Geometrical arguments concerning
two-sided submanifolds, flat submanifolds
and pinched bicollars *

by

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1. Expository discussion of two-sidedness and main results. A connected
$n$-manifold $M$ without boundary in the interior of an $(n+1)$-manifold
$X^{n+1}$ is two-sided if there is a connected open neighborhood $U$ of $M$ in $N$
such that $U - M$ has exactly two components each of which is open in $N$
and each of which has $M$ as its frontier relative to $U$. A connected
$n$-manifold $M$ with boundary contained in the interior of an $(n+1)$-manifold
$N^{n+1}$ is two-sided if Int $M$ is two-sided. It is easy to obtain submanifolds
which are not two-sided. For instance, the center 1-sphere of a Möbius band
is not two-sided in the Möbius band and a Möbius band
in $S^3$ is not two-sided. An embedding $f$ of a connected $n$-manifold $M$
into the interior of an $(n+1)$-manifold $X^{n+1}$ will be said to be two-sided
if $f(M)$ is two-sided in $X^{n+1}$. Of course, it is possible to have one embedding
of a manifold $M$ into a manifold $X^{n+1}$ which is two-sided and another
which is not. (For instance, there are two such embeddings of $S^3$ into
a Möbius band.) However, if the manifolds $M$ and $N$ possess certain
properties, one naturally suspects that every embedding of $M$ into $N$ is
two-sided. One classical result of this nature is the Jordan–Brouwer
separation theorem which has as a corollary that every embedding of $S^n$
into $S^{n+1}$ is two-sided. Brouwer's techniques can be generalized to show
that every embedding of a closed (i.e., compact and without boundary),
orientable $n$-manifold into $S^{n+1}$ is two-sided. Of course, the orientability
condition in the previous sentence can be removed since a non-orientable,
closed $n$-manifold cannot be embedded in $S^{n+1}$ (c.f., p. 179 of [3]).
However, it is false that every embedding of a nonclosed $n$-manifold
into $S^{n+1}$ is two-sided, since such embeddable manifolds do not have
to be orientable. The duality techniques of algebraic topology used to
establish the above results do not seem to suffice even to show that every

* Supported by National Science Foundation grant GP–19707.
embedding of a non-closed, orientable \( n \)-manifold into \( S^{n+1} \) is two-sided.
In fact, about the most general result of this nature that one can obtain with those techniques seems to be that an orientable, closed \( n \)-manifold
in an orientable, closed \((n+1)\)-manifold with \( n \)-dimensional Betti number zero is two-sided (c.f., p. 294 of [10]) (1). Many times, however, one is only interested
in locally flat embeddings, as we shall see later in this paper.
Our first theorem states that locally flat embeddings of simply-connected
\( n \)-manifolds into arbitrary \((n+1)\)-manifolds are two-sided. (Of course,
simply connected manifolds are orientable (c.f., p. 116 of [5]).) The proof
is completely geometrical and involves no algebraic topology. Thus, for
the special case of locally flat embeddings of closed, simply-connected
manifolds, the classical result mentioned above can be proved without
using algebraic topology or the formal notion of orientability. In fact,
for such embeddings, that result is generalized by dropping the require-
ment that the ambient manifold be orientable and have \((n)\)-Betti number
zero. It is quite important to notice that our first theorem is not true
if the simply-connectivity hypothesis is replaced by an orientability
hypothesis. To see an example, take a Möbius band with the boundary
excluded and cross it with \((0, 1)\). Then, the center \(1\)-sphere crossed with
\((0, 1)\) is a locally flat, orientable (non-simply-connected) submanifold of
the product which is not two-sided. By taking further products with \((0, 1)\),
this example generalizes to higher dimensions. Similar examples, where
the manifolds are closed, are obtained by taking the natural inclusion
of each odd dimensional projective space into the next higher even
dimensional projective space.

Because of the innocent appearance of the first result, especially
for the case the ambient manifold is \( E^{n+1} \), it is natural for one to feel
that he can find a rigorous half-page proof in a few minutes. Perhaps
it would be wise to comment on what the author believes is the "obvious
proof" of Theorem 1 that would quickly occur to almost any topologist.

