Concerning indecomposable continua and upper semi-continuous collections of nondegenerate continua*

by

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The first theorem of this paper gives a condition sufficient to ensure that the closure of the union of the terms of a sequence of continua be an indecomposable continuum. The second gives a set of conditions sufficient to ensure that the closure of the union of some continua be filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua. These two theorems hold in any metric space. A corollary to the second theorem is that every compact, metric, hereditarily indecomposable continuum is filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua. The remainder of the paper is concerned with the description, in the plane, of a compact indecomposable continuum which is filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua and which contains a decomposable continuum. The terminology and notation used in this paper is, with a few exceptions, that of R. L. Moore [2].

THEOREM 1. *Suppose $M_1, M_2, \ldots$ is a sequence of continua such that for each positive integer n, M_n is a proper subset of M_{n+1}, $\overline{M_{n+1}-M_n}$ is an irreducible continuum from M_n to some point of $M_{n+1}-M_n$, and*

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$u(M_n, M_{n+1}-M_n) < 1/n$ [1]. *Then if* $\overline{M_1 \cup M_2 \cup \ldots}$ *is hereditarily unicoherent, it is an indecomposable continuum.*

Proof. Clearly, $\overline{M_1 \cup M_2 \cup \ldots}$ is a nondegenerate continuum. Suppose it is hereditarily unicoherent. Suppose H and K are two continua whose union is $\overline{M_1 \cup M_2 \cup \ldots}$ Suppose that, for some n, M_n is a subset neither of H nor of K. Let P and Q denote points of $M_n \cap (H-H \cap K)$ and $M_n \cap (K-H \cap K)$ respectively. There is an $\varepsilon > 0$ such that $l(P, K) > \varepsilon$ and $l(Q, H) > \varepsilon$. There is an integer $i > n$ such that $1/i < \varepsilon$. Then $M_{i+1}-M_i$ intersects both $H-H \cap K$ and $K-H \cap K$, since $l(P, \overline{M_{i+1}-M_i}) < 1/i$ and $l(Q, M_{i+1}-M_i) < 1/i$. For some point X of $M_{i+1}-M_i$, $\overline{M_{i+1}-M_i}$ is irreducible from X to M_i. Either H or K contains X. Suppose H does. Then $\overline{(M_{i+1}-M_i)} \cap H$ is a proper subcontinuum of $\overline{M_{i+1}-M_i}$ containing X. Since $M_i \cap H$ is a proper subcontinuum of $\overline{M_{i+1} \cap H}$, $\overline{M_i \cap H}$ contains a point of $\overline{M_{i+1} \cap H-M_i \cap H}$. Since $\overline{M_{i+1} \cap H-M_i \cap H}$ is a subset of $\overline{(M_{i+1}-M_i)} \cap H$, $\overline{(M_{i+1}-M_i)} \cap H$ intersects $M_i \cap H$. Thus $\overline{(M_{i+1}-M_i)} \cap H$ is a proper subcontinuum of $\overline{M_{i+1}-M_i}$ containing X and intersecting M_i. This involves a contradiction. The supposition that K contains X leads to a similar contradiction. Thus, for each n, M_n is a subset either of H or of K. Therefore either H or K contains $\overline{M_1 \cup M_2 \cup \ldots}$

THEOREM 2. *Suppose H is a collection of continua such that*

(1) $\overline{H^*}$ *is compact,*

(2) *if a is a convergent sequence, each term of which is an element of H, the limiting set of a is a nondegenerate proper subset of* $\overline{H^*}$, *and*

(3) *if $\varepsilon > 0$, there is a $\delta > 0$ such that if h' and h'' are two elements of H and $l(h', h'') < \delta$, then either $u(h', h'') < \varepsilon$ or $u(h'', h') < \varepsilon$.*

Then $\overline{H^*}$ is the union of the elements of an upper semi-continuous collection of mutually exclusive nondegenerate continua.

