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Added in proof. A generalization of Theorem 2.8 is contained in H. H. Wicke and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order,* Duke Math. Journ. 34 (1967), pp. 255–272. One can prove that the theorem remains valid if Y is complete.

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Tate resolutions for commutative graded algebras over a local ring

by

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Introduction. Let R be a commutative Noetherian ring with a unit element.

Tate has constructed in [11] for cyclic R-modules free resolutions with additional algebra structure and used them for the study of the functor Tor^R. Only recently (see [4], [8], [10]) it has turned out that for a local ring R and residue class field k the Tate resolution has an important property: it is minimal. A minimal resolution F determines completely the algebra $\mathrm{Tor}^R(k, k)$: we have $\mathrm{Tor}^R(k, k) \simeq F \otimes k$. These two properties: the algebra structure and minimality facilitate the investigation of the structure of the homology of the ring R.

The main purpose of the present paper is to build the theory of Tate resolutions for graded commutative algebras over a local ring R (called R-algebras in this paper, cf. (1.1)).

From the existence of the Tate resolution of an R-algebra A we obtain the following formula for the Poincaré series of A:

$$\mathcal{P}(A) = \frac{(1+t)^{n_1}(1+t^2)^{n_2}(1+t^3)^{n_3}\ldots}{(1-t)^{m_1}(1-t^2)^{m_2}(1-t^3)^{m_3}\ldots}.$$

The organization of the paper is as follows:

In § 1 we recall the definition of an R-algebra and some basic properties of the category of graded modules over such an R-algebra.

§ 2 contains the definition and properties of a normal sequence in an R-algebra. The main result of this section is a characterization of those R-algebras whose unique maximal homogeneous ideal is generated by a normal sequence.

In § 3 we define bigraded Γ-algebras and differential Γ-algebras. Furthermore we present the basic construction of the differential Γ-algebra $A\langle M; \varphi\rangle$ obtained from the differential Γ-algebra A by the adjunction of the R-module M by means of the map φ: $M \to Z(A)$.

In § 4 we construct the Tate resolution X of an R-algebra A as the sum of an ascending chain of differential Γ-algebras F_qX, $q = 0, 1, 2, \ldots$ The basic properties of X, minimality and invariance, are also proved.

In § 5 we give the full characterization of those R-algebras for which $X = F_1X$. These are exactly algebras isomorphic with the tensor product of exterior and symmetric algebras of free modules.

In § 6 we apply the theory of the Tate resolutions to the computation of the Poincaré series. The last theorem of this section contains the relationship between the Poincaré series for A and $\bar{A}$, where $\bar{A}$ is the residue algebra of A by a non-zero divisor of A.

Some results of the present paper were announced in [6].

§ 1. R-algebras. Let R be a commutative local ring with a unit element.

(1.1) Definition. An associative, graded algebra $A = \bigoplus_{i\geqslant 0} A_i$ over R will be called an *R-algebra* in this paper if the following conditions are satisfied:

(i) A_i is a finitely generated R-module for $i \geqslant 0$,

(ii) A has a unit element $1 \in A_0$ such that $A_0 = R\cdot 1 \simeq R$,

(iii) $a\cdot b = (-1)^{ij} b\cdot a$, for $a \in A_i$, $b \in A_j$,

(iv) $a^2 = 0$, for $a \in A_i$, i odd.

An element $a \in A_i$ is called a *homogeneous element of degree i*. We shall write $i = \partial(a)$.

(1.2) Definition. By a *graded A-module* $M = \bigoplus_{q\geqslant 0} M_q$ we shall mean in this paper a locally finite graded A-module, i.e. each M_q is a finitely generated R-module.

We denote by A-Mod the category of graded A-modules and their homogeneous homomorphisms of degree 0.

For the convenience of the reader and for references we recall below some basic properties of the category A-Mod. For the details see [2] and [3].

Let $\mathfrak{m}$ be the unique maximal ideal in R and let $k = R/\mathfrak{m}$. We write $I = \mathfrak{m} \oplus \bigoplus_{i\geqslant 1} A_i$. The ideal I is the unique maximal homogeneous ideal in A and $A/I \simeq k$.

(1.3) Nakayama Lemma. *If $M \in \operatorname{ob} A$-Mod, then $IM = M$ implies $M = 0$.*

(1.4) Definition. An epimorphism $f\colon M \to N$, $f \in \operatorname{morph} A$-Mod, is called a *minimal epimorphism* if the following condition is satisfied: for arbitrary $Y \in \operatorname{ob} A$-Mod, $g \in \operatorname{morph} A$-Mod, $g\colon Y \to M$, if the morphism $Y \xrightarrow{g} M \xrightarrow{f} N$ is an epimorphism, then $Y \xrightarrow{g} M$ is also an epimorphism.

If M, N are objects in A-Mod and $f\colon M \to N$ is a morphism in A-Mod, then we write $\bar{M} = M/IM$, and denote by $\bar{f}\colon \bar{M} \to \bar{N}$ the mapping induced by f.

