direction the minus part of the flattening neighborhood. Now after cutting down the flattening neighborhoods a bit, it is easy to see that their union is a connected neighborhood of M which is separated by M into the part made up of the union of all the plus parts of the flattening neighborhoods and the part made up of the union of all the minus parts of the flattening neighborhoods".

Even if this argument were put in precise language, it would not work, because it does not use any properties of the ambient manifold nor does it use the fact that M is simply connected. The two examples at the end of the paragraph before last demonstrate the problem with this argument.

(¹) John Bryant and David Galewski have pointed out that more general results than this can be obtained by non-geometrical methods which involve applying a rather sophisticated sequence argument in a covering space.

Let X be a topological space and Y a subset of X. Then, Y is *collared* in X if there is a homeomorphism h carrying $Y\times[0,1]$ onto a neighborhood of Y such that $h(y,0)=y$ for all $y\in Y$. We call $h(Y\times[0,1])$ a *collar* of Y in X. If there is a homeomorphism h carrying $Y\times[-1,1]$ onto a neighborhood of Y such that $h(y,0)=y$ for all $y\in Y$, then Y is *bicollared* in X. We call $h(Y\times[-1,1])$ a *bicollar* of Y in X. Let $f\colon M^n\to N^{n+1}$ be an embedding of an n-manifold without boundary M^n into the interior of an $(n+1)$-manifold N^{n+1}. Then, f is *flat* if $f(M)$ is bicollared in N. Now, suppose that M^n is an n-manifold with boundary and let $\widetilde{M}^n = M^n\cup_g(\operatorname{Bd} M^n\times[0,1))$ where $g\colon \operatorname{Bd} M\to\to \operatorname{Bd} M\times 0$ is defined by $g(x)=(x,0)$. Then, an embedding $f\colon M^n\to\operatorname{Int}(N^{n+1})$ is *flat* if f extends to an embedding $\widetilde{f}$ of $\widetilde{M}^n$ into $\operatorname{Int}(N^{n+1})$ which is flat. In Theorem 2, we will show that every locally flat embedding of a connected, simply-connected n-manifold M^n into the interior of an arbitrary $(n+1)$-manifold N^{n+1} is flat.

Suppose that M^n is an n-submanifold of the $(n+1)$-manifold N^{n+1} and that Z is a closed subset of M^n. If there is a homeomorphism

$$h\colon (M\times[-1,1])/[(z,t)\approx(z,0) \text{ if } z\in Z,\ -1\leqslant t\leqslant 1]\to N$$

such that $h([(z,0)])=z$ and

$$P_h(Z,M,N)=h\big((M\times[-1,1])/[(z,t)\approx(z,0) \text{ if } z\in Z,\ -1\leqslant t\leqslant 1]\big)$$

is a neighborhood of $M-Z$ in N, then we call $P_h(Z,M,N)$ a *bicollar of M in N pinched at Z*. Pinched collars are defined similarly. Theorem 3 asserts that if M^n is an n-manifold which is contained in $\operatorname{Int}(N^{n+1})$ and Z is a closed subset of M^n such that $M-Z$ is connected, simply connected and locally flat in N^{n+1}, then there is a bicollar of M in N pinched at Z. One simple application of this result would be to establish, for $n-1\geqslant 4$, the existence of the bicollar of a certain cell pinched at an interior point of the cell which is assumed to exist in the first paragraph of the proof of Theorem 3 of [6]. Evidently, the author of that excellent paper based its existence on his assertion in the last paragraph of Section 1 that a locally flat embedding is flat in codimension one, which, of course, is false unless the embedding is two-sided.

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2. Formal statements and proofs of results. Throughout this section, except for the lemma, we assume that $n \geqslant 4$. For the case $n = 4$, we also assume that the manifold M^n can support a PL structure.

LEMMA. *Let D^k be a k-cell with bicollared boundary in a connected k-manifold M^k. Then, for any point $x \in M^k$, there is a k-cell D^k_x with bicollared boundary such that $x \in \operatorname{Int} D^k_x$ and $D^k \cup D^k_x$ lies in the interior of a k-cell with bicollared boundary.*

Proof. Consider the set X of all points x of M^k for which there is a k-cell D^k_x with bicollared boundary such that $x \in \operatorname{Int} D^k_x$ and $D^k \cup D^k_x$ lies in the interior of a k-cell with bicollared boundary. Certainly, $\operatorname{Int} D^k \subset X$ and so X is nonempty. It is routine to show that X is both open and closed and so the lemma follows since M is connected.

THEOREM 1. *A connected, simply-connected n-manifold M^n which is contained locally flatly in the interior of an arbitrary $(n+1)$-manifold N^{n+1} is two-sided.*

Of course, it is only necessary to establish the case that $\operatorname{Bd} M^n = \emptyset$. Since the proof is somewhat long, we break it down into the following steps. (These steps give an outline of the proof.)

Step 1. For a fixed point p of M, we construct a flattening $(n+1)$-cell neighborhood D^{n+1}_p of M in N at p and define the plus part, $+D^{n+1}_p$, and the minus part, $-D^{n+1}_p$ of D^{n+1}_p.