"Well, we have this orientable \( n \)-manifold \( M^n \) sitting in \( E^{n+1} \), so
at every point \( z \) of \( M \) we have some local orientation. Now, take a
flattening neighborhood of \( M \) at \( z \) and point your index finger in the
direction of the orientation and call the part of the flattening
neighborhood in the direction your thumb is pointing the plus part
and the part in the other direction the minus part of the flattening
neighborhood. Now after cutting down the flattening neighborhoods
a bit, it is easy to see that their union is a connected neighborhood
of \( M \) which is separated by \( M \) into the part made
up of the union of all the plus parts of the flattening neighborhoods,
and the part made up of the union of all the minus parts of the flattening
neighborhoods."

Even if this argument were put in precise language, it would not
work, because it does not use any properties of the ambient manifold
nor does it use the fact that \( M \) is simply connected. The two examples
at the end of the paragraph before last demonstrate the problem with
this argument.

Let \( X \) be a topological space and \( Y \) a subset of \( X \). Then, \( X \) is collared
in \( X \) if there is a homeomorphism \( h \) carrying \( Y \times [0, 1] \) onto a
neighborhood of \( Y \) such that \( h(y, 0) = y \) for all \( y \in Y \).
We call \( h(Y \times [0, 1]) \) a collar of \( Y \) in \( X \). If there is a
homeomorphism \( h \) carrying \( Y \times [-1, 1] \) onto a neighborhood of \( Y \)
such that \( h(y, 0) = y \) for all \( y \in Y \), then \( Y \) is biocollared in \( X \).
We call \( h(Y \times [-1, 1]) \) a biocollar of \( Y \) in \( X \).
Let \( f : M^n \to N^{n+1} \) be an embedding of an \( n \)-manifold without boundary \( M^n \)
into the interior of an \((n+1)\)-manifold \( N^{n+1} \). Then, \( f \) is flat if \( f(M) \)
is biocollared in \( N \). Now, suppose that \( M^n \) is an \( n \)-manifold with boundary
and let \( \partial M^n = \partial M \) where \( g : \partial M \to \partial M \times [0, 1] \)
and \( g(x) = (x, 0) \). Then, an embedding \( f : M^n \to \text{Int}(N^{n+1}) \)
is flat if \( f \) extends to an embedding \( \tilde{f} \) of \( M^n \) into \( \text{Int}(N^{n+1}) \)
which is flat. In Theorem 2, we will show that every locally flat embedding
of a connected, simply-connected \( n \)-manifold \( M^n \) into the interior of a
non-empty \((n+1)\)-manifold \( N^{n+1} \) is flat.

Suppose that \( M^n \) is an \( n \)-submanifold of the \((n+1)\)-manifold \( N^{n+1} \)
and that \( Z \) is a closed subset of \( M^n \). If there is a homeomorphism
\( h : (M \times [-1, 1])/(\{z, 0\}) \to Z \) if \( z \in Z \), \(-1 \leq t \leq 1 \)

(1) John Bryant and David Galewski have pointed out that more general results
than this can be obtained by non-geometrical methods which involve applying a rather
sophisticated sequence argument in a covering space.
The author would like to thank Eliot Chamberlin and Les Glaser for interesting conversations regarding this paper.

2. Formal statements and proofs of results. Throughout this section, except for the lemma, we assume that \( n \geq 4 \). For the case \( n = 4 \), we also assume that the manifold \( M \) supports a PL structure.

**Lemma.** Let \( D_s^k \) be a k-cell with bicolared boundary in a connected \( k \)-manifold \( M^k \). Then, for any point \( x \in M^k \), there is a k-cell \( D_x^k \) with bicolled boundary such that \( x \in \text{Int}(D_x^k) \) and \( D_x^k \setminus D_s^k \) lies in the interior of a k-cell with bicolled boundary.

Proof. Consider the set \( X \) of all points \( x \) of \( M^k \) for which there is a k-cell \( D_x^k \) with bicolled boundary such that \( x \in \text{Int}(D_x^k) \) \&\( D_x^k \setminus D_s^k \) lies in the interior of a k-cell with bicolled boundary. Certainly, \( \text{Int}(D_x^k) \subseteq X \) and so \( X \) is nonempty. It is routine to show that \( X \) is both open and closed and so the lemma follows since \( M \) is connected.

