is not exact. But the coefficient group $H_0(pt; \boldsymbol{Z})$ for our theory is $\hat{\boldsymbol{Z}}$; and it turns out, interestingly, that the limit of the system corresponding to (6.3) (with $\boldsymbol{Z}$ replaced by $\hat{\boldsymbol{Z}}$) is exact.

It is clear from §§ 2–5 that for each abelian group G we get a cohomology theory for closed pairs with

$$H^n(X, A; G) = H^n\big(\mathrm{Hom}\,(\bar{C}(X; \boldsymbol{Z})/\bar{C}(A; \boldsymbol{Z}), G)\big).$$

(6.4). What is the relation of this theory to Čech cohomology?

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Reçu par la Rédaction le 14. 4. 1970

The product of certain measurable spaces

by

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Let X and Y be topological spaces and let $S(X)$ and $S(Y)$ be the σ-rings of Baire sets of X and Y respectively, i.e. the σ-rings of subsets generated by the compact G_δ sets of X and Y. $S(X)\times S(Y)$, the cartesian product of $S(X)$ and $S(Y)$, is the σ-ring of subsets of the topological product $X\times Y$ generated by the family $G=\{E\times F|\ E\in S(X), F\in S(Y)\}$ of subsets of $X\times Y$. It is a well known fact that for locally compact Hausdorff spaces X and Y, $S(X)\times S(Y)=S(X\times Y)$. It is the purpose of this note to examine the corresponding situation for the σ-rings $Z(X)$ and $\mathrm{wb}(X)$ generated respectively by the zero sets and the closed G_δ sets of X. Following Berberian [1] we will call the elements of $\mathrm{wb}(X)$ the *weakly Baire sets of* X. In the following sufficient conditions on the product $X\times Y$ and on the spaces X and Y will be given to insure that $Z(X)\times Z(Y)=Z(X\times Y)$ and that $\mathrm{wb}(X)\times\mathrm{wb}(Y)=\mathrm{wb}(X\times Y)$. In the latter case the results give a partial answer to a question posed by Berberian ([1], p. 183).

Definitions and notation will be introduced as they become necessary. All topological spaces under consideration will be assumed to be completely regular and Hausdorff. The Stone–Čech compactification of such a topological space X will be denoted by βX.

A subset Z of a topological space X is said to be a *zero set of* X if there exists a continuous real valued function f on X such that $Z=Z(f)=\{x\in X|\ f(x)=0\}$. It is obvious that f may be so chosen such that $0\leqslant f(x)\leqslant 1$ for every x in X. A subset of X is called a *co-zero set* if it is the complement of a zero set of X. Since X itself is always a zero set (and a co-zero set), the σ-ring $Z(X)$ is in fact always a σ-algebra. It follows that $Z(X)$ is generated also by the co-zero sets of X. Similarly, since any topological space is a closed G_δ set we have the same situation for the σ-ring $\mathrm{wb}(X)$. Every compact G_δ set in a completely regular Hausdorff space X is a zero set and every zero set is a closed G_δ so that for these spaces we always have the relation $S(X)\subseteq Z(X)\subseteq\mathrm{wb}(X)$. If a topological space X is normal then every closed G_δ set is a zero set so that in this case $\mathrm{wb}(X)=Z(X)$.

A topological space X is said to be *σ-compact* if $X=\bigcup K_n$ where each K_n is compact. If these sets can be so chosen so that each K_n is contained in the interior of K_{n+1} then X is said to be *regularly σ-compact.* If X is locally compact, Hausdorff and σ-compact then it is an easy argument to show that X is regularly σ-compact. A σ-compact Hausdorff is certainly paracompact and is therefore normal. We may now state

(1) THEOREM. *If X and Y are locally compact, σ-compact Hausdorff spaces then* $Z(X)\times Z(Y)=Z(X\times Y)$.

Proof. First note that the topological product $X\times Y$ must be also locally compact, σ-compact and Hausdorff so that it is regularly σ-compact. Now if C is a compact subset of a locally compact Hausdorff space and U is an open set containing C then there is a compact G_δ set D such that $U\supset D\supset C$. It follows that a locally compact, σ-compact Hausdorff space X is a countable union of compact G_δ sets (and consequently is a Baire set). It is now immediate that any zero set of X, Y, or $X\times Y$ is a countable union of compact G_δ^σ sets and hence is a Baire set so that $S(X)=Z(X)$, $S(Y)=Z(Y)$ and $S(X\times Y)=Z(X\times Y)$ so that $Z(X)\times Z(Y)=Z(X\times Y)$. ■

(2) COROLLARY. *If X and Y are locally compact, σ-compact Hausdorff spaces then* $\mathrm{wb}(X)\times\mathrm{wb}(Y)=\mathrm{wb}(X\times Y)$.