Step 2. We cover M with a collection of flattening $(n+1)$-cell neighborhoods by putting each point x of M in a certain flattening $(n+1)$-cell, denoted by D^{n+1}_x. Then, we generate another cover of flattening $(n+1)$-cells which contains the previous cover as a proper subset by defining for each $\varepsilon > 0$ and each $y \in D^{n+1}_x \cap M$ a sub-flattening $(n+1)$-cell, ${}^xD^{n+1}_{y,\varepsilon}$, of D^{n+1}_x which contains y. Finally, we define the plus part, $+{}^xD^{n+1}_{y,\varepsilon}$, and the minus part, $-{}^xD^{n+1}_{y,\varepsilon}$, of $D^{n+1}_{y,\varepsilon}$ relative to the $+D^{n+1}_p$ and $-D^{n+1}_p$ mentioned in Step 1.

Step 3. In order to know that $+{}^xD^{n+1}_{y,\varepsilon}$ and $-{}^xD^{n+1}_{y,\varepsilon}$ are well-defined, we must show that their determination is independent of the choice of

Part a) a certain homeomorphism $h_{\mathfrak{D}}$, and

Part b) a certain n-cell $\mathfrak{D}^n$ (the simple connectivity is used here) each of which will have been constructed in completing Step 2.

Step 4. We take a countable subcover $\{D^{n+1}_{x_i}\}_{i=1}^{\infty}$ of the cover $\{D^{n+1}_x\}_{x \in M}$ mentioned in Step 2 and "cut down" its members to obtain a new collection of flattening $(n+1)$-cells denoted by $\{d^{n+1}_{x_i}\}_{i=1}^{\infty}$.

Step 5. We take the U in the definition of two-sideness to be $\bigcup_{i=1}^{\infty} \operatorname{Int} d^{n+1}_{x_i}$ and show that M separates it into the two complementary domains $\bigcup_{i=1}^{\infty} \operatorname{Int}(+d^{n+1}_{x_i})$ and $\bigcup_{i=1}^{\infty} \operatorname{Int}(-d^{n+1}_{x_i})$. (This step involves use of the second cover constructed in Step 2.)

Notation. Let $I^1 = [-1, 1]$ and $I^{k+1} = I^k \times I^1$. For a point $y \in I^k$, let $N_\varepsilon(I^k, y) = \{x \in I^k \mid \operatorname{dist}(x, y) \leqslant \varepsilon\}$, which is a closed k-cell, and let $I^{k+1}_{y,\varepsilon} = N_\varepsilon(I^k, y) \times I^1$. Let $+I^{k+1} = I^k \times [0, 1]$, $-I^{k+1} = I^k \times [-1, 0]$, $+I^{k+1}_{y,\varepsilon} = I^{k+1}_{y,\varepsilon} \cap +I^{k+1}$ and $-I^{k+1}_{y,\varepsilon} = I^{k+1}_{y,\varepsilon} \cap -I^{k+1}$.

Proof of Step 1. Fix a point p of M^n and let W_p be a neighborhood of p in N^{n+1} for which there is a homeomorphism $h\colon (W_p, W_p \cap M) \to\to (E^{n+1}, E^n)$. Let $D^{n+1}_p = h^{-1}(I^{n+1})$ and $D^n_p = D^{n+1}_p \cap M$. Then, D^n_p is an n-cell with bicollared boundary in the connected n-manifold M^n, as in the statement of the lemma. Let $h_p\colon (D^{n+1}_p, D^{n+1}_p \cap M = D^n_p) \to\to (I^{n+1}, I^n)$ denote $h|D^{n+1}_p$ and denote $h_p^{-1}(+I^{n+1})$ by $+D^{n+1}_p$ and $h_p^{-1}(-I^{n+1})$ by $-D^{n+1}_p$.

Proof of Step 2. Now let x be an arbitrary point of M. By the lemma, there is an n-cell D^n_x with bicollared boundary such that $x \in \operatorname{Int} D^n_x$ and $D^n_p \cup D^n_x$ lies in the interior of an n-cell with bicollared boundary. It is easy to see (by an argument similar to the one in the preceding step that we may assume that there is an $(n+1)$-cell $D^{n+1}_x \subset N^{n+1}$ such that $D^{n+1}_x \cap M = D^n_x$ and a homeomorphism $h_x\colon (D^{n+1}_x, D^{n+1}_x \cap M = D^n_x) \to\to (I^{n+1}, I^n)$. For $y \in \operatorname{Int} D^n_x$, we will use the notation ${}^xD^{n+1}_{y,\varepsilon} = h_x^{-1}(I^{n+1}_{h_x(y),\varepsilon})$ and ${}^xD^n_{y,\varepsilon} = h_x^{-1}\big(N_\varepsilon(I^n, h_x(y))\big) = {}^xD^{n+1}_{y,\varepsilon} \cap M$. As observed above, $D^n_p \cup D^n_x$ is contained in the interior of an n-cell with bicollared boundary, hence, for each $y \in \operatorname{Int} D^n_x$ and $\varepsilon > 0$, $D^n_p \cup {}^xD^n_{y,\varepsilon}$ is contained in the interior of an n-cell with bicollared boundary.