**Theorem 1.** A connected, simply-connected \( n \)-manifold \( M^n \) which is contained locally flatly in the interior of an arbitrary \((n+1)\)-manifold \( X^{n+1} \) is two-sided.

Of course, it is only necessary to establish the case that \( Bd \ M^n = \emptyset \). Since the proof is somewhat long, we break it down into the following steps. (These steps give an outline of the proof.)

**Step 1.** For a fixed point \( p \) of \( M \), we construct a flattening \((n+1)\)-cell neighborhood \( D^{n+1}_x \) of \( M \) \( \cap \) \( X \) at \( p \) and define the plus part, \( D^{n+1}_s \), and the minus part, \( -D^{n+1}_s \).

**Step 2.** We cover \( M \) with a collection of flattening \((n+1)\)-cell neighborhoods by putting each point \( x \) of \( M \) in a certain flattening \((n+1)\)-cell, denoted by \( D^{n+1}_x \). Then, we generate another cover of flattening \((n+1)\)-cells which contains the previous cover as a proper subset by defining for each \( x \geq 0 \) and each \( y \in D^{n+1}_x \), \( D^{n+1}_y \), \( D^{n+1}_y \) which contains \( y \). Finally, we define the plus part, \( D^{n+1}_s \), and the minus part, \( -D^{n+1}_s \), of \( D^{n+1}_s \) relative to the plus part and \( -D^{n+1}_s \), mentioned in Step 1.

**Step 3.** In order to know that \( D^{n+1}_s \) and \( -D^{n+1}_s \) are well-defined, we must show that their determination is independent of the choice of the above steps.

**Part a.** A certain homeomorphism \( h_D \) and \( h_M \) of \( D_s \) and \( M \), respectively, each of which will have been constructed in completing Step 2.

**Part b.** A certain \( n \)-cell \( D^s \) (the simple connectivity is used here) each of which will have been constructed in completing Step 2.

**Step 4.** We take a countable subcover \( \left\{ D^{n+1}_x \right\}_{x \in M} \) of the cover \( \left\{ D^{n+1}_x \right\}_{x \in M} \) mentioned in Step 2 and "cut down" its members to obtain a new collection of flattening \((n+1)\)-cells denoted by \( \left\{ D^{n+1}_x \right\}_{x \in M} \).

**Step 5.** We take the \( U \) in the definition of two-sidedness to be \( \bigcup_{x \in M} \text{Int}(D^{n+1}_x) \) and show that \( M \) separates it into the two complementary domains \( \bigcup_{x \in M} \text{Int}(+D^{n+1}_x) \) and \( \bigcup_{x \in M} \text{Int}(-D^{n+1}_x) \). (This step involves use of the second cover constructed in Step 2.)

**Notation.** Let \( I^k = [-1,1] \) and \( I^{k+1} = I^k \times I^k \). For a point \( y \in I^k \), let \( N(I^k) = \{ x \in I^k \} \} dist(x,y) \leq \epsilon \}, \) which is a closed k-cell, and let \( \psi_{y,\epsilon} = \pi(x,y) \times I^k \). Let \( +I^{k+1} = I^k \times [0,1] \), \( -I^{k+1} = I^k \times [-1,0] \), \( +I^{k+1} \cap +I^{k+1} \), \( -I^{k+1} \cap -I^{k+1} \).

**Proof of Step 1.** Fix a point \( p \) of \( M \) and let \( W_p \) be a neighborhood of \( p \) in \( M \) for which there is a homeomorphism \( h : (W_p, W_p \cap M) \to (\mathbb{R}^{n+1}, \mathbb{R}^n) \). Let \( D^s \) be a neighborhood of \( p \) in \( M \) and \( D_s = D^s \cap M \). Then, \( D_s \) is an n-cell with bicolled boundary in the connected \( n \)-manifold \( M \), as in the statement of the lemma. Let \( h_p : (\mathbb{R}^{n+1}, D^s) \to (\mathbb{R}^{n+1}, \mathbb{R}^n) \) denote \( h_*(D^s) \) and denote \( h_p^*(+I^{n+1}) \) by \( +I^{n+1} \) and \( h_p^*(-I^{n+1}) \) by \( -I^{n+1} \).