Proof. The corollary follows trivially from the theorem and the preceding remarks. ■

A topological space X is said to be *pseudocompact* if every continuous real valued function on X is bounded. If a completely regular Hausdorff space is pseudocompact then it is easy to show that every zero set Z in βX must have a non-empty intersection with X and in fact this property characterizes pseudocompact spaces. If Z is a zero set of X we will denote the closure in βX of Z by $\mathrm{Cl}_{\beta X}Z$. We will need the following fact.

(3) LEMMA. *If X is a completely regular Hausdorff space which is pseudocompact and Z is a zero set of X then $\mathrm{Cl}_{\beta X}Z$ is a zero set of βX.*

Proof. Given a zero set Z of X let f be any continuous function on X such that $Z=Z(f)$ and such that $0\leqslant f(x)\leqslant 1$ for every x in X. Let $\hat{f}$ be the continuous extension of f to βX and let $Z(\hat{f})$ be the zero set of $\hat{f}$ in βX. We will show that $\mathrm{Cl}_{\beta X}Z=Z(\hat{f})$. It is obvious that $\mathrm{Cl}_{\beta X}Z\subseteq Z(\hat{f})$. Let x_0 be any point of $Z(\hat{f})$ and suppose that x_0 is not in $\mathrm{Cl}_{\beta X}Z$. Since βX is a compact Hausdorff space there exists a continuous function g on βX such that g is identically 1 on $\mathrm{Cl}_{\beta X}Z$, $g(x_0)=0$ and $0\leqslant g(x)\leqslant 1$ for every x in βX. Consider the function $g+\hat{f}$. $(g+\hat{f})(x_0)=0$ and $g+\hat{f}$ is identically 1 on $\mathrm{Cl}_{\beta X}Z$ so that the zero set $Z(g+\hat{f})$ does not intersect $\mathrm{Cl}_{\beta X}Z$. Since X is pseudocompact $Z(g+\hat{f})\cap X$ is not empty so that there is a point x in X such that $(g+\hat{f})(x)=0$ and hence $f(x)=\hat{f}(x)=0$ so that x is in $Z(f)\cap Z(g+\hat{f})$ which is a contradiction. Therefore $\mathrm{Cl}_{\beta X}Z=Z(\hat{f})$. ■

If the product $X\times Y$ of the topological spaces X and Y is pseudocompact then both X and Y are necessarily pseudocompact. The converse, however, is not true. For any completely regular space X the family of co-zero sets of X is a base for its topology and consequently if X and Y are completely regular the family of cartesian products of co-zero sets of X with co-zero sets of Y is a base for the topology of $X\times Y$. It is easy to show that for any topological spaces X and Y the relations $Z(X)\times Z(Y)\subseteq Z(X\times Y)$ and $\mathrm{wb}(X)\times\mathrm{wb}(Y)\subseteq\mathrm{wb}(X\times Y)$ always hold. We give next a sufficient condition on the product $X\times Y$ that $Z(X)\times Z(Y)=Z(X\times Y)$ which depends on a theorem due to Glicksberg [4] which states that for completely regular Hausdorff spaces X and Y, if $X\times Y$ is pseudocompact then $\beta X\times\beta Y$ is identical to $\beta(X\times Y)$, that is the identity mapping of $X\times Y$ onto itself extends to a homeomorphism of $\beta(X\times Y)$ onto $\beta X\times\beta Y$.

(4) THEOREM. *If X and Y are completely regular Hausdorff spaces and $X\times Y$ is pseudocompact then* $Z(X)\times Z(Y)=Z(X\times Y)$.

Proof. Since $Z(X)\times Z(Y)\subseteq Z(X\times Y)$ it is sufficient to show that any zero set Z of $X\times Y$ is in $Z(X)\times Z(Y)$. Let Z be such a zero set. Then by (3) $\mathrm{Cl}_{\beta(X\times Y)}Z$ is a zero set of $\beta(X\times Y)$ and thus since $X\times Y$ is pseudocompact of $\beta X\times\beta Y$. It follows that $\mathrm{Cl}_{\beta(X\times Y)}Z$ is a compact G_δ set in $\beta X\times\beta Y$ so that $\mathrm{Cl}_{\beta(X\times Y)}Z=\bigcap_n O_n$ where O_n is an open subset of $\beta X\times\beta Y$ for each n. Now each O_n is a union of sets of the form $E\times F$ where E is a co-zero set of βX and F is a co-zero set of βY. Since $\mathrm{Cl}_{\beta(X\times Y)}Z$ is compact a finite number k of these rectangular co-zero sets must cover $\mathrm{Cl}_{\beta(X\times Y)}Z$ so that we have for each n

$$\mathrm{Cl}_{\beta(X\times Y)}Z\subset\bigcup_{i=1}^{k}(E_{n,i}\times F_{n,i})\subseteq O_n\,.$$