(1.5) *Let $f\colon M \to N$ be a morphism in A-Mod. The following conditions are equivalent:*

(i) *f is a minimal epimorphism,*

(ii) *$\bar{f}$ is an isomorphism,*

(iii) *f is an epimorphism and* $\operatorname{Ker} f \subset IM$.

(1.6) Definition. A set V of homogeneous generators of the graded A-module M is said to be a *minimal set of generators of M* if any proper subset of V does not generate M.

(1.7) *Any graded A-module possesses a minimal set of generators. The set $V = \{v_i\}$ of homogeneous elements of the A-module M is a minimal set of generators of M if and only if the set $\{\bar{v}_i\}$ of residue classes modulo IM forms a base of a k-vector space $\bar{M}$.*

(1.8) *In the category A-Mod there are enough minimal epimorphisms, i.e. for arbitrary $M \in \operatorname{ob} A$-Mod there exist a free module $F \in \operatorname{ob} A$-Mod and a minimal epimorphism $f\colon F \to M$.*

(1.9) *Any projective object of the category A-Mod is a free module.*

(1.10) Definition. A free resolution (F, d) of the graded A-module M is called a *minimal resolution* provided $dF \subset IF$.

(1.11) *An arbitrary object of the category A-Mod possesses a free minimal resolution.*

(1.12) *Any two minimal free resolutions of a graded A-module M are isomorphic.*

§ 2. Normal sequences. Let A be an R-algebra. We write $I' = \bigoplus_{i=1}^{\infty} A_i$. Recall that $I = \mathfrak{m} \oplus I'$, where $\mathfrak{m}$ denotes the unique maximal ideal in R.

(2.1) Definition. A homogeneous element ξ of I is called a *non-zero divisor in A* if the following condition is satisfied:

$$(0):(\xi) = 0 \quad \text{if} \quad \partial(\xi) \text{ is even},$$
$$(0):(\xi) = (\xi) \quad \text{if} \quad \partial(\xi) \text{ is odd}.$$

(2.2) Definition. Let $\xi_1, \ldots, \xi_n$ be a sequence of homogeneous elements of the ideal I. The sequence $\xi_1, \ldots, \xi_n$ is called a *normal sequence in A* if for an arbitrary i, $1 \leqslant i \leqslant n$, the image of ξ_i in $A/(\xi_1, \ldots, \xi_{i-1})A$ is a non-zero divisor in $A/(\xi_1, \ldots, \xi_{i-1})A$.

(2.3) THEOREM. *If $\xi_1, \ldots, \xi_n$ is a normal sequence in A and σ is an arbitrary permutation of the set $\{1, 2, \ldots, n\}$, then the sequence $\xi_{\sigma(1)}, \xi_{\sigma(2)}, \ldots, \xi_{\sigma(n)}$ is also normal.*

Proof. Since each permutation is a product of transpositions, it suffices to prove the theorem for the transposition changing 1 and 2.

Consider four cases: 1) $\partial(\xi_1)$ even, $\partial(\xi_2)$ even, 2) $\partial(\xi_1)$ even, $\partial(\xi_2)$ odd, 3) $\partial(\xi_1)$ odd, $\partial(\xi_2)$ even, 4) $\partial(\xi_1)$ odd, $\partial(\xi_2)$ odd.

ad 1). We must prove the equalities $(0):(\xi_2) = 0$, $(\xi_2):(\xi_1) = (\xi_2)$. Write $M = (0):(\xi_2)$. If $a \in M$ then $a\xi_2 = 0$. Since the sequence $\xi_1, \ldots, \xi_n$ is normal, it follows that $a \in (\xi_1)$, $a = b\xi_1$; it implies $b\xi_1\xi_2 = 0$. But ξ_1 is a non-zero divisor, and so we have $b\xi_2 = 0$ and $b \in M$; consequently, $a \in IM$. We have proved that $M = IM$ and from Nakayama Lemma it follows that $M = 0$.

Since the sequence $\xi_1, \ldots, \xi_n$ is normal, we have a chain of implications:

$$c \in (\xi_2):(\xi_1) \Rightarrow c\xi_1 \in (\xi_2) \Rightarrow c\xi_1 = b\xi_2 \Rightarrow b \in (\xi_1)$$
$$\Rightarrow b = c_1\xi_1 \Rightarrow c\xi_1 = c_1\xi_1\xi_2 \Rightarrow c - c_1\xi_2 = 0 \Rightarrow c = c_1\xi_2 \in (\xi_2).$$

Therefore $(\xi_1):(\xi_2) = (\xi_2)$.