We wish now to define $+{}^xD^{n+1}_{y,\varepsilon}$ and $-{}^xD^{n+1}_{y,\varepsilon}$. In order to do so let $\mathfrak{D}^n$ be any n-cell with bicollared boundary which contains $D^n_p \cup {}^xD^n_{y,\varepsilon}$ in its interior. (We know that such $\mathfrak{D}^n$'s exist by the last sentence of the preceding paragraph.) Let $h_{\mathfrak{D}}\colon U_{\mathfrak{D}} \to\to E^{n+1}$ be a homeomorphism of a neighborhood $U_{\mathfrak{D}}$ of $\mathfrak{D}^n$ such that $h_{\mathfrak{D}}(\mathfrak{D}^n) \subset E^n$. ($\mathfrak{D}^n$ is locally flat by transitivity of local flatness, hence $U_{\mathfrak{D}}$ and $h_{\mathfrak{D}}$ exist by [7].) It is easy to show that there is some $\varepsilon > 0$ such that $h_{\mathfrak{D}} h_p^{-1}(I^n \times [0, \varepsilon])$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int} D^n_p)$ in either E^{n+1}_+ or E^{n+1}_-. (Without loss of generality, we may assume that $h_{\mathfrak{D}} h_p^{-1}(I^n \times [0, \varepsilon])$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int} D^n_p)$ in E^{n+1}_+ since if not, we could follow $h_{\mathfrak{D}}$ by a reflection of E^{n+1} about E^n.) Now there is a $\delta > 0$ such that either $h_{\mathfrak{D}} h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta]\big)$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int} {}^xD^n_{y,\varepsilon})$ in E^{n+1}_+ or E^{n+1}_-. If the former is the case let $+{}^xD^{n+1}_{y,\varepsilon} = h_x^{-1}(+I^{n+1}_{h_x(y),\varepsilon})$ and $-{}^xD^{n+1}_{y,\varepsilon} = h_x^{-1}(-I^{n+1}_{h_x(y),\varepsilon})$, but if the latter is the case let $+{}^xD^{n+1}_{y,\varepsilon} = h_x^{-1}(-I^{n+1}_{h_x(y),\varepsilon})$ and $-{}^xD^{n+1}_{y,\varepsilon} = h_x^{-1}(+I^{n+1}_{h_x(y),\varepsilon})$.

Proof of Step 3. Part a). We now show independence of the choice of $h_{\mathfrak{D}}$ of Step 2. In order to do so fix an n-cell $\mathfrak{D}$ with bicollared boundary in M such that $D_p^n \cup {}^xD_{y,\varepsilon}^n \subset \operatorname{Int}\mathfrak{D}$ and suppose that $h_{\mathfrak{D}}$ and $h'_{\mathfrak{D}}$ are two homeomorphisms of neighborhoods $U_{\mathfrak{D}}$ and $U'_{\mathfrak{D}}$ of $\mathfrak{D}_n$ onto E^{n+1} such that $h_{\mathfrak{D}}(\mathfrak{D})$ and $h'_{\mathfrak{D}}(\mathfrak{D})$ are contained in E^n and for which there exists some $\varepsilon > 0$ such that $h_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon])$ and $h'_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon])$ are neighborhoods of $h_{\mathfrak{D}}(\operatorname{Int}D_p^n)$ and $h'_{\mathfrak{D}}(\operatorname{Int}D_p^n)$, respectively, in E_+^{n+1}. Let $\mathfrak{D}'$ be a bicollared n-cell contained in $\operatorname{Int}\mathfrak{D}$ concentric with $\mathfrak{D}$ such that $D_p^n \cup {}^xD_{y,\varepsilon}^n \subset \operatorname{Int}\mathfrak{D}'$. Then, obviously $h_{\mathfrak{D}}(\mathfrak{D}')$ is collared in E_+^{n+1}. In fact, we can easily obtain a collar C of $h_{\mathfrak{D}}(\mathfrak{D}')$ in E_+^{n+1} such that

1. $h_{\mathfrak{D}}^{-1}(C) \subset U_{\mathfrak{D}} \cap U'_{\mathfrak{D}}$,
2. l.u.b. $\{\operatorname{dist}(x, y) |\ x \in C, y \in h_{\mathfrak{D}}(\mathfrak{D}')\} < \operatorname{dist}\big(h_{\mathfrak{D}}(\mathfrak{D}'), E^n - h_{\mathfrak{D}}(\mathfrak{D})\big)$,

and

3. l.u.b. $\{\operatorname{dist}(x, y) |\ x \in h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C), y \in h'_{\mathfrak{D}}(\mathfrak{D}')\} < \operatorname{dist}\big(h'_{\mathfrak{D}}(\mathfrak{D}'), E^n - h'_{\mathfrak{D}}(\mathfrak{D})\big)$.

It is not hard to conclude that $h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C)$ is a collar of $h'_{\mathfrak{D}}(\mathfrak{D}')$ in E_+^{n+1}. First, we can find some point in $h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C) \cap E_+^{n+1}$ since by definition of $h_{\mathfrak{D}}$ for small $t > 0$, there are points $(x, t) \in I^n \times [0, 1]$ such that $h_{\mathfrak{D}}h_p^{-1}\big((x, t)\big) \subset C \cap E_+^{n+1}$ and by definition of $h'_{\mathfrak{D}}$, $h'_{\mathfrak{D}}h_p^{-1}\big((x, t)\big) \subset E_+^{n+1}$ from which it follows that $h'_{\mathfrak{D}}h_p^{-1}\big((x, t)\big) \in E_+^{n+1} \cap h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C)$. Now, if all of $h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C - E^n)$ were not contained in E_+^{n+1}, we could easily obtain the contradiction that $h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C - E^n)$ is not connected.