Proof. Suppose that $g_1, g_2, \ldots$ and $h_1, h_2, \ldots$ are two convergent sequences of elements of H having limiting sets g and h respectively, and g intersects h. Suppose neither g nor h is a subset of the other. Then there exist an $\varepsilon > 0$ and a positive integer n such that, if $i > n$, there are two points P_i and Q_i of g_i and h_i respectively such that $l(P_i, h_i) > \varepsilon$ and $l(Q_i, g_i) > \varepsilon$. There is a $\delta > 0$ such that if h' and h'' are two elements of H and $l(h', h'') < \delta$, then either $u(h', h'') < \varepsilon$ or $u(h'', h') < \varepsilon$. There is an integer $k > n$ such that $u(g, g_k) < \delta/2$ and $u(h, h_k) < \delta/2$. Then $u(g \cap h, g_k) < \delta/2$ and $u(g \cap h, h_k) < \delta/2$, therefore $l(g_k, h_k) < \delta$. Therefore either $u(g_k, h_k) < \varepsilon$ or $u(h_k, g_k) < \varepsilon$, so either $l(P_k, h_k) < \varepsilon$ or $l(Q_k, g_k) < \varepsilon$. Since $k > n$, this involves a contradiction. Therefore one of the sets g and h is a subset of the other.

For each point X of $\overline{H^*}$, let J_X denote the collection to which j belongs only if j contains X and is the limiting set of a convergent sequence of elements of H. Suppose X is a point of $\overline{H^*}$. Since J_X is a monotonic collection of closed and compact point sets, there is a sequence $j_1, j_2, \ldots$ of elements of J_X such that, for each n, j_n is a subset of j_{n+1}, and every element of J_X is a subset of some set of this sequence. For each n, j_n is the limiting set of a convergent sequence of elements of H, so there is an element h_n of H such that $u(j_n, h_n)+u(h_n, j_n) < 1/n$. The sequence $h_1, h_2, \ldots$ has $\overline{J_X^*}$ as a sequential limiting set. Thus $\overline{J_X^*}$ is itself an element of J_X. Thus $\overline{J_X^*}$ is J_X^*. Suppose Y and Z are two points of $\overline{H^*}$, and J_Y^* intersects J_Z^*. Then, since each is the limiting set of a convergent sequence of elements of H, one of J_Y^* and J_Z^* is a subset of the other. Suppose J_Y^* is a subset of J_Z^*. Then J_Z^* contains Y, so it is an element of J_Y. Thus J_Z^* is a subset of J_Y^*. Thus J_Z^* is J_Y^*. Similarly, if J_Z^* is a subset of J_Y^*, J_Y^* is J_Z^*.

Let G denote the collection to which g belongs only if, for some point X of $\overline{H^*}$, g is J_X^*. Then if g_1 and g_2 are two elements of G, g_1 and g_2 do not intersect. Each element of G is both a nondegenerate continuum and a proper subset of $\overline{H^*}$, and G^* is $\overline{H^*}$, so G is a nondegenerate collection.

Suppose $g_1, g_2, \ldots$ is a sequence of elements of G, for each n, A_n and B_n are points of g_n, and $A_1, A_2, \ldots$ converges to a point A of the element g of G. Suppose there is an infinite subsequence of $B_1, B_2, \ldots$ such that no infinite subsequence of it has a sequential limit point lying in g. Then, since G^* is closed and compact, there is an increasing sequence $n_1, n_2, \ldots$ of positive integers such that $g_{n_1}, g_{n_2}, \ldots$ converges to a set L that is not a subset of g. Since $A_{n_1}, A_{n_2}, \ldots$ converges to A, L contains A. For each positive integer k, there is a set h_k of H such that $u(g_{n_k}, h_k)+u(h_k, g_{n_k}) < 1/k$. The sequence $h_1, h_2, \ldots$ has L as a sequential limiting set. Thus L is an element of J_A. Since J_A^* is g, L is therefore a subset of g. This is a contradiction. Thus every infinite subsequence of $B_1, B_2, \ldots$ has an infinite subsequence that converges to a point of g. Hence G is an upper semi-continuous collection.

COROLLARY. *Every compact, hereditarily indecomposable continuum is filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua.*

Proof. It may be shown that if M is a compact, hereditarily indecomposable continuum, the collection G of all nondegenerate subcontinua of M satisfies condition 3 of the hypothesis of Theorem 2. There is a subcollection H of G filling up M such that every element of H has a diameter

[1] If M is a point set and P is a point, then by $l(P, M)$ is meant the lower bound of the distances from P to all the different points of M. If M and N are two point sets, then by $l(M, N)$ is meant the lower bound of the values $[l(P, N)]$ for all points P of M, while by $u(M, N)$ is meant the upper bound of these values for all points P of M. It is to be observed that $u(M, N)$ may be different from $u(N, M)$.

at least 1/3 of and no greater than 2/3 of the diameter of M. The collection H satisfies the hypothesis of Theorem 2.