ad 2). We shall prove that $(0):(\xi_2) = (\xi_2)$ and $(\xi_2):(\xi_1) = (\xi_2)$. Let $M = (0):(\xi_2)$, $N = (\xi_2)$. From $a \in M$ and from the normality of $\xi_1, \ldots, \xi_n$ we obtain $a = b_1\xi_1 + b_2\xi_2$. Multiplying this by ξ_2, we have $b_1\xi_1\xi_2 = a\xi_2 = 0$ so $b_1\xi_2 = 0$ but this means that $b_1 \in M$. Consequently, $M = IM + N$ and by Nakayama Lemma $M = N$. The proof of the formula $(\xi_2):(\xi_1) = (\xi_2)$ is similar to the proof of the appropriate formula of 1).

ad 3). We must prove $(0):(\xi_2) = 0$, $(\xi_2):(\xi_1) = (\xi_1, \xi_2)$. If $a\xi_2 = 0$ then $a = b\xi_1$ and so $b\xi_1\xi_2 = 0$. Since $\xi_1, \ldots, \xi_n$ is normal, we obtain successively $b\xi_2 \in (\xi_1)$, $b \in (\xi_1)$. Therefore $a = 0$ and the equality $(0):(\xi_2) = (0)$ has been proved. Now let $a \in (\xi_2):(\xi_1)$; this means that $a\xi_1 = b\xi_2$ and, further, from the normality of the sequence $\xi_1, \ldots, \xi_n$ we get $b = c\xi_1$, $a\xi_1 = c\xi_1\xi_2$, $(a - c\xi_2)\xi_1 = 0$, $a = e\xi_1 + c\xi_2 \in (\xi_1, \xi_2)$.

ad 4). We shall prove that $(0):(\xi_2) = (\xi_2)$, $(\xi_2):(\xi_1) = (\xi_1, \xi_2)$. Let $a\xi_2 = 0$; since $\xi_1, \ldots, \xi_n$ is normal, we have $a = b\xi_1 + c\xi_2$. Multiplying it by ξ_2, we obtain $b\xi_1\xi_2 = 0$ and using once more the normality of $\xi_1, \ldots, \xi_n$, we get $b \in (\xi_1, \xi_2)$ and finally $a \in (\xi_2)$. The proof of the equality $(\xi_2):(\xi_1) = (\xi_1, \xi_2)$ is similar to the proof of the appropriate formula in 3).

Observe that Theorem (2.3) follows simply from Proposition (5.1).

By Theorem (2.3) we may speak of a normal set.

The following proposition gives a characterization of algebras generated by a normal set.

(2.4) PROPOSITION. *Let A be a finitely generated R-algebra. The fowolling conditions are equivalent:*

(i) *$A \simeq \Lambda M \otimes SN$, where M and N are free R-modules and ΛM, SN denotes exterior and symmetric algebras, respectively,*

(ii) *any minimal set of generators of I' is a normal set in A,*

(iii) *there exists a minimal normal set of generators of the ideal I'.*

For the proof of Proposition (2.4) we shall need

(2.5) LEMMA. *Let $\xi_1, \ldots, \xi_n$, $\xi_i \in I'$, be a normal sequence in A and let $\partial(\xi_i)$, $i = 1, \ldots, r$, be odd numbers, and $\partial(\xi_i)$, $i = r+1, \ldots, n$, be even numbers. Denote by E the algebra $\Lambda(Rx_1 \oplus \ldots \oplus Rx_r) \otimes R[x_{r+1}, \ldots, x_n]$ and define grading in E by $\partial(x_i) = \partial(\xi_i)$. The homomorphism $\varphi\colon E \to A$ such that $\varphi(x_i) = \xi_i$ is injective.*

Proof. It is evident that for $n = 1$ the homomorphism φ is an injection. The proof will be by induction on n, so it will be supposed that the lemma is true for an arbitrary R-algebra and for all normal sequences of length $< n$, $n > 1$. Consider two cases:

1) $r > 0$, i.e. $\partial(\xi_1)$ is odd.

Each element x of the algebra E can be written uniquely in the form $x = x_1a + b$, where a and b are polynomials in $x_2, \ldots, x_n$ with coefficients in R. The homomorphism φ induces the mapping $\overline{\varphi}\colon E/x_1E \to A/\xi_1A$. From $x \in \operatorname{Ker}\varphi$ it follows that $\overline{\varphi}(b + x_1E) = 0$. Applying the induction hypothesis to the algebra A/ξ_1A and to the sequence $\xi_2 + \xi_1A, \ldots, \xi_n + \xi_1A$ we find that $\overline{\varphi}$ is an injection, i.e. $b + x_1E = 0$ and consequently $b = 0$. Hence $x = x_1a$; thus we have $\xi_1\varphi(a) = 0$. Since ξ_1 is a non-zero divisor in A, we have $\varphi(a) \in \xi_1A$. Hence $\overline{\varphi}(a + x_1E) = 0$ and finally $a = 0$. This proves $\operatorname{Ker}\varphi = 0$.

2) $r = 0$, i.e. $\partial(\xi_1)$ is even.