**Proof of Step 2.** Now let \( x \) be an arbitrary point of \( M \). By the lemma, there is an n-cell \( D^s \) with bicolled boundary such that \( x \in \text{Int}(D^s) \) and \( D^s \setminus D^s \) lies in the interior of an n-cell with bicolled boundary. It is easy to see (by an argument similar to the one in the preceding step that we may assume that there is an \((n+1)\)-cell \( D^{n+1}_x \) such that \( D^{n+1}_x \cap M = D^s \) and a homeomorphism \( h_x : (D^{n+1}_x, D^{n+1}_x \cap M = D^s) \to (\mathbb{R}^{n+1}, \mathbb{R}^n) \). For \( y \in \text{Int}(D^s) \), we will use the notation \( D^{n+1}_y = h_x(D^{n+1}_y) \) and \( D^{n+1}_y = h_x^{-1}(\mathbb{R}^{n+1} \cap M) \) is contained in the interior of an n-cell with bicolled boundary, hence, for each \( y \in \text{Int}(D^s) \) and \( \epsilon > 0 \), \( D^{n+1}_y \) is contained in the interior of an n-cell with bicolled boundary.

We wish now to define \( D^{n+1}_s \) and \( D^{n+1}_s \). In order to do so let \( D^s \) be any n-cell with bicolled boundary which contains \( D^s \cap D^s \) in its interior. (We know that such \( D^s \)'s exist by the last sentence of the preceding paragraph.) Let \( h_0 : U_0 \to \mathbb{R}^{n+1} \) be a homeomorphism of a neighborhood of \( U_0 \) of \( D^s \) such that \( h_0(\mathbb{R}^{n+1} \cap \mathbb{R}^n) \) (\( \mathbb{R}^{n+1} \) is locally flat by transitivity of local flatness, hence \( h_0 \) and \( h_0 \) exist by \( \{\} \).) It is easy to show that there is some \( \epsilon > 0 \) such that \( h_0(D^s) \cap (\mathbb{R}^{n+1} \cap \mathbb{R}^n) \) is a neighborhood of \( h_0(D^s) \) in \( \mathbb{R}^{n+1} \). (Without loss of generality, we may assume that \( h_0(D^s) \cap (\mathbb{R}^{n+1} \cap \mathbb{R}^n) \) is a neighborhood of \( h_0(D^s) \) in \( \mathbb{R}^{n+1} \).) Now there is a \( \delta > 0 \) such that \( h_0(D^s) \cap (\mathbb{R}^{n+1} \cap \mathbb{R}^n) \) is a neighborhood of \( h_0(D^s) \) in \( \mathbb{R}^{n+1} \), \( \mathbb{R}^{n+1} \), \( \mathbb{R}^{n+1} \), \( \mathbb{R}^{n+1} \), \( \mathbb{R}^{n+1} \).
Proof of Step 3. Part a). We now show independence of the choice of \( h_0 \) of \( h_0 \) in Step 2. In order to do so fix an \( n \)-cell \( D \) with bicollared boundary in \( M \) such that \( D_0 \cup D_0^c \subset \text{Int } D \) and suppose that \( k_0 \) and \( k_0^c \) are two homeomorphisms of neighborhoods \( U_0 \) and \( U_0^c \) of \( D_0 \) onto \( E^{n+1} \) such that \( h_0(D) \) and \( h_0(D) \) are contained in \( E^n \) and for which there exists some \( \varepsilon > 0 \) such that \( k_0 h_0^{-1}(I^{n}[0, \varepsilon]) \) and \( k_0^c h_0^{-1}(I^{n}[0, \varepsilon]) \) are neighborhoods of \( k_0(\text{Int } D_0) \) and \( k_0^c(\text{Int } D_0^c) \), respectively, in \( E^{n+1} \). Let \( D' \) be a bicollared \( n \)-cell contained in \( \text{Int } D \) concentric with \( D \) such that \( D_0 \cup D_0^c \subset \text{Int } D' \). Then, obviously \( h_0(D') \) is collared in \( E^{n+1} \). In fact, we can easily choose a collar \( C \) of \( h_0(D') \) in \( E^{n+1} \) such that

1. \( h_0^{-1}(C) \cap D_0 \cap D_0^c \),
2. \( \text{l.u.b.} \{ \text{dist}(x, y) \mid x \in C, y \in h_0(D') \} < \text{dist}(h_0(D'), E^n - h_0(D)) \),
3. \( \text{l.u.b.} \{ \text{dist}(x, y) \mid x \in C, y \in h_0(D') \} < \text{dist}(h_0(D'), E^n - h_0(D)) \).