DEFINITION. Let g be the graph of

$$f(x)=\begin{cases}\frac{1}{2}\sin\frac{1}{x} & \text{if} \quad 0<x\leqslant\frac{1}{2},\\ \frac{1}{2}\sin\frac{1}{1-x} & \text{if} \quad \frac{1}{2}\leqslant x<1,\end{cases}$$

and let I_1 and I_2 be the vertical intervals whose union is $\bar{g}-g$. The continuum M will be said to be a *Q-set* if and only if there is a homeomorphism h of $\bar{g}$ onto M such that $h(I_1)$ and $h(I_2)$ are vertical intervals of length 1, and no vertical line contains two points of $h(g)$. Loosely speaking, a Q-set would be a copy of a continuum which could be obtained in two reversibly continuous steps, the first step consisting of either leaving $\bar{g}$ alone, or expanding or contracting $\bar{g}$ horizontally (while keeping $\bar{g}-g$ vertical), and the second step consisting of moving I_1 or I_2 or some points of g either straight up or straight down or not at all. If M is a Q-set, the vertical intervals of M corresponding to I_1 and I_2 will be called the *ends of* M.

EXAMPLE. Let AB denote an interval of the X-axis having length 4, and let C denote a Cantor set lying in AB and containing A and B such that every component of $AB-C$ has length less than 1. For each component T of $AB-C$, let R_T denote the vertical rectangular disc (that is, a rectangular disc with two of its sides vertical) of height 1 which has $\bar{T}$ as its lower horizontal side, and let Q_T denote a Q-set lying in R_T whose ends are the vertical sides of R_T. Let M_1 denote the closure of the union of all the point sets Q_T for all components T of $AB-C$. Then M_1 is a compact continuum which is the union of the elements of an upper semi-continuous collection H_{M_1} of mutually exclusive nondegenerate continua such that h belongs to H_{M_1} only if either h is an element of the collection Q of all points sets Q_T for all components T of $AB-C$, or h does not intersect Q^* but is the limiting set of a convergent sequence of elements of Q. With respect to its elements, H_{M_1} is an arc, and the end elements of H_{M_1} are the vertical intervals of length 1 whose lower endpoints are A and B. Every maximal vertical interval of M_1 has length 1 and is either an element of H_{M_1} or an end of a Q-set element of H_{M_1}, and is a component of the union of all maximal vertical intervals of M_1. Every element of H_{M_1} is either a maximal vertical interval of M_1 or a Q-set.

DEFINITION. The statement that the continuum M is an *A-continuum* means that M is a compact plane continuum which is the union of the elements of an upper semi-continuous collection H_M of mutually exclusive continua, such that

(1) H_M is, with respect to its elements, an arc, and its end elements are vertical intervals,

(2) each element of H_M is either a maximal vertical interval of M or a Q-set,

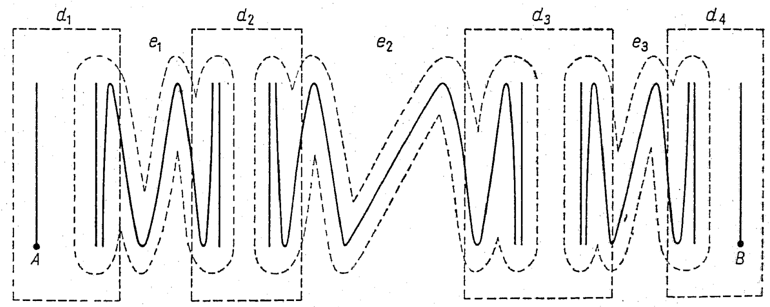
(3) each maximal vertical interval of M has length 1 and is either an element of H_M or an end of a Q-set element of H_M, and is a component of the union of all maximal vertical intervals of M, and

(4) no vertical line intersects two elements of H_M.

If M is an A-continuum, the end elements of H_M will be called the *ends of* M.