Let $\overline{\varphi}$ denote the mapping induced by φ, $\overline{\varphi}\colon E/x_1E \to A/\xi_1A$. If $x = a_0 + a_1x_1 + \ldots + a_kx_1^k$, $a_i \in R[x_2, \ldots, x_n]$, and $x \in \operatorname{Ker}\varphi$, then $\overline{\varphi}(a_0 + x_1E) = 0$. From the induction hypothesis it follows that $\overline{\varphi}$ is injective, i.e. $a_0 + x_1E = 0$ and so $a_0 = 0$. This implies $x = x_1(a_1 + a_2x_1 + \ldots)$. But ξ_1 is a non-zero divisor in A, and so we have $x' = a_1 + a_2x_1 + \ldots + a_kx_1^{k-1} \in \operatorname{Ker}\varphi$. Applying similar arguments to the element x' as above, we prove $a_1 = 0$ and further $a_2 = \ldots = a_k = 0$. Hence $x = 0$ and $\operatorname{Ker}\varphi = 0$.

Proof of Proposition (2.4). In view of (1.7) the implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let $\xi_1, \ldots, \xi_n$ be a minimal normal set of generators of the ideal I'. By Theorem (2.3) we may assume that $\partial(\xi_i)$ are odd for $i = 1, \ldots, r$ and even for $i = r+1, \ldots, n$. From Lemma (2.5) we get an injective map $\varphi\colon E \to A$. Since the elements $\xi_1, \ldots, \xi_n$ generate I', it follows that φ is surjective. Hence $E \simeq A$.

(i) ⇒ (ii). If $A = \Lambda(Rx_1 \oplus \ldots \oplus Rx_r) \otimes R[x_{r+1}, \ldots, x_n]$, then the sequence $x_1, \ldots, x_n$ is normal in A. Let $\xi_1, \ldots, \xi_m$ be an arbitrary minimal set of generators of the ideal I'. From (1.7) it follows that $m = n$, and it may be assumed that $\partial(x_i) = \partial(\xi_i)$ for $i = 1, \ldots, n$. We define the homomorphism $\psi\colon A \to A$ by putting $\psi(x_i) = \xi_i$. It is surjective of course. On the other hand, each homogeneous component of A is a free R-module of finite rank and therefore the surjectivity of ψ implies injectivity. Hence ψ is a an isomorphism and the sequence $\xi_1, \ldots, \xi_m$ is normal in A, being the image of the normal sequence $x_1, \ldots, x_n$ by the isomorphism ψ.

(2.6) THEOREM. *Let A be a finitely generated R-algebra. The following conditions are equivalent:*

(i) *the ring R is a regular local ring and $A \simeq \Lambda M \otimes SN$, where M and N are free R-modules,*

(ii) *every minimal set of generators of the ideal I is normal in A,*

(iii) *there exists a minimal normal set of generators of the ideal I.*

Proof. The implication (ii) ⇒ (iii) is trivial.

(iii) ⇒ (i). Let $\xi_1, \ldots, \xi_n$ be a normal sequence which is a minimal set of generators of the ideal I. It may be assumed that $\partial(\xi_i) \leqslant \partial(\xi_j)$ for $i < j$. Hence there exists a natural number s, $s \leqslant n$, such that $\xi_1, \ldots, \xi_s \in \mathfrak{m}$ and $\xi_{s+1}, \ldots, \xi_n \in I'$. The sequence $\xi_1, \ldots, \xi_s$ is a minimal normal set of generators of the ideal $\mathfrak{m}$ in R and therefore R is regular. Denote by P the ideal in A generated by the elements $\xi_{s+1}, \ldots, \xi_n$. Each element from I' can be written as a sum $\alpha + \beta$, where $\alpha = \sum_{i=1}^{s} a_i \xi_i$, $a_i \in I'$, $\beta = \sum_{i=s+1}^{n} \beta_i \xi_i$, $\beta_i \in A$. Since $\alpha \in II'$, $\beta \in P$, we have $II' + P = I'$ and from Nakayama Lemma it follows that $P = I'$. Hence the sequence $\xi_{s+1}, \ldots, \xi_n$ is a minimal normal set of generators of the ideal I', and applying Proposition (2.4) we finish the proof.

(i) ⇒ (ii). If $\xi_1, \ldots, \xi_n$ is a minimal set of generators of I and $\xi_1, \ldots, \xi_s \in \mathfrak{m}$, $\xi_{s+1}, \ldots, \xi_n \in I'$, then the sequence $\xi_1, \ldots, \xi_s$ is a minimal set of generators of $\mathfrak{m}$ in R and by the regularity of R it is a normal set in R. From Proposition (2.4) it follows that the sequence $\xi_{s+1}, \ldots, \xi_n$ is normal in A. From the definition of normality we immediately infer that the sequence $\xi_1, \ldots, \xi_n$ is also normal in A.

The notion of a normal set can be generalized to sets of arbitrary cardinality.