We have assumed that $h_{\mathfrak{D}}$ and $h'_{\mathfrak{D}}$ have the property that there exists some $\varepsilon > 0$ such that $h_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon])$ and $h'_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon])$ are neighborhoods of $h_{\mathfrak{D}}(\operatorname{Int}D_p^n)$ and $h'_{\mathfrak{D}}(\operatorname{Int}D_p^n)$, respectively, in E_+^{n+1}. By choosing ε perhaps smaller, we can add the conditions that

$$h_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon]) \subset C \quad \text{and} \quad h'_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon]) \subset h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C)\,.$$

Suppose that there is a $\delta > 0$ such that $h_{\mathfrak{D}}h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta]\big)$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int}{}^xD_{y,\varepsilon}^n)$ in E_+^{n+1}. Then, we defined $+{}^xD_{y,\varepsilon}^{n+1}$ to be $h_x^{-1}(+I_{h_x(y),\varepsilon}^{n+1})$. By choosing the above δ perhaps smaller, we can assume that

$$h_{\mathfrak{D}}h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta]\big) \subset C\,.$$

Thus, since $h'_{\mathfrak{D}}h_{\mathfrak{D}}^{-1}(C)$ is a collar of $h'_{\mathfrak{D}}(\mathfrak{D}')$ in E_+^{n+1}, it follows that $h'_{\mathfrak{D}}h_x^{-1}\big(N_\varepsilon(I_n, h_x(y)) \times [0, \delta]\big)$ is a neighborhood of $h'_{\mathfrak{D}}(\operatorname{Int}D_{y,\varepsilon}^{xn})$ in E_+^{n+1}, and so if $h'_{\mathfrak{D}}$ were used to define $+{}^xD_{y,\varepsilon}^{n+1}$, the same set as determined by $h_{\mathfrak{D}}$ would result.

In the case that there exists a $\delta > 0$ such that $h_{\mathfrak{D}}h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta]\big)$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int}{}^xD_{y,\varepsilon}^n)$ in E_-^{n+1}, an argument similar to the above would show that the determination of $+{}^xD_{y,\varepsilon}^{n+1}$ and $-{}^xD_{y,\varepsilon}^{n+1}$ would be the same whether we use $h_{\mathfrak{D}}$ or $h'_{\mathfrak{D}}$ as the defining homeomorphism.

Part b). We are now ready to show that the determination of $+{}^xD_{y,\varepsilon}^{n+1}$ and $-{}^xD_{y,\varepsilon}^{n+1}$ is independent of the choice of $\mathfrak{D}^n$. In order to do so, suppose that we have two n-cells $\mathfrak{D}^n$ and $\mathfrak{D}_*^n$ in M^n with bicollared boundaries each of which contains $D_p^n \cup {}^xD_{y,\varepsilon}^n$ in its interior. First, we wish to establish the following claim: *the independence of the determination of* $+{}^xD_{y,\varepsilon}^{n+1}$ *and* $-{}^xD_{y,\varepsilon}^{n+1}$ *will follow if we can show that there is a homeomorphism* h *of the pair* (N^{n+1}, M^n) *onto itself such that* $h(\mathfrak{D}_*^n) \subset \mathfrak{D}^n$ *and* $h|N_\varepsilon(N^{n+1}, p) \cup N_\varepsilon(N^{n+1}, y) = 1$ *for some* $\varepsilon > 0$, *where* $N_\varepsilon(N^{n+1}, y)$ *represents the* ε*-neighborhood of* y *in* N^{n+1}.

Let $h_{\mathfrak{D}}$ be a homeomorphism of $U_{\mathfrak{D}}$, constructed in Step 2, onto E^{n+1} such that $h_{\mathfrak{D}}(\mathfrak{D}^n) \subset E^n$ and for which there is an $\varepsilon > 0$ such that $h_{\mathfrak{D}}h_p^{-1}(I^n \times [0, \varepsilon])$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int}D_p^n)$ in E_+^{n+1}. Let us assume that there is a $\delta > 0$ such that $h_{\mathfrak{D}}h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta]\big)$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int}{}^xD_{y,\varepsilon}^n)$ in E_+^{n+1}. (The proof of the case where $h_{\mathfrak{D}}h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta]\big)$ is a neighborhood of $h_{\mathfrak{D}}(\operatorname{Int}{}^xD_{y,\varepsilon}^n)$ in E_-^{n+1} is similar to this case.) Now consider the homeomorphism $h_{\mathfrak{D}_*} = h_{\mathfrak{D}}h$ of $U_{\mathfrak{D}_*} = h^{-1}(U_{\mathfrak{D}})$ onto E^{n+1}. Since $h(\mathfrak{D}_*^n) \subset \mathfrak{D}^n$, we have that $h_{\mathfrak{D}_*}(\mathfrak{D}_*^n) \subset E^n$. It follows easily from the fact that $h|N_\varepsilon(N^{n+1}, p) = 1$, that there exists a $\varepsilon' > 0$ such that $h_{\mathfrak{D}_*}h_p^{-1}(I^n \times [0, \varepsilon'])$ is a neighborhood of $h_{\mathfrak{D}_*}(\operatorname{Int}D_p^n)$ in E_+^{n+1}. Also, since $h|N_\varepsilon(N^{n+1}, y) = 1$, it follows that there is a $\delta' > 0$ such that $h_{\mathfrak{D}_*}h_x^{-1}\big(N_\varepsilon(I^n, h_x(y)) \times [0, \delta']\big)$ is a neighborhood of $h_{\mathfrak{D}_*}(\operatorname{Int}{}^xD_{y,\varepsilon}^n)$ in E_+^{n+1} and the claim is established.

