The sequence $M_1, M_2, \ldots$ satisfies the hypothesis of Theorem 1, thus M is indecomposable. For each i, M_i is decomposable, hence M contains a decomposable continuum. Let H denote the collection to which h belongs only if, for some i, h is an element of H_i. Every element of H has a diameter at least 1 and not greater than 4. Since $\overline{H^*}$ contains M_1, which has a diameter greater than 4, $\overline{H^*}$ has a diameter greater than 4. Thus if a is a convergent sequence, each term of which is an element of H, the limiting set of a is a nondegenerate proper subset of $\overline{H^*}$. If $\varepsilon > 0$, there is an i such that $1/i < \varepsilon$, and there is a $\delta > 0$ such that $\delta < l(\overline{d}_1, \overline{d}_2)$ for each two elements d_1 and d_2 of D_i, $\delta < l(\overline{e_1}, \overline{e_2})$ for each two elements e_1 and e_2 of E_i, and $\delta < l(\bar{e}, \bar{d})$ for each element e of E_i and each element d of D_i that does not contain an end of the element of H_i lying in e but not in $\overline{D_i^*}$; δ is a positive number such that if h' and h'' are two elements of H and $l(h', h'') < \delta$, then either $u(h', h'') < \varepsilon$ or $u(h'', h') < \varepsilon$. Thus the collection H satisfies the hypothesis of Theorem 2. Since M is $\overline{H^*}$, it follows that M is filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua.

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One-dimensional n-leaved continua

by

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It is well-known ([3], p. 60) that all one-dimensional continua are embeddable in Euclidean 3-dimensional space. A continuum is a compact connected separable metric space. Continua which are embeddable in Euclidean 2-dimensional space are called planar continua; one-dimensional planar continua have been extensively studied, see for example [8]. In this note we study certain one-dimensional continua that generalize the notion of planar continua. All planar continua are embeddable in a geometric 2-simplex. An *n-book*, $B(n)$ for $n \in Z$ (Z denoting the positive integers), is the union of n geometric 2-simplexes such that each pair of 2-simplexes meets precisely on a single geometric 1-simplex B on the face of each. The 2-simplexes are called the *leaves* of $B(n)$ and B is its *back*. Planar one-dimensional continua are said to be 1-*leaved*. A one-dimensional continuum X is said to be n-*leaved* ($n \geqslant 3$) if X embeds in $B(n)$ but does not embed in $B(k)$ for $0 < k < n$. Of course, there are one-dimensional continua that are not n-leaved for any $n \in Z$, for example the universal curve [1].

Utilizing Sierpiński's universal plane curve [6], we construct a universal n-leaved continuum. It is shown that all one-dimensional subcontinua of a surface (a compact connected 2-manifold) are n-leaved where $0 < n \leqslant 3$. Borsuk ([2], p. 79) has given an example of a locally plane and locally connected one-dimensional continuum which is not embeddable in any surface. This continuum is shown to be 3-leaved.

First, we construct a universal n-leaved continuum ($n \neq 2$). Let $D_1, D_2, \ldots$ be a sequence of closed disks in $B(n)$ such that D_i, for all $i \in Z$, does not intersect a 1-simplex in the face of any of the 2-simplexes in $B(n)$, $\bigcup_{i=1}^{\infty} D_i$ is dense in $B(n)$, and the diameters of the disks D_i converge to zero. Let $S(n) = B(n) - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ (Int = interior in the sense of manifolds). It follows from results of Whyburn [7] that $S(n)$ intersected with a leaf of $B(n)$ is homeomorphic to Sierpiński's universal plane curve and that if another sequence of disks $E_1, E_2, \ldots$ satisfy the same conditions as the disks $D_1, D_2, \ldots$ then $S(n)$ is homeomorphic to $B(n)-\bigcup_{i=1}^{\infty}\operatorname{Int}E_i$. We next prove that $S(n)$ is a universal n-leaved continuum $(n \neq 2)$.

Let J denote the one-dimensional continuum which is the union of all edges of a tetrahedron and of a segment joining two points lying in the interiors of two opposite edges of it. Then J is one of Kuratowski's primitive skew curves and is not embeddable in the plane [4]. This fact is needed in the proof of the following theorem.