(2.7) DEFINITION. A set Ξ of homogeneous elements of an R-algebra A is called a *normal set* if every finite subset of Ξ forms a normal sequence (in the sense of Definition (2.2)).

(2.8) Using this definition of a normal set, we can easily show by simple direct limit arguments that Theorem (2.6) is still valid for R-algebras which are not finitely generated.

§ 3. Bigraded Γ-algebras.

Let A be an R-algebra.

(3.1) DEFINITION. An A-module $\boldsymbol{M}$ is called a *bigraded A-module* if $\boldsymbol{M} = \bigoplus_{q \geqslant 0} \boldsymbol{M}_{*,q}$ where $\boldsymbol{M}_{*,q}$ is a graded A-module, $q \geqslant 0$.

If $\boldsymbol{M}_{*,q} = \bigoplus_{p \geqslant 0} \boldsymbol{M}_{p,q}$ is a grading in $\boldsymbol{M}_{*,q}$, then an element $x \in \boldsymbol{M}_{p,q}$ is called *homogeneous of degree* (p, q). We shall write $p = \partial(x)$, $q = \omega(x)$.

(3.2) DEFINITION. A bigraded A-module $\boldsymbol{A}$ is called a *bigraded commutative A-algebra* if

(i) for each pair (p, q) of natural numbers the biadditive, associative mapping of graded A-modules of degree 0 is given,

$$\boldsymbol{A}_{*,p} \times \boldsymbol{A}_{*,q} \to \boldsymbol{A}_{*,p+q}, \quad (x, y) \mapsto x \cdot y,$$

which is bilinear in the following sense: we have

$$(ax) \cdot y = (-1)^{\partial(a)\partial(x)} x \cdot (ay) = a(x \cdot y),$$

for homogeneous $a \in A$, $x \in \boldsymbol{A}_{*,p}$, $y \in \boldsymbol{A}_{*,q}$,

(ii) for homogeneous $x, y \in \boldsymbol{A}$

$$x \cdot y = (-1)^{\partial(x)\partial(y) + \omega(x)\omega(y)} y \cdot x,$$

(iii) $x^2 = 0$ if $\partial(x) + \omega(x)$ is odd,

(iv) $\boldsymbol{A}$ has a unit element $1 \in \boldsymbol{A}_{*,0}$.

(3.3) DEFINITION. A bigraded commutative A-algebra $\boldsymbol{A}$ is called a *bigraded Γ-algebra over A* if it is endowed with the structure of an A-algebra with divided powers, i.e. there are defined mappings $\gamma_k\colon \boldsymbol{A}_{p,q} \to \boldsymbol{A}_{pk,qk}$, $p \geqslant 0$, $q > 0$, $p + q$ even, $k = 0, 1, 2, \ldots$, such that the following conditions are satisfied:

(i) $\gamma_0(x) = 1$, $\gamma_1(x) = x$,

(ii) $\gamma_k(x) \cdot \gamma_h(x) = (k, h) \cdot \gamma_{k+h}(x)$, where $(k, h) = (k+h)!/k! \cdot h!$,

(iii) $\gamma_k(x + y) = \sum_{k_1 + k_2 = k} \gamma_{k_1}(x) \cdot \gamma_{k_2}(y)$,

(iv) $\gamma_k(\lambda x) = \lambda^k \gamma_k(x)$, for $\lambda \in A_r$, r even,

(v) $\gamma_k(x \cdot y) = \begin{cases} 0 & \text{if } \partial(x) + \omega(x) \text{ is odd, } k > 1, \\ k!\, \gamma_k(x) \gamma_k(y) & \text{if } \partial(x) + \omega(x) \text{ is even and } \omega(x) > 0. \end{cases}$

Observe that if $A_i = 0$ for $i > 0$ and $\boldsymbol{A}_{p,q} = 0$ for $p > 0$, then the algebra $\boldsymbol{A}$ may be identified with a graded Γ-algebra over R (see [9]).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be bigraded Γ-algebras over A and let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism of A-algebras of degree $(0, 0)$. It is called a *homo-*

morphism of bigraded Γ-algebras if $f(\gamma_k(x)) = \gamma_k(f(x))$ for $x \in A_{p,q}$, $p+q$ even, $q > 0$.

Let Γ_A-Bialg denote the category whose objects are bigraded Γ-algebras over A and whose morphisms are homomorphisms of bigraded Γ-algebras.

At first we will be interested in bigraded Γ-algebras over a commutative local ring R.

(3.4) *In the category* Γ_R-Bialg *there exist arbitrary coproducts* (see [1]).