THEOREM 3. *There exists in the plane, a compact indecomposable continuum which contains a decomposable continuum and is filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua.*

Proof. Consider the example given above. The set M_1 is an A-continuum. There exists a finite collection D_1 of vertical rectangular domains (interiors of vertical rectangular discs) having height less than $1+1/2$ and width less than 1/2, that properly covers the union of all maximal vertical intervals of M_1, such that the closure of each two elements of D_1 are mutually exclusive. For each element d of D_1, neither horizontal side of $\bar{d}$ contains a point of M_1. Let $d_1, \dots, d_k$ denote the elements of D_1, numbered from left to right. For each i from 1 to $k-1$, let h_i denote the element of H_{M_1} that intersects both d_i and d_{i+1}. Let δ_1 denote a positive number less than 1/2 such that if $1\leqslant i<j<k$, $l(h_i, h_j)>\delta_1$, and for



Indication of M_1, D_1 and E_1

each element d of D_1 and each point P of a horizontal side of the boundary of d, $l(P, M_1)>\delta_1$. For each i from 1 to $k-1$, let e_i denote the set to which a point X belongs only if either X is a point of $d_i\cup d_{i+1}$ such that $l(X, h_i)<\delta_1/2$, or the vertical line λ_X containing X separates d_i from d_{i+1} and $l(X, \lambda_X\cap h_i)<\delta_1/2$. Let E_1 denote the collection of all e_i for all integers i from 1 to $k-1$.

For each i, $1 \leqslant i < k$, there exist vertical rectangular discs r_i and s_{i+1} of height 1 lying in $d_i \cap e_i$ and $d_{i+1} \cap e_i$ respectively and between the vertical lines containing the ends of h_i, such that r_i and s_{i+1} lie beneath M_1 (that is, for every vertical line λ intersecting $r_i \cup s_{i+1}$, λ intersects M_1 and $\lambda \cap (r_i \cup s_{i+1})$ lies in the component of $\lambda - \lambda \cap M_1$ that lies beneath $\lambda \cap M_1$). There exists a sequence $t_1, t_2, \ldots$ of vertical rectangular discs of height 1 lying in d_1 and beneath M_1 such that (1) t_1 is r_1, (2) for each n, each vertical line intersecting t_{n+1} is to the left of each vertical line intersecting t_n, and (3) $t_1, t_2, \ldots$ converges to L, the left end of M_1. For each i from 1 to $k-1$, there is a Q-set U_i lying in $e_i - e_i \cap M_1$, such that the ends of U_i are the right side of r_i and the left side of s_{i+1}, and there are two A-continua R_i and S_{i+1} lying in r_i and s_{i+1} respectively and having as ends the vertical sides of r_i and the vertical sides of s_{i+1} respectively. For each i from 2 to $k-1$, there is a Q-set V_i lying in $d_i - d_i \cap M_1$ such that the ends of V_i are the right side of s_i and the left side of r_i. For each $i > 1$, there is an A-continuum T_i lying in t_i whose ends are the vertical sides of t_i. For each i, there is a Q-set W_i lying in $d_1 - d_1 \cap M_1$ such that the ends of W_i are the right side of t_{i+1} and the left side of t_i. The union of L, $\bigcup_{i=1}^{\infty}(W_i \cup T_{i+1})$, $\bigcup_{j=1}^{k-2}(R_j \cup U_j \cup S_{j+1} \cup V_{j+1})$, and $R_{k-1} \cup U_{k-1} \cup S_k$ is an A-continuum N_2 such that $M_1 \cap N_2$ is the left end of both M_1 and N_2.

There is a vertical rectangular domain g containing $M_1 \cap N_2$ and having height less than $1+1/4$ and width less than $1/4$, such that $\bar{g}$ lies in d_1, neither horizontal side of $\bar{g}$ contains a point of $M_1 \cup N_2$, and M_1 and N_2 each contain only one point of the boundary of g. There is a positive number ε_2 less than $1/4$ and less than $l(M_1 - M_1 \cap g, N_2 - N_2 \cap g)$. There is a finite collection G_1 of vertical rectangular domains that properly covers the union of all maximal vertical intervals of $M_1 - M_1 \cap g$, such that each element of G_1 has height less than $1+\varepsilon_2/4$ and width less than $\varepsilon_2/4$, the closures of each two elements of G_1 are mutually exclusive, the closure of each element of G_1 lies in some element of D_1, and neither horizontal side of the closure of an element of G_1 contains a point of M_1. There exists a similar collection G_2 for the union of all maximal vertical intervals of $N_2 - N_2 \cap g$. The closure of G_1^* does not intersect $N_2 \cup \overline{G_2^*}$, and $\overline{G_2^*}$ does not intersect M_1. Let D_2 denote the collection to which d belongs only if d is either g, or an element of G_1, or an element of G_2.