Now we will establish the above claim. First, run an arc a_y from y out the radial structure on ${}^xD_{y,\varepsilon}^n$ and on out the radial structure of a collar C_y of $\operatorname{Bd}{}^xD_{y,\varepsilon}^n$, on the outside of ${}^xD_{y,\varepsilon}^n$, to a point y' in $\operatorname{Int}\mathfrak{D}^n \cap \operatorname{Int}\mathfrak{D}_*^n$. Similarly, run an arc a_p from p out the radial structure on D_p^n and on out the radial structure of a collar C_p of $\operatorname{Bd}D_p^n$, on the outside of D_p^n, to a point p' in $\operatorname{Int}\mathfrak{D}^n \cap \operatorname{Int}\mathfrak{D}_*^n$. Now p' and y' can be connected by a locally flat arc $a_{\mathfrak{D}}$ in $\operatorname{Int}\mathfrak{D}^n - (D_p^n \cup {}^xD_{y,\varepsilon}^n)$ and p' and y' can be connected by a locally flat arc $a_{\mathfrak{D}_*}$ in $\operatorname{Int}\mathfrak{D}_*^n - (D_p^n \cup {}^xD_{y,\varepsilon}^n)$. By taming $a_y \cup a_{\mathfrak{D}} \cup a_p \cup (a_{\mathfrak{D}_*} \cap \operatorname{Int}\mathfrak{D}^n)$ in a PL structure on $\operatorname{Int}\mathfrak{D}^n$ and then applying general position, we may assume that $a_{\mathfrak{D}} \cap a_{\mathfrak{D}_*} = p' \cup y'$, that $a_p \cap (a_{\mathfrak{D}} \cup a_{\mathfrak{D}_*} \cup a_y) = p'$, and that $a_y \cap (a_{\mathfrak{D}} \cup a_{\mathfrak{D}_*} \cup a_p) = y'$. (To do the taming one could use [4], for instance.) Since M is simply-connected, $M' = M - ({}^xD_{y,\varepsilon}^n \cup C_y \cup D_p^n \cup C_p)$ is simply-connected and so the simple closed curve $a_{\mathfrak{D}} \cup a_{\mathfrak{D}_*}$ bounds a singular disk D_s in M'. We would like for $a_{\mathfrak{D}} \cup a_{\mathfrak{D}_*}$ to bound a real, locally flat disk in M'. If M could support a triangulation, this would follow by general position for $n \geqslant 5$. However, for the case M is a topological manifold, we can obtain the desired disk d^2 by applying taming and general position on small patches of D_s and M^n (c.f. [2]). Now in the case $n = 4$, we can apply general position to get a singular disk D'_s having point singularities. Choose a small real disk D''

in $\operatorname{Int} D'_s$. Then, $\operatorname{Bd} D''$ and $\operatorname{Bd} D'_s = a_{\mathfrak{D}} \cup a_{\mathfrak{D}_*}$ are homotopic and since for $n = 4$ we are assuming M can support a PL structure, we can carry $\operatorname{Bd} D''$ onto $\operatorname{Bd} D'_s$ with a space homeomorphism by standard unknotting theorems (c.f., Theorem 24 of [11]). Hence, again $a_{\mathfrak{D}} \cup a_{\mathfrak{D}_*}$ bounds a locally flat real disk, d^2.

Let $a'_{\mathfrak{D}}$ be a locally flat arc in $d^2 \cap \operatorname{Int} \mathfrak{D}^n$ with endpoints p' and y' such that $\operatorname{Int} a'_{\mathfrak{D}} \subset \operatorname{Int} d^2$ and let $a'_{\mathfrak{D}_*}$ be a locally flat arc in $d^2 \cap \operatorname{Int} \mathfrak{D}^n_*$ with endpoints p' and y' such that $\operatorname{Int} a'_{\mathfrak{D}_*} \subset \operatorname{Int} d^2$. Since d^2 is locally flat in M^n, it follows from [7] that there is a neighborhood U in M^n such that $(U, d^2) \approx (E^n, I^2)$. Thus, there is a locally flat n-cell d^n in U such that (d^n, d^2) is a trivial cell pair. By using Lemma 3.6 of [8], we may assume that $d^n \cap ({}^xD^n_{y,\varepsilon} \cup a_y \cup D^n_p \cup a_p) = y' \cup p'$. Notice that $(d^n, a'_{\mathfrak{D}})$ and $(d^n, a'_{\mathfrak{D}_*})$ are also trivial cell pairs, hence, by unknotting locally flat cell pairs, there is a homeomorphism h_1 of d^n onto itself such that $h_1|\operatorname{Bd} d^n = 1$ and $h_1(a'_{\mathfrak{D}_*}) = a'_{\mathfrak{D}}$. (To unknot cell pairs see [3]. The case $n = 4$ follows from standard taming theorems and PL unknotting theorems.) Now, by transitivity of local flatness, d^n is locally flat in N^{n+1}, hence, by applying the lemma in the appendix of [7] and Lemma 3.6 of [8], we can obtain a locally flat $(n+1)$-cell d^{n+1} in N^{n+1} such that $d^{n+1} \cap M^n = d^n$. It is easy to extend h_1 to a homeomorphism h_2 of d^{n+1} such that $h_2|\operatorname{Bd} d^{n+1} = 1$. Let h_3 represent the extension of h_2 over N^{n+1} by way of the identity. Then, h_3 is a homeomorphism of (N^{n+1}, M) onto itself such that $h_3(a_y \cup a'_{\mathfrak{D}_*} \cup a_p) = a_y \cup a'_{\mathfrak{D}} \cup a_p$ and $h_3|N_\varepsilon(N^{n+1}, y) \cup \cup N_\varepsilon(N^{n+1}, p) = 1$ for some $\varepsilon > 0$. (However, h_3 is not the homeomorphism h which we seek for it is not necessarily true that $h_3(\mathfrak{D}^n_*) \subset \mathfrak{D}^n$.)