If A, B are bigraded Γ-algebras over R, then the algebra $A\otimes B$ is their coproduct with multiplication given by

$$(a\otimes b)(a_1\otimes b_1) = (-1)^{\partial(b)\partial(a_1)+\omega(b)\omega(a_1)} aa_1\otimes bb_1 .$$

Divided powers are defined as follows: if $x = \sum_{i=1}^{p} a_i\otimes b_i \in A_{i,j}\otimes B_{r,s}$, then

$$\gamma_k(x) = \begin{cases} \sum\limits_{1\leqslant i_1<\ldots<i_k\leqslant p} (a_{i_1}\otimes b_{i_1})\ldots(a_{i_k}\otimes b_{i_k}) & \text{if } i+j,\ r+s \text{ are odd},\\ \sum\limits_{k_1+\ldots+k_p=k} (a_1^{k_1}\otimes 1)\ldots(a_p^{k_p}\otimes 1)\cdot(1\otimes\gamma_{k_1}(b_1))\ldots(1\otimes\gamma_{k_p}(b_p)) & \\ & \text{if } i+j,\ r+s \text{ are even.}\end{cases}$$

(3.5) From (3.4) it follows that if $A \in \operatorname{ob}\Gamma_A$-Bialg, $B \in \operatorname{ob}\Gamma_R$-Bialg, then $A\otimes B \in \operatorname{ob}\Gamma_A$-Bialg.

(3.6) Let M be a bigraded R-module. Write $M^+ = \bigoplus_{p+q \text{ even}} M_{p,q}$, $M^- = \bigoplus_{p+q \text{ odd}} M_{p,q}$. We have $M = M^+\oplus M^-$. If $M = M^-$, we put $\Lambda M = \bigotimes_{p,q\geqslant 0} \Lambda M_{p,q}$ and define the grading of $x = x_1\wedge\ldots\wedge x_n$, $x\neq 0$, $x_i \in M_{p,q}$ by $\partial(x) = np$, $\omega(x) = nq$. The algebra ΛM with such grading becomes a commutative bigraded R-algebra.

(3.7) The algebra ΛM can be endowed with a unique Γ-algebra structure such that for $x = \sum_{i=1}^{p} x^{(i)}$, $x^{(i)} = x_1^{(i)}\wedge\ldots\wedge x_n^{(i)}$, $x_k^{(i)} \in M_{p,q}$, n even, we have

$$\gamma_k(x) = \sum_{1\leqslant i_1<\ldots<i_k\leqslant p} x^{(i_1)}\wedge\ldots\wedge x^{(i_k)}.$$

Denote by A-Bimod the category of bigraded A-modules M such that $M_{*,0} = 0$ and their homogeneous homomorphisms of degree $(0,0)$.

We define the functor I: Γ_R-Bialg$\to R$-Bimod, $I(A) = \bigoplus_{p\geqslant 0, q\geqslant 1} A_{p,q}$ for $A \in \operatorname{ob}\Gamma_R$-Bialg and $I(f) = f|I(A)$ for f: $A\to B$.

(3.8) *There exists a functor* Γ: R-Bimod$\to\Gamma_R$-Bialg *which is a left adjoint to the functor* I, *i.e.*

$$\Gamma_R\text{-Bialg}(\Gamma\langle M\rangle, A) \simeq R\text{-Bimod}(M, I(A)).$$

An outline of the construction of the functor Γ (see also [7]). For $M = M^+\oplus M^-$ we define $\Gamma\langle M\rangle = \Gamma\langle M^+\rangle\otimes\Gamma\langle M^-\rangle$ and $\Gamma\langle M^-\rangle = \Lambda M^-$, $\Gamma\langle f\rangle = \Lambda f$; from (3.7) it follows that it is in fact a bigraded Γ-algebra. To define $\Gamma\langle M^+\rangle$ we begin with the assignement to each pair (x, k) an indeterminate $X_{(x,k)}$, where x is a homogeneous element of M^+, $\partial(x)+\omega(x)$ is even, and k is natural. Then we form the algebra $\Omega = R[X_{(x,k)}]$ of polynomials in $X_{(x,k)}$ over R and bigrade it by setting $\partial(X_{(x,k)}) = k\cdot\partial(x)$, $\omega(X_{(x,k)}) = k\cdot\omega(x)$. The ideal $\mathfrak{a}$ generated in Ω by elements of the form

$$X_{(x,0)}-1,\quad X_{(\lambda x,k)}-\lambda^k X_{(x,k)},$$
$$X_{(x,k)}\cdot X_{(x,h)}-(k,h)X_{(x,k+h)},$$
$$X_{(x+y,k)}-\sum_{i+j=k} X_{(x,i)}\cdot X_{(y,j)},$$

$\lambda \in R$, is homogeneous and we put $\Gamma\langle M^+\rangle = \Omega/\mathfrak{a}$. We have the mapping γ: $M^+\to\Gamma\langle M^+\rangle$, $\gamma(x) = X_{(x,1)} \bmod \mathfrak{a}$, and it can be proved that the algebra $\Gamma\langle M^+\rangle$ can be supplied with a unique Γ-algebra structure such that $\gamma_k(\gamma(x)) = X_{(x,k)} \bmod \mathfrak{a}$. The natural map γ: $M\to\Gamma\langle M\rangle$ is injective if M is R-free. We shall identify x with $\gamma(x)$ by means of the map γ.