It is not hard to conclude that \( k_0 h_0^{-1}(C) \cap \text{Int } E^{n+1} \) is a collar of \( h_0(D') \) in \( E^{n+1} \). First, we can find some point in \( h_0^{-1}(C) \cap \text{Int } E^{n+1} \) since by definition of \( h_0 \) for small \( t > 0 \), there are points \( x, t \in I^{n}[0, \varepsilon] \) such that \( \text{dist}(x, t) < \text{dist}(h_0(D'), E^n - h_0(D)) \). By choosing \( \varepsilon \) smaller, we can add the conditions that

\[ k_0 h_0^{-1}(I^{n}[0, \varepsilon]) \subset C \quad \text{and} \quad k_0^c h_0^{-1}(I^{n}[0, \varepsilon]) \subset C. \]

Suppose that there is a \( \delta > 0 \) such that \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \), respectively, in \( E^{n+1} \). By choosing the above \( \delta \) smaller, we can assume that \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \subset C \). Thus, since \( k_0 h_0^{-1}(C) \) is a collar of \( h_0(D') \) in \( E^{n+1} \), it follows that \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \), respectively, in \( E^{n+1} \), and so if \( k_0 h_0^{-1} \) were defined to be \( E^{n+1} \) the same set determined by \( h_0 \) would result.

In the case that there exists a \( \delta > 0 \) such that \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \), an argument similar to the above would show that the determination of \( -E^{n+1} \) and \( -E^{n+1} \) is independent of the choice of \( D' \). In order to do so, suppose that we have two \( n \)-cells \( D_0 \) and \( D_0^c \) in \( M \) with bicollared boundaries each of which contains \( D_0 \cup D_0^c \subset \text{Int } M \). First, we wish to establish the following claim: the independence of the determination of \( +E^{n+1} \) and \( -E^{n+1} \) will follow if we can show that the homeomorphism \( h \) of the pair \((D_0, M) \) onto itself such that \( h(D_0) \subset D_0^c \) and \( h(D_0^c) \subset D_0 \) is independent of the \( \varepsilon \)-neighborhood of \( y \) in \( +E^{n+1} \). Let \( k_0 \) be a homeomorphism of \( U_0 \), constructed in Step 2, onto \( E^{n+1} \) such that \( k_0(D_0) \subset E^n \) and for which there exists a \( \varepsilon > 0 \) such that \( k_0 h_0^{-1}(I^{n}[0, \varepsilon]) \) is a neighborhood of \( k_0(\text{Int } D_0^c) \) in \( E^{n+1} \). Let us assume that there is a \( \delta > 0 \) such that \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \) in \( E^{n+1} \). (The proof of the case where \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \) in \( E^{n+1} \) is similar to this case.) Now consider the homeomorphism \( h \circ k_0 \) of \( U_0 \) onto \( E^{n+1} \). Since \( k_0(D_0) \subset E^n \), we have that \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \) in \( E^{n+1} \). Also, since \( k_0 h_0^{-1}(I^{n}[\delta, \varepsilon]) \times [0, \delta] \) is a neighborhood of \( k_0(\text{Int } D_0^c) \) in \( E^{n+1} \) and the claim is satisfied.