We have that the boundary of $h_3(\mathfrak{D}^n_*)$ is bicollared in M^n and so let $\bar{D}^n$ represent a locally flat n-cell in M^n which is the union of $h_3(\mathfrak{D}^n_*)$ and a closed collar of $\operatorname{Bd} h_3(\mathfrak{D}^n_*)$ on the outside of $h_3(\mathfrak{D}^n_*)$. By stretching out the collar, we can get a homeomorphism h_4 of $\bar{D}^n$ onto itself which is the identity on $\operatorname{Bd} \bar{D}^n$ and on a small neighborhood of $a'_{\mathfrak{D}}$ and is such that $h_4(h_3(\mathfrak{D}^n_*)) \subset \mathfrak{D}^n$. Again, since $\bar{D}^n$ is locally flat in N^{n+1}, we can apply the lemma in the appendix of [7] and Lemma 3.6 of [8] to obtain an $(n+1)$-cell $\bar{D}^{n+1}$ in N^{n+1} such that $\bar{D}^{n+1} \cap M = \bar{D}^n$. Now, by taking a suspension extension of h_4, we can obtain a homeomorphism h_5 of $\bar{D}^{n+1}$ onto itself such that $h_5|\operatorname{Bd} \bar{D}^{n+1} = 1$ and $h_5|N_\varepsilon(N^{n+1}, p) \cup N_\varepsilon(N^{n+1}, y) = 1$ for some $\varepsilon > 0$. Let h_6 represent the extension of h_5 to N^{n+1} by way of the identity. Then, $h = h_6 h_3$ is clearly the homeomorphism that we seek, and this concludes the proof that $+{}^xD^{n+1}_{y,\varepsilon}$ and $-{}^xD^{n+1}_{y,\varepsilon}$ are well-defined.

Proof of Step 4. Let $\{D^{n+1}_x\}_{x \in M}$ denote the first cover of M mentioned in Step 2. Since M is separable, we can get a countable subcover which we denote by $\{D^{n+1}_{x_i}\}_{i=1}^{\infty}$. Then, there is associated with each x_i a homeomorphism h_{x_i}: $(D^{n+1}_{x_i}, D^{n+1}_{x_i} \cap M = D^n_{x_i}) \rightarrow\!\!\rightarrow (I^{n+1}, I^n)$. Let $(x_1, D^{n+1}_{x_1}, D^n_{x_1}, h_{x_1})$ correspond to $(p, D^{n+1}_p, D^n_p, h_p)$ constructed in Step 1 and in terms of which $+{}^xD^{n+1}_{y,\varepsilon}$ and $-{}^xD^{n+1}_{y,\varepsilon}$ were defined in Step 2. Then, corresponding to each $D^{n+1}_{x_i}$, we have a $+D^{n+1}_{x_i} = {}^{x_i}D^{n+1}_{x_i,2}$ and a $-D^{n+1}_{x_i} = -{}^{x_i}D^{n+1}_{x_i,2}$. It would be nice if we could take $\bigcup_{i=1}^{\infty} \operatorname{Int} D^{n+1}_{x_i}$ as the connected open set U in the definition of two-sideness and argue that M separates U into the two complementary domains $\bigcup_{i=1}^{\infty} \operatorname{Int}(+D^{n+1}_{x_i})$ and $\bigcup_{i=1}^{\infty} \operatorname{Int}(-D^{n+1}_{x_i})$. This will not work though, since it is easy to see that the later two sets do not have to be disjoint. However, this is the raw idea of the remainder of the proof, but it will be necessary to cut the $D^{n+1}_{x_i}$'s down to make it work.

Let $d^{n+1}_{x_1} = D^{n+1}_{x_1}$, $+d^{n+1}_{x_1} = +D^{n+1}_{x_1}$, $-d^{n+1}_{x_1} = -D^{n+1}_{x_1}$ and $\bar{h}_{x_1} = h_{x_1}$. Now, we have the homeomorphism h_{x_2}: $(D^{n+1}_{x_2}, D^n_{x_2}) \rightarrow\!\!\rightarrow (I^{n+1}, I^n)$. Without loss of generality, we may suppose that $+D^{n+1}_{x_2} = h^{-1}_{x_2}(+I^{n+1})$ and $-D^{n+1}_{x_2} = h^{-1}_{x_2}(-I^{n+1})$. It is not difficult to show that there exists an $\varepsilon_2 > 0$ such that $h^{-1}_{x_2}(I^n \times (0, \varepsilon_2]) \cap -d^{n+1}_{x_1} = \emptyset$ and $h^{-1}_{x_2}(I^n \times [-\varepsilon_2, 0)) \cap +d^{n+1}_{x_1} = \emptyset$. Let $d^{n+1}_{x_2} = h^{-1}_{x_2}(I^n \times [-\varepsilon_2, \varepsilon_2])$. Let h'_{x_2}: $I^n \times [-\varepsilon_2, \varepsilon_2] \rightarrow\!\!\rightarrow I^{n+1}$ be defined by taking $x \times [-\varepsilon_2, \varepsilon_2]$ linearly onto $x \times [-1, 1]$ for each $x \in I^n$ and let $\bar{h}_{x_2} = h'_{x_2} h_{x_2}$.