(3.9) From (3.8) it follows that the functor Γ preserves coproducts, i.e. $\Gamma\langle M\oplus N\rangle \simeq \Gamma\langle M\rangle\otimes\Gamma\langle N\rangle$.

(3.10) Let M be a free bigraded R-module on one generator x of degree (p,q). If $p+q$ is odd, then $\Gamma\langle M\rangle$ is a free R-module of rank 2, $\Gamma\langle M\rangle = R\oplus Rx$, $x^2 = 0$. If $p+q$ is even, then $\Gamma\langle M\rangle$ is a free R-module with a countable basis $1, \gamma_1(x), \gamma_2(x), \ldots, \gamma_k(x), \ldots$, multiplication being determined by

$$\gamma_k(x)\cdot\gamma_h(x) = (k,h)\gamma_{k+h}(x).$$

If M is a free bigraded R-module with basis $\{x_i\}_{i\in\Lambda}$, then we shall often write $\Gamma\langle M\rangle = \Gamma\langle\{x_i\}_{i\in\Lambda}\rangle$.

(3.11) Definition. A bigraded Γ-algebra A over Λ is called a *differential Γ-algebra* if an A-homomorphism d: $A\to A$ of degree $(0,-1)$ is defined such that the following conditions are satisfied:

(i) $d^2 = 0$,

(ii) $d(xy) = dx\cdot y+(-1)^{\omega(x)} x\cdot dy$, for homogeneous $x, y \in A$,

(iii) $d\cdot\gamma_k(x) = \gamma_{k-1}(x)dx$, for homogeneous $x \in A$, $\partial(x)+\omega(x)$ even, $\omega(x) > 0$.

The map d will be called a *differential on* $\mathbf{A}$. We will sometimes denote it by $d_{\mathbf{A}}$.

Let $Z(\mathbf{A})$ be the kernel of d and let $B(\mathbf{A})$ be the image of d. Then $Z(\mathbf{A})$ is a bigraded Γ-algebra and $B(\mathbf{A})$ is a homogeneous ideal in $Z(\mathbf{A})$. The residue class algebra $H(\mathbf{A}) = Z(\mathbf{A})/B(\mathbf{A})$ is called the *homology algebra of* $\mathbf{A}$ and has the structure of a commutative bigraded A-algebra.

(3.12) Let $\mathbf{A}$ be a differential Γ-algebra over A and let $\mathbf{M}$ be an object of the category R-Bimod. By (3.5) the algebra $\mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle$ is a bigraded Γ-algebra over A. Let $\varphi\colon \mathbf{M}\to Z(\mathbf{A})$ be a homomorphism of bigraded R-modules of degree $(0, -1)$.

We shall prove the following

(3.13) THEOREM. *The algebra* $\mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle$ *can be endowed with a unique differential* d *such that*

(i) $\mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle$ *is differential* Γ-*algebra*,

(ii) $d(1\otimes x) = \varphi(x)\otimes 1$ *for* $x\in\mathbf{M}$,

(iii) $d|\mathbf{A} = d_{\mathbf{A}}$.

Proof. Since $\Gamma\langle\mathbf{M}\rangle = \Gamma\langle\mathbf{M}^-\rangle\otimes\Gamma\langle\mathbf{M}^+\rangle$ it is sufficient to prove the theorem for $\mathbf{M} = \mathbf{M}^+$ and for $\mathbf{M} = \mathbf{M}^-$, separately.

1) $\mathbf{M} = \mathbf{M}^+$. From the definition of the algebra $\Gamma\langle\mathbf{M}\rangle$ it follows that there exists a unique R-linear mapping $\bar{d}\colon \Gamma\langle\mathbf{M}\rangle\to\mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle$ such that

(1) $\bar{d}\big(\gamma_k(x)\big) = \varphi(x)\otimes\gamma_{k-1}(x)$ for homogeneous $x\in\mathbf{M}$,

(2) $\bar{d}(yz) = \bar{d}y(1\otimes z)+(-1)^{\omega(y)}(1\otimes y)\bar{d}z$ for $y, z\in\Gamma\langle\mathbf{M}\rangle$.

2) $\mathbf{M} = \mathbf{M}^-$. In this case we define $\bar{d}\colon \Gamma\langle\mathbf{M}\rangle\to\mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle$ as follows $\bar{d}(x_1\wedge\ldots\wedge x_r) = \sum_{i=1}^{r}(-1)^{i+1}\varphi(x_i)\otimes x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge x_r$, $x_i\in\mathbf{M}$. Standard computation shows that formulas (1), (2) are fulfilled.