If we let $\hat{E}_{n,i}=E_{n,i}\cap X$ and $\hat{F}_{n,i}=F_{n,i}\cap X$ for each i and n, then $\hat{E}_{n,i}$ and $\hat{F}_{n,i}$ are co-zero sets of X and Y respectively and we have

$$Z\subset\bigcup_{i=1}^{k}(\hat{E}_{n,i}\times\hat{F}_{n,i})\subseteq O_n\cap(X\times Y)\,.$$

Since $Z=\bigcap_n\big(O_n\cap(X\times Y)\big)$ it now follows immediately that Z is in $Z(X)\times Z(Y)$ and hence that $Z(X)\times Z(Y)=Z(X\times Y)$. ■

(5) COROLLARY. *If X and Y are completely regular Hausdorff spaces, $X\times Y$ is pseudocompact and X, Y, and $X\times Y$ are normal then* $\mathrm{wb}(X)\times\mathrm{wb}(Y)=\mathrm{wb}(X\times Y)$.

Proof. The proof follows directly from (4) and from the fact that for a normal space X, $Z(X)=\mathrm{wb}(X)$. ■

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The condition of normality on the product space $X\times Y$ in the above corollary is not necessary as can be seen from the following example. Let Ω_0 be the space of all ordinals less than the first uncountable ordinal with its usual order topology and let Ω be its one point compactification. Ω and Ω_0 are normal and pseudocompact and $\Omega_0\times\Omega$ is pseudocompact but is not normal. Hence $\mathrm{wb}(\Omega_0)\times\mathrm{wb}(\Omega) = Z(\Omega_0)\times Z(\Omega) = Z(\Omega_0\times\Omega)$. However, every closed G_δ set in $\Omega_0\times\Omega$ is a zero set (see for example [3], p. 129) so that $\mathrm{wb}(\Omega_0\times\Omega) = Z(\Omega_0\times\Omega)$.

In (4) and (5) the sufficiency conditions are given in terms of the product space $X\times Y$. For a certain class of completely regular spaces we may state the conditions in terms of X and Y. We will need a corollary to a theorem of Frolík [2] which states that if $X\times Y$ is a completely regular, pseudocompact Hausdorff space and f is a continuous function on $X\times Y$ then the function F defined on X by $F(x) = \sup_{y\in Y} f(x,y)$ is a continuous function on X.

(6) Lemma. *Let X be a locally compact Hausdorff space and let Y be a completely regular, pseudocompact Hausdorff space. If f is a continuous function on $X\times Y$ then the function F defined on X by $F(x) = \sup_{y\in Y} f(x,y)$ is a continuous function on X.*

Proof. It is easy to show that the product of a compact space and a pseudocompact space must be pseudocompact (see [3], p. 134). Let x_0 be an arbitrary point of X and C a compact neighborhood of x_0. Then $C\times Y$ is pseudocompact, f is a continuous function on $C\times Y$ and hence by Frolík's theorem the function F is continuous at x_0. Since x_0 was arbitrary, F is continuous on X. ■

(7) Lemma. *Let X and Y be pseudocompact, completely regular Hausdorff spaces and let X be locally compact, then $X\times Y$ is pseudocompact.*

Proof. The lemma is easy consequence of (6). It also appears as a special case of Theorem 3.3 in [2]. ■

We may now state the previous results in the following form:

(8) Theorem. *If X and Y are completely regular, pseudocompact Hausdorff spaces and either X or Y is locally compact then $Z(X)\times Z(Y) = Z(X\times Y)$.*

Proof. The theorem is a routine consequence of (4) and (7). ■

Theorem (8) has the obvious corollary for weakly Baire sets which is analogous to (5) and which we omit here.

Remarks. In Theorem 1 and its corollary the results follow clearly from the identification of both the zero sets of X and the closed G_δ sets of X as Baire sets and the fact that $S(X)\times S(Y) = S(X\times Y)$. It is less obvious that a similar situation obtains in Theorem (4) and its corollary,

that is when $X\times Y$ is pseudocompact. We have noted that when X is pseudocompact every zero set in βX must intersect X and shown that if Z is a zero set of X then $\mathrm{Cl}_{\beta X}Z$ is a zero set and hence a compact G_δ in βX. It is equally easy to demonstrate that if Z is a zero set (a compact G_δ) in βX then $\mathrm{Cl}_{\beta X}(Z\cap X) = Z$ so that there is a natural one to one correspondence between the zero sets of X and the compact G_δ sets of βX and hence between $Z(X)$ and $S(\beta X)$. Thus Theorem (4) will follow from the fact that $S(\beta X)\times S(\beta Y) = S(\beta X\times\beta Y) = S(\beta(X\times Y))$. It is therefore natural to ask whether we may have the relation $Z(X)\times Z(Y) = Z(X\times Y)$ when the zero sets of the respective spaces cannot be identified with the Baire sets of the spaces or of their Stone–Čech compactifications. The answer to this question is not known to the author.

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Reçu par la Rédaction le 1. 7. 1970