Now we will establish the above claim. First, we run an arc \( a_0 \) from \( y \) out the radial structure on \( E^n \), and on out the radial structure of a collar \( C_0 \) of \( E^n \), on the outside of \( D_0 \), to a point \( y' \) in \( \text{Int } D_0 \). Similarly, we run an arc \( a_0 \) from \( y \) out the radial structure on \( D_0^c \) and on out the radial structure of a collar \( C_0 \) of \( E^n \), on the outside of \( D_0 \), to a point \( y' \) in \( \text{Int } D_0^c \). Now \( y' \) and \( y' \) can be connected by a locally flat arc \( a_0 \) in \( C_0 \) of \( D_0 \setminus (D_0 \cup D_0^c) \), and \( y' \) and \( y' \) can be connected by a locally flat arc \( a_0 \) in \( C_0 \) of \( D_0 \setminus (D_0 \cup D_0^c) \). By taming \( a_0 \) and \( a_0 \) and \( a_0 \) and \( a_0 \) in a PL structure on \( E^n \) and then applying general position, we may assume that \( a_0 \cap a_0 = y' \), and that \( a_0 \cap a_0 = y' \), and that \( a_0 \cap a_0 = y' \). (To do the taming one could use [41], for instance.) Since \( M \) is simply-connected, \( M = M \cup (D_0 \cup D_0^c) \) is simply-connected and so the simple closed curve \( a_0 \cup a_0 \cup a_0 \) bounds a singular disk \( D_0 \) in \( M \). We would like for \( a_0 \cup a_0 \cup a_0 \) to bound a real, locally flat disk in \( M \). If \( M \) could support a triangulation, this would follow by general position for \( s > 5 \). However, for the case \( M \) is a topological manifold, we can obtain the desired disk \( D_0 \) by applying taming and general position on small patches of \( D_0 \) and \( M^n \) (c.f. [23]). Now in the case \( n = 4 \), we can apply general position to get a singular disk \( D_0 \) with point singularities. Choose a small real disk \( D_0 \)
in $\text{Int}D_n$. Then, $\text{Bd}D^n_0$ and $\text{Bd}D_0 = \partial D_n \cup \partial D_0$ are homotopic and since for $n = 4$ we are assuming $M$ can support a PL structure, we can carry
\[ \text{Bd}D^n_0 \to \text{Bd}D_0 \] with a space homomorphism by standard unknotting theorems (c.f., Theorem 24 of [11]). Hence, again $\partial D_n \cup \partial D_0$ bounds a locally flat disk, $D$. Let $\partial D$ be a locally flat arc in $\partial \cap \text{Int}D_n$ with endpoints $p'$ and $y'$ such that $\text{Int} \partial D \subset \text{Int}D_n$ and let $\partial D_0$ be a locally flat arc in $\partial \cap \text{Int}D_0$ with endpoints $p'$ and $y'$ such that $\text{Int} \partial D_0 \subset \text{Int}D_0$. Since $\partial D$ is locally flat in $M$, it follows from [7] that there is a neighborhood $U$ in $M$ such that $\{U, \partial D\} \cong (\mathbb{S}^n, \partial \mathbb{S}^n)$. Thus, there is a locally flat $n$-cell $D$ in $U$ such that $(\partial D, \partial D_0)$ is a trivial cell pair. By using Lemma 3.6 of [8], we may assume that $\partial D \cap (\partial D_0 \cup \partial D_0 \cup \partial D_0) = y' \cup p'$. Notice that $(\partial D, \partial D_0)$ and $(\partial D, \partial D_0)$ are also trivial cell pairs, hence, by unknotting locally flat cell pairs, there is a homomorphism $h_0$ of $D$ onto itself such that $h_0|\text{Bd}D = 1$ and $h_0|\partial D_0 = \partial D_0$. (To unknot cell pairs see [3].) The case $n = 4$ follows from standard tameness theorems and PL unknotting theorems.) Now, by transitivity of local flatness, $D$ is locally flat in $M^{n+1}$, hence, by applying the lemma in the appendix of [7] and Lemma 3.6 of [8], we can obtain a locally flat $(n+1)$-cell $D^{n+1}$ in $M^{n+1}$ such that $\partial D^{n+1} = D^n$. It is easy to extend $h_0$ to a homomorphism $h_0$ of $D^{n+1}$ such that $h_0|\text{Bd}D^{n+1} = 1$. Let $h_0$ be the extension of the homomorphism over $M^{n+1}$ by way of the identity. Then $h_0$ is a homomorphism of $(M^{n+1}, M)$ onto itself such that $h_0\partial D \cup \partial D_0 \cup \partial D_0 = \partial D \cup \partial D_0 \cup \partial D_0$ and $h_0|\text{Int}D^{n+1} = 1$ for some $\varepsilon > 0$. (However, $h_0$ is not the homomorphism $h$ which we seek for it is not necessarily true that $h_0(\text{D}_2^{n+1}) \subset \mathbb{D}^n$.)