Let H_2 denote the collection to which h belongs only if h is an element of either H_{M_1} or H_{N_2}. There are only finitely many elements of H_2 that do not lie in D_2^*. Let H_2' denote the collection of all such elements of H_2. Each element of H_2' lies either in D_1^* or in E_1^*. There is a positive number δ_2 such that $\delta_2 < \varepsilon_2$, $\delta_2 < l(h, \overline{D_1^*} - D_1^*)$ for each element h of H_2' that lies in D_1^*, $\delta_2 < l(h, \overline{E_1^*} - E_1^*)$ for each element h of H_2' that lies in E_1^*, $\delta_2 < l(h', h'')$ for each two elements h' and h'' of H_2', $\delta_2 < l(h, \bar{d})$

for each element h of H_2' and each element d of D_2 that does not contain an end of h, and $\delta_2 < l(P, M_1 \cup N_2)$ for each point P of a horizontal side of the closure of an element of D_2. For each element h of H_2', let e_h denote the set to which X belongs only if either X is a point of an element of D_2 containing an end of h and $l(X, h) < \delta_2/3$, or the vertical line λ_X containing X lies between the two elements of D_2 containing the ends of h and $l(X, \lambda_X \cap h) < \delta_2/3$. Let E_2 denote the collection of all the sets e_h for all elements h of H_2'. Each element of E_2 is a subset of either D_1^* or E_1^*.

Using methods similar to those used above, sequences $M_1, M_2, \ldots$ $\ldots N_1, N_2, \ldots, D_1, D_2, \ldots$, and $E_1, E_2, \ldots$ may be described such that

(1) M_1, D_1 and E_1 are as described above,

(2) for each i,

(a) N_i is an A-continuum, $N_i \cap N_{i+1}$ is an end of both N_i and N_{i+1}, and if $i+1 < j$, N_i and N_j are mutually exclusive,

(b) M_i is $N_1 \cup \ldots \cup N_{2^{i-1}}$, and H_i is the collection to which h belongs only if for some j such that $1 \leqslant j \leqslant 2^{i-1}$, h is an element of H_{N_j},

(c) D_i is a finite collection of vertical rectangular domains that properly covers the sum of all maximal vertical intervals of M_i, the closures of each two elements of D_i are mutually exclusive, each element of D_i has height less than $1+1/2i$ and width less than $1/2i$; and if d is an element of D_i no horizontal side of $\bar{d}$ intersects M_i, $\bar{d}$ does not intersect three of $N_1, \ldots, N_{2^{i-1}}$ and if, for each j from 1 to $2^{i-1}-1$, d does not contain $N_j \cap N_{j+1}$ then there is a j' from 1 to 2^{i-1} such that $\bar{d} \cap M_i$ is a subset of $N_{j'}$,

(d) E_i is a finite collection of connected domains that properly covers the sum of all elements of H_i that do not lie in D_i^*, the closures of each two elements of E_i are mutually exclusive, each element of E_i contains only one element of H_i that does not lie in D_i^*, and if e is an element of E_i and h_e is the element of H_i lying in e but not in D_i^*, then (i) $\bar{e}$ does not intersect the closure of an element d of D_i unless d contains an end of h_e, and (ii) if λ is a vertical line intersecting e and $\lambda \cap e$ is not a subset of D_i^*, then λ lies between the two elements of D_i that contain an end of h_e and $\lambda \cap e$ lies in an interval of length less than $1/2i$,

(e) for each non-negative integer $j < 2^{i-1}$, $u(N_{2^{i-1}-j}, N_{2^{i-1}+j+1}) < 1/i$ and

(f) $D_i^* \cup E_i^*$ does not disconnect the plane, $\overline{D_{i+1}^* \cup E_{i+1}^*}$ is a subset of $D_i^* \cup E_i^*$, D_{i+1}^* lies in D_i^*, and every element of E_{i+1} lies in either D_i^* or E_i^*.