Inductively, we have the homeomorphism h_{x_k}: $(D^{n+1}_{x_k}, D^n_{x_k}) \rightarrow\!\!\rightarrow (I^{n+1}, I^n)$ and without loss of generality we may suppose that $+D^{n+1}_{x_k} = h^{-1}_{x_k}(+I^{n+1})$ and $-D^{n+1}_{x_k} = h^{-1}_{x_k}(-I^{n+1})$. It is not difficult to show that there exists an $\varepsilon_k > 0$ such that $h^{-1}_{x_k}(I^n \times (0, \varepsilon_k]) \cap (\bigcup_{i=1}^{k-1} -d^{n+1}_{x_i}) = \emptyset$ and $h^{-1}_{x_k}(I^n \times [-\varepsilon_2, 0)) \cap (\bigcup_{i=1}^{k-1} +d^{n+1}_{x_i}) = \emptyset$. Let $d^{n+1}_{x_k} = h^{-1}_{x_k}(I^n \times [-\varepsilon_2, \varepsilon_2])$ and define $\bar{h}_{x_k}$ similar to the definition of $\bar{h}_{x_2}$ above.

Proof of Step 5. We take the U in the definition of two-sideness to be $\bigcup_{i=1}^{\infty} \operatorname{Int} d^{n+1}_{x_i}$. Certainly, U is connected since M is connected. It is also clear by construction that the two open sets $\bigcup_{i=1}^{\infty} \operatorname{Int}(+d^{n+1}_{x_i})$ and $\bigcup_{i=1}^{\infty} \operatorname{Int}(-d^{n+1}_{x_i})$ are disjoint. Hence, the proof will be complete after we show that the two sets $\bigcup_{i=1}^{\infty} \operatorname{Int}(+d^{n+1}_{x_i})$ and $\bigcup_{i=1}^{\infty} \operatorname{Int}(-d^{n+1}_{x_i})$ are connected.

We will only show that $\bigcup_{i=1}^{\infty} \operatorname{Int}(+d^{n+1}_{x_i})$ is connected for the proof that $\bigcup_{i=1}^{\infty} \operatorname{Int}(-d^{n+1}_{x_i})$ is connected would be similar. Let z be an arbitrary point of $\bigcup_{i=1}^{\infty} \operatorname{Int}(+d^{n+1}_{x_i})$. Then, it will suffice to show that there is an arc in $\bigcup_{i=1}^{\infty} \operatorname{Int}(+d^{n+1}_{x_i})$ which connects z to some point of $\operatorname{Int}(+d^{n+1}_p)$

since $\mathrm{Int}(+d_p^{n+1})$ is connected. Certainly, there exists some positive integer k such that $z \in \mathrm{Int}(+d_{x_k}^{n+1})$. Let d_p^n denote $d_p^{n+1} \cap M = D_{x_1}^n$, and $d_{x_i}^n$ denote $d_{x_i}^{n+1} \cap M = D_{x_1}^n$. Then, there is some n-cell $\mathfrak{D}^n$ in M with bicollared boundary such that $d_p^n \cup d_{x_k}^n \subset \mathrm{Int}\,\mathfrak{D}^n$. By transitivity of local flatness $\mathfrak{D}^n$ is locally flat in N^{n+1} and so by [7] there is a homeomorphism $h_{\mathfrak{D}}$ of some neighborhood $U_{\mathfrak{D}}$ of $\mathfrak{D}^n$ in N^{n+1} onto E^{n+1} such that $h_{\mathfrak{D}}(\mathfrak{D}^n) = I^n$. We may also assume that $h_{\mathfrak{D}}$ is such that there exists an $\varepsilon > 0$ such that $h_{\mathfrak{D}}\bar{h}_p^{-1}(I^n \times [0, \varepsilon])$ is a neighborhood of $h_{\mathfrak{D}}(d_p^n)$ in E_+^{n+1}. Now, for each point $y \in \mathrm{Int}\,\mathfrak{D}^n$, there is some $d_{x_i}^n$ in M such that $y \in d_{x_i}^n$. Hence, there is some ${}^{x_i}d_{y,\varepsilon}^n \subset \mathrm{Int}\,\mathfrak{D}^n$ such that $y \in {}^{x_i}d_{y,\varepsilon}^n$. Let $V = \bigcup_{y \in \mathrm{Int}\,\mathfrak{D}} \mathrm{Int}(+{}^{x_i}d_{y,\varepsilon}^{n+1})$, where

$$+{}^{x_i}d_{p,\varepsilon}^{n+1} = +d_p^{n+1} \quad \text{and} \quad +{}^{x_i}d_{x_k,\varepsilon}^{n+1} = +d_{x_k}^{n+1}. \quad \text{Then,} \quad V \subset \bigcup_{i=1}^{\infty} \mathrm{Int}(+d_{x_i}^{n+1}).$$

Hence, it will suffice to show that there is an arc in V which connects z to a point of $\mathrm{Int}(+d_p^{n+1})$.