We have that the boundary of $h_0(\mathbb{D}^2)$ is bi-coxeral in $M^n$ so let $D$ represent a locally flat $n$-cell in $M^n$ which is the union of $h_0(\mathbb{D}^2)$ and a closed collar of $\text{Bd}h_0(\mathbb{D}^2)$ on the outside of $h_0(\mathbb{D}^2)$. By stretching out the collar, we can get a homomorphism $h_0(\text{D}_2^{n+1})$ onto itself which is the identity on $\text{Bd}D^n$ and $\text{Bd}D^n$ on a small neighborhood of $\partial D_n$ and is such that $h_0|\text{Bd}D^{n+1} \subset \partial D_0$. Again, since $D$ is locally flat in $M^{n+1}$, we can apply the lemma in the appendix of [7] and Lemma 3.6 of [8] to obtain an $(n+1)$-cell $D^{n+1}$ in $M^{n+1}$ such that $\partial D^{n+1} \cap M = D^n$. Now, by taking a suspension extension of $h_0$, we can get a homomorphism $h_0$ of $D^{n+1}$ onto itself such that $h_0|\text{Bd}D^{n+1} = 1$ and $h_0|\text{Int}D^{n+1} = 1$ for some $\varepsilon > 0$. Let $h_0$ represent the extension of $h_0$ to $M^{n+1}$ by way of the identity. Then $h_0 = h_0h_0$ is clearly the homomorphism that we seek, and this concludes the proof that $+D^n_0$ and $-D^n_0$ are well-defined.

Proof of Step 4. Let $(D^n_0)^{\#} \subset M$ denote the first cover of $M$ mentioned in Step 2. Since $M$ is separable, we can get a countable subcover which we denote by $(D^n_0)^{\#} \subset M$. Then, there is associated with each $x$ a homomorphism $h_x^1(D^n_0, D^n_0 \cap M = D^n_0) \to (\mathbb{S}^n, \partial \mathbb{S}^n)$. Let $(x_1, D^n_0, D^n_0, h_x^1)$ correspond to $(x, D^n_0, D^n_0, h_x^1)$ constructed in Step 1 and in terms of which $+D^n_0$ and $-D^n_0$ were defined in Step 2. Then, corresponding to each $D^n_0$, we have a $+D^n_0 = h_x^1(D^n_0)$ and a $-D^n_0 = h_x^1(D^n_0)$. It would be nice if we could take $\bigcup (\text{Int}D^n_0)$ as the connected open set $U$ in the definition of two-sidedness and argue that $M$ separates $U$ into the two complementary domains $\bigcup \text{Int}(+D^n_0)$ and $\bigcup \text{Int}(-D^n_0)$. This will not work though, since it is easy to see that the later two sets do not have to be disjoint. However, this is the raw idea of the remainder of the proof, but it will be necessary to cut the $D^n_0$'s down to make it work.

Let $D^n_0 = D^n_0 + D^n_0$, $+D^n_0 = +D^n_0$, $-D^n_0 = -D^n_0$, and $h_x^1 = h_x^1$. Now, we have the homomorphism $h_x^1(D^n_0, D^n_0) \to (\mathbb{S}^n, \partial \mathbb{S}^n)$ and in terms of $+D^n_0$ and $-D^n_0$ and without loss of generality we may suppose that $+D^n_0 = h_x^1(D^n_0, D^n_0)$ and $-D^n_0 = h_x^1(D^n_0, D^n_0)$. It is not difficult to show that there exists an $\varepsilon > 0$ such that $h_x^1(\mathbb{S}^n \times (0, \varepsilon)) \cap -D^n_0 = \emptyset$ and $h_x^1(\mathbb{S}^n \times (0, \varepsilon)) \cap +D^n_0 = \emptyset$. Let $d^n_0 = h_x^1(\mathbb{S}^n \times (-\varepsilon, \varepsilon))$. Let $h_x^1 : \mathbb{S}^n \times (-\varepsilon, \varepsilon) \to D^n_0$ be defined by taking $x \times [-\varepsilon, \varepsilon]$ linearly onto $x \times [-1, 1]$ for each $x \in \mathbb{S}^n$ and let $h_x^1 = h_x^1$. Inductively, we have the homomorphism $h_x^1(D^n_0, D^n_0) \to (\mathbb{S}^n, \partial \mathbb{S}^n)$ and in terms of $+D^n_0$ and $-D^n_0$ and without loss of generality we may suppose that $+D^n_0 = h_x^1(D^n_0, D^n_0)$ and $-D^n_0 = h_x^1(D^n_0, D^n_0)$. It is not difficult to show that there exists an $\varepsilon > 0$ such that $h_x^1(\mathbb{S}^n \times (0, \varepsilon)) \cap (\bigcup \text{Int}(+D^n_0) \cap \text{Int}(-D^n_0)) = \emptyset$ and $h_x^1(\mathbb{S}^n \times (0, \varepsilon)) \cap (\bigcup \text{Int}(+D^n_0) \cap \text{Int}(-D^n_0)) = \emptyset$. Let $d^n_0 = h_x^1(\mathbb{S}^n \times (-\varepsilon, \varepsilon))$, and define $h_x^1$ similar to the definition of $h_x^1$ above.