Let M denote $\overline{M_1 \cup M_2 \cup \ldots}$. Then M is a compact continuum that does not disconnect the plane and M contains no domain, hence no subset of M disconnects the plane. Thus [1] M is hereditarily unicoherent.

The sequence $M_1, M_2, \ldots$ satisfies the hypothesis of Theorem 1, thus M is indecomposable. For each i, M_i is decomposable, hence M contains a decomposable continuum. Let H denote the collection to which h belongs only if, for some i, h is an element of H_i. Every element of H has a diameter at least 1 and not greater than 4. Since $\overline{H^*}$ contains M_1, which has a diameter greater than 4, $\overline{H^*}$ has a diameter greater than 4. Thus if α is a convergent sequence, each term of which is an element of H, the limiting set of α is a nondegenerate proper subset of $\overline{H^*}$. If $\varepsilon > 0$, there is an i such that $1/i < \varepsilon$, and there is a $\delta > 0$ such that $\delta < l(\overline{d}_1, \overline{d}_2)$ for each two elements d_1 and d_2 of D_i, $\delta < l(\overline{e_1}, \overline{e_2})$ for each two elements e_1 and e_2 of E_i, and $\delta < l(\overline{e}, \overline{d})$ for each element e of E_i and each element d of D_i that does not contain an end of the element of H_i lying in e but not in $\overline{D_i^*}$; δ is a positive number such that if h' and h'' are two elements of H and $l(h', h'') < \delta$, then either $u(h', h'') < \varepsilon$ or $u(h'', h') < \varepsilon$. Thus the collection H satisfies the hypothesis of Theorem 2. Since M is $\overline{H^*}$, it follows that M is filled up by an upper semi-continuous collection of mutually exclusive nondegenerate continua.

References

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One-dimensional n-leaved continua

by

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It is well-known ([3], p. 60) that all one-dimensional continua are embeddable in Euclidean 3-dimensional space. A continuum is a compact connected separable metric space. Continua which are embeddable in Euclidean 2-dimensional space are called planar continua; one-dimensional planar continua have been extensively studied, see for example [8]. In this note we study certain one-dimensional continua that generalize the notion of planar continua. All planar continua are embeddable in a geometric 2-simplex. An *n-book*, $B(n)$ for $n \in Z$ (Z denoting the positive integers), is the union of n geometric 2-simplexes such that each pair of 2-simplexes meets precisely on a single geometric 1-simplex B on the face of each. The 2-simplexes are called the *leaves* of $B(n)$ and B is its *back*. Planar one-dimensional continua are said to be 1-*leaved*. A one-dimensional continuum X is said to be *n-leaved* ($n \geqslant 3$) if X embeds in $B(n)$ but does not embed in $B(k)$ for $0 < k < n$. Of course, there are one-dimensional continua that are not n-leaved for any $n \in Z$, for example the universal curve [1].

Utilizing Sierpiński's universal plane curve [6], we construct a universal n-leaved continuum. It is shown that all one-dimensional subcontinua of a surface (a compact connected 2-manifold) are n-leaved where $0 < n \leqslant 3$. Borsuk ([2], p. 79) has given an example of a locally plane and locally connected one-dimensional continuum which is not embeddable in any surface. This continuum is shown to be 3-leaved.

First, we construct a universal n-leaved continuum ($n \neq 2$). Let $D_1, D_2, \ldots$ be a sequence of closed disks in $B(n)$ such that D_i, for all $i \in Z$, does not intersect a 1-simplex in the face of any of the 2-simplexes in $B(n)$, $\bigcup_{i=1}^{\infty} D_i$ is dense in $B(n)$, and the diameters of the disks D_i converge to zero. Let $S(n) = B(n) - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ (Int = interior in the sense of manifolds). It follows from results of Whyburn [7] that $S(n)$ intersected with a leaf of $B(n)$ is homeomorphic to Sierpiński's universal plane curve and that if another sequence of disks $E_1, E_2, \ldots$ satisfy the same conditions