In both cases we define $d\colon \mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle\to\mathbf{A}\otimes\Gamma\langle\mathbf{M}\rangle$ by putting

$$d(a\otimes x) = da\otimes x+(-1)^{\omega(a)}(a\otimes 1)\bar{d}x$$

for homogeneous elements $a\in\mathbf{A}$, $x\in\Gamma\langle\mathbf{M}\rangle$. We shall verify that d has properties (i)–(iii) of Theorem (3.13). It is evident that d is an A-homomorphism. Now let a, b, x, y be homogeneous elements, $a, b\in\mathbf{A}$, $x, y\in\Gamma\langle\mathbf{M}\rangle$. Write $\partial(x)\partial(b)+\omega(x)\omega(b) = \alpha$. We have

$$\begin{aligned} d\big((a\otimes x)(b\otimes y)\big) &= d\big((-1)^{\alpha}ab\otimes xy\big) \\ &= (-1)^{\alpha}da\cdot b\otimes xy+(-1)^{\alpha+\omega(a)}a\cdot db\otimes xy+ \\ &\quad +(-1)^{\alpha+\omega(a)+\omega(b)}(ab\otimes 1)\bar{d}x(1\otimes y)+ \\ &\quad +(-1)^{\alpha+\omega(a)+\omega(b)+\omega(x)}(ab\otimes 1)(1\otimes x)\bar{d}y\,. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d(a\otimes x)(b\otimes y)&+(-1)^{\omega(a\otimes x)}(a\otimes x)d(b\otimes y) \\ &= \big(da\otimes x+(-1)^{\omega(a)}(a\otimes 1)\bar{d}x\big)(b\otimes y)+ \\ &\quad +(-1)^{\omega(a)+\omega(x)}(a\otimes x)\big(db\otimes y+(-1)^{\omega(b)}(b\otimes 1)\bar{d}y\big) \\ &= (-1)^{\alpha}da\cdot b\otimes xy+ \\ &\quad +(-1)^{\omega(a)+\partial(b)\partial(dx)+\omega(b)\omega(dx)}(ab\otimes 1)\bar{d}x(1\otimes y)+ \\ &\quad +(-1)^{\omega(a)+\omega(x)+\partial(x)\partial(db)+\omega(x)\omega(db)}a\cdot db\otimes xy+ \\ &\quad +(-1)^{\omega(a)+\omega(x)+\omega(b)+\alpha}(ab\otimes 1)(1\otimes x)\bar{d}y\,. \end{aligned}$$

From this computation we have obtained the formula (ii) of (3.11). The validity of the formula (iii) of (3.11) follows from (1) and from the appropriate property of the algebra $\mathbf{A}$. The properties (ii) and (iii) of the theorem follow immediately from the definition of d.

We shall denote the differential Γ-algebra described in Theorem (3.13) by $\mathbf{A}\langle\mathbf{M};\varphi\rangle$ and we shall call $\mathbf{A}\langle\mathbf{M};\varphi\rangle$ the differential Γ-algebra obtained from $\mathbf{A}$ by the adjunction of the module $\mathbf{M}$ by means of the map $\varphi\colon \mathbf{M}\to Z(\mathbf{A})$. If $\mathbf{M}$ is a free R-module with base $\{S_i\}_{i\in\Lambda}$, we shall often write $\mathbf{A}\langle\{S_i\}_{i\in\Lambda};\varphi\rangle$ for $\mathbf{A}\langle\mathbf{M};\varphi\rangle$.

From (3.9) and from Theorem (3.13) we obtain

(3.14) COROLLARY. *If* $\varphi\colon \mathbf{M}\to Z(\mathbf{A})$, $\eta\colon \mathbf{N}\to Z(\mathbf{A})$ *are homomorphisms of bigraded* R-*modules of degree* $(0, -1)$ *then we have an isomorphism of differential* Γ-*algebras*

$$\mathbf{A}\langle\mathbf{M};\varphi\rangle\langle\mathbf{N};\eta\rangle\simeq\mathbf{A}\langle\mathbf{M}\oplus\mathbf{N};\varphi\oplus\eta\rangle\,.$$

The above corollary will be used freely in the next sections.

(3.15) LEMMA. *Let* $\mathbf{A}, \mathbf{A}'$ *be differential* Γ-*algebras over* A *and let* $\alpha\colon \mathbf{A}\to\mathbf{A}'$ *be an isomorphism of differential* Γ-*algebras. Further, suppose that* $\mathbf{M}, \mathbf{M}'$ *are free* R-*modules in* R-Bimod *and* $\psi\colon \mathbf{M}\to\mathbf{M}'$ *is an isomorphism. If* $\varphi\colon \mathbf{M}\to Z(\mathbf{A})$, $\varphi'\colon \mathbf{M}'\to Z(\mathbf{A}')$ *are* R-*homomorphisms of degree* $(0, -1)$ *making the diagram*

$$(3)\qquad \begin{array}{ccccc} \mathbf{M} & \xrightarrow{\varphi} & Z(\mathbf{A}) & \to & H(\mathbf{A}) \\ {\scriptstyle\psi}\downarrow & & & & \downarrow{\