THEOREM 1. *$S(n)$ is a one-dimensional n-leaved continuum* $(n \neq 2)$. *All n-leaved one-dimensional continua are embeddable in $S(n)$.*

Proof. By construction $S(n)$ embeds in $B(n)$. Assume that there exists an embedding h of $S(n)$ into $B(k)$ for $n \geqslant 3$ and $0 < k < n$ and reach a contradiction. Let B' denote the back of $B(n)$ and B denote the back of $B(k)$. If $z \in B' \subset S(n)$ then z is contained in arbitrarily small subsets of $S(n)$ homeomorphic to J. Thus $h(B') \subset B$. Since h is a uniform homeomorphism, it follows that there is a neighborhood N of z in $S(n)$ and three leaves B_1, B_2, B_3 of $B(n)$ such that $h\big(N \cap (B_1 \cup B_2 \cup B_3)\big)$ is contained in precisely two leaves of $B(k)$. But $N \cap (B_1 \cup B_2 \cup B_3)$ contains a subset homeomorphic to J which is not embeddable in E^2. Thus such a homeomorphism h does not exist and $S(n)$ is n-leaved.

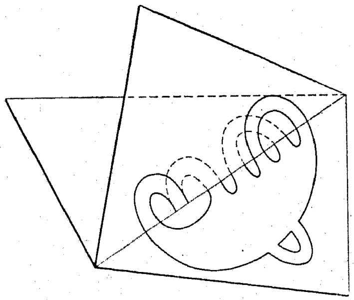

Fig. 1

If X is a one-dimensional n-leaved continuum then it follows from reasoning analogous to that of the Sierpiński universal plane curve that X embeds in $S(n)$.

LEMMA. *All surfaces with non-empty boundary embed in a 3-book.*

Proof. It is well-known that all compact connected 2-manifolds with non-empty boundary are homeomorphic to (i) a disk with $r \geqslant 0$ single loops and $h \geqslant 0$ double loops or (ii) a disk with $r \geqslant 0$ single loops and $q \geqslant 0$ twisted loops ([5], p. 43). Thus to prove that all compact connected 2-manifolds embed in a 3-book, it suffices to consider only disks with various types of loops. Figure 1 indicates that such "disks" can be embedded in a 3-book.

The next theorem follows immediately from the above lemma.

THEOREM 2. *One-dimensional subcontinua of a surface are n-leaved,* $0 < n \leqslant 3$.

K. Borsuk has given an example of a locally plane and locally connected one-dimensional continuum Y which is not embeddable in any surface ([2], p. 79). We use the notation of Borsuk. Figure 2 part (a)

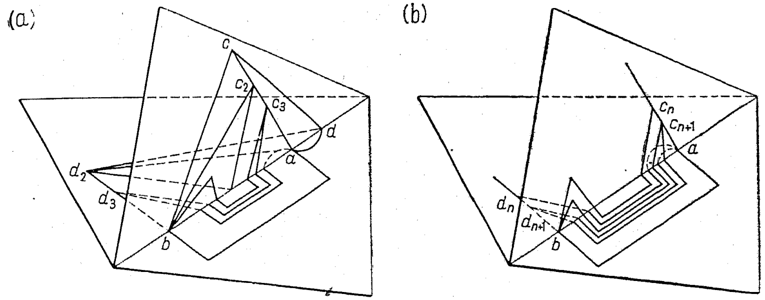

Fig. 2

shows an embedding of $X_1 \cup X_2 \cup X_3 - \overline{c_4 d_4}$ in $B(3)$. In general, if $\bigcup_{i=1}^{n} X_i - \overline{c_{n+1}d_{n+1}}$ has been embedded in $B(3)$, then $X_{n+1} - \overline{c_{n+2}d_{n+2}}$ is embedded in $B(3)$ as in Figure 2 part (b). Thus continuing in this manner it is clear that X embeds in $B(3)$ and also that Y embeds in $B(3)$. Hence there exist locally plane and locally connected one-dimensional continua that are 3-leaved but do not embed in any surface.

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