It follows from the construction of the $+{}^{x_i}d_{y,\varepsilon}^{n+1}$'s that $h_{\mathfrak{D}}(V)$ is a neighborhood of $\mathrm{Int}\,h_{\mathfrak{D}}(\mathfrak{D}^n) = I^n$ in E_+^{n+1}. Thus, it easy to run an arc a from $h_{\mathfrak{D}}(z)$ down the product structure of $h_{\mathfrak{D}}(\mathrm{Int}(+d_{x_k}^{n+1}))$ until it is very close to $\mathrm{Int}\,I^n$ in E_+^{n+1} and then through the neighborhood $h_{\mathfrak{D}}(V)$ until it reaches a point of $h_{\mathfrak{D}}(\mathrm{Int}(+d_p^{n+1}))$. Then, $h_{\mathfrak{D}}^{-1}(a)$ is the arc that we sought and the proof is complete.

THEOREM 2. *Every locally flat embedding f of a connected, simply-connected n-manifold M^n into the interior of an arbitrary $(n+1)$-manifold N^{n+1} is flat.*

Proof. If the boundary of M is empty, then Theorem 2 follows immediately from Theorem 1 and Theorem 3 of [1]. Suppose $\mathrm{Bd}\, M \neq \emptyset$.

We will first establish the fact that there is a homeomorphism q of N onto itself such that $q(f(M))$ is a locally flat subset of $f(\mathrm{Int}\, M)$. By local flatness, choose a countable collection $\{D_i^{n+1}\}$ of $(n+1)$-cells in N such that there is a homeomorphism h_i: $(D_i^{n+1}, D_i^{n+1} \cap f(M)) \to\to (I^{n+1}, I_+^n)$ and such that $f(\mathrm{Bd}\, M)$ is contained in the union of the interiors of members of $\{D_i^{n+1}\}$. Let $\{D_i^{n-1}\}$ be a collection of $(n-1)$-cells whose interiors cover $f(\mathrm{Bd}\, M)$ such that $D_i^{n-1} \subset \mathrm{Int}\, D_i^{n+1} \cap f(\mathrm{Bd}\, M)$. Let k_1 be a homeomorphism of (I^{n+1}, I^n) onto itself such that

$$k_1 | \mathrm{Bd}\, I^{n+1} = 1, \quad k_1(I_+^n) \subset I_+^n \quad \text{and} \quad k_1(h_1(D_1^{n-1})) \subset \mathrm{Int}\, I_+^n.$$

Define k_1': $D_1^{n+1} \to\to D_1^{n+1}$ by $k_1' = h_1^{-1}k_1h_1$ and define q_1 to be the extension of k_1' to all of N by the identity.

Now to obtain q_2, repeat the above construction where $(q_1 f, q_1(D_i^{n+1}), q_1(D_i^{n-1}), h_i q_i^{-1})$ plays the role of $(f, D_i^{n+1}, D_i^{n-1}, h_i)$.

Given $q_1, \ldots, q_k$, obtain q_{k+1} by going through the construction where $(q_k \ldots q_1 f, q_k \ldots q_1(D_i^{n+1}), q_k \ldots q_1(D_i^{n-1}), h_i q_1^{-1} \ldots q_k^{-1})$ plays the role of $(f, D_i^{n+1}, D_i^{n-1}, h_i)$.

Then, $q = \ldots q_2 q_1 f$ is the homeomorphism desired.

Since $q(f(M))$ is locally flat in $f(\mathrm{Int}\, M)$, it follows that $f(M) - q(f(\mathrm{Int}\, M))$ is a manifold and as such has a collared boundary by [1]. Thus, there is a homeomorphism λ: $(\mathrm{Bd}\, M \times [0, 1)) \to (f(M) - q(f(\mathrm{Int} M)))$ such that $\lambda(x, 0) = qf(x)$ for $x \in \mathrm{Bd}\, M$. By the first sentence of this proof, $f(\mathrm{Int}\, M)$ is bicollared in N; hence, $gf(M) \cup \lambda([0, 1))$ is bicollared in N. Finally, $q^{-1}(qf(M) \cup \lambda([0, 1))) = f(M) \cup q^{-1}\lambda([0, 1))$ is bicollared in N. But, if $\widetilde{M}^n = M^n \cup_g (\mathrm{Bd}\, M^n \times [0, 1))$ where g: $\mathrm{Bd}\, M \to\to \mathrm{Bd}\, M \times 0$ is defined by $g(x) = (x, 0)$, then f: $M^n \to \mathrm{Int}(N^{n+1})$ extends to $\tilde{f}$: $\widetilde{M} \to\to \mathrm{Int}(N^{n+1})$ be letting $\tilde{f}(x, t) = q^{-1}\lambda(x, t)$. Since we have just seen $\tilde{f}(\widetilde{M}) = f(M) \cup q^{-1}\lambda([0, 1))$ to be bicollared, the conclusion follows.

Remark. A special case of the above proof yields that a locally flat $(n-1)$-sphere in S^n is bicollared. Hence, it is not necessary to use the Jordan–Brower Separation Theorem to obtain this result as is usually done.

THEOREM 3. *If M^n is an n-manifold which is contained in the interior of an $(n+1)$-manifold N^{n+1} and Z is a closed subset of M^n such that $M - Z$ is connected, simply-connected and locally flat in N^{n+1}, then there is a bicollar of M in N pinched at Z.*

Proof. Theorem 3 follows immediately from Theorem 1 and the Pinched Bicollar Lemma of [9].

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