Proof of Step 5. We take the $U$ in the definition of two-sidedness to be $\bigcup \text{Int}(+d^n_0)$. Certainly, $U$ is connected since $M$ is connected. It is also clear by construction that the two open sets $\bigcup \text{Int}(+d^n_0)$ and $\bigcup \text{Int}(-d^n_0)$ are disjoint. Hence, the proof will be complete after we show that the two sets $\bigcup \text{Int}(+d^n_0)$ and $\bigcup \text{Int}(-d^n_0)$ are connected.

We will only show that $\bigcup \text{Int}(+d^n_0)$ is connected for the proof that $\bigcup \text{Int}(-d^n_0)$ is connected would be similar. Let $x$ be an arbitrary point of $\bigcup \text{Int}(+d^n_0)$. Then, it will suffice to show that there is an arc in $\bigcup \text{Int}(+d^n_0)$ which connects $x$ to some point of $\text{Int}(+d^n_0)$. Fundamenta Mathematicae, T. XXVI.
Two sided submanifolds, flat submanifolds and pinched biolars

Given $q_1, \ldots, q_k$, obtain $q_{k+1}$ by going through the construction where $(q_1 f, q_1f(D_n^{-1}), q_2f, \ldots, q_kf(D_n^{-1}), h_1q_1f \ldots q_kf)$ plays the role of $(f, D_n^{-1}, D_n^{-1}, h_1)$. Then, $q_1f \ldots q_kf$ is the homeomorphism desired.

Since $f(M)$ is locally flat in $f(\text{Int} M)$, it follows that $f(M) - \{\text{Int} M \}$ is a manifold and as such has a collared boundary by [1]. Thus, there is a homeomorphism $\lambda : (\text{Bd} M \times [0, 1]) \to f(M) - f(\text{Int} M)$ such that $\lambda(x, 0) = q(x)$ for $x \in \text{Bd} M$. By the first sentence of this proof, $f(\text{Int} M)$ is bicollied in $N$; hence, $q(f(M) \cup \{0, 1\})$ is bicollied in $N$. Finally, $q^{-1}[f(M) \cup \{0, 1\}] = M \cup q^{-1}[-\lambda([0, 1])]$ is bicollied in $N$. But, if $M' = M' \cup q(\text{Bd} M \times [0, 1])$, where $g : \text{Bd} M \to \text{Bd} M \times 0$ is defined by $g(x) = (x, 0)$, then $f(M) \to f(\text{Int} M)$ extends to $f(M' \cup \{0, 1\})$ be letting $f(x, 0) = q^{-1}(-\lambda(x, 0))$. Since we have just seen $f(M) = f(M) \cup q^{-1}[-\lambda([0, 1])]$ to be bicollied, the conclusion follows.

Remark. A special case of the above proof yields that a locally flat $(n-1)$-sphere in $S^n$ is bicollied. Hence, it is not necessary to use the Jordan–Brouwer Separation Theorem to obtain this result as is usually done.

THEOREM 3. If $M$ is an $n$-manifold which is contained in the interior of an $(n+1)$-manifold $X^{n+1}$ and $Z$ is a closed subset of $M$ such that $M - Z$ is connected, simply-connected and locally flat in $X^{n+1}$, then there is a bicollier of $M$ in $X^{n+1}$ pinched at $Z$.

Proof. Theorem 3 follows immediately from Theorem 1 and the Pinched Bioller Lemma of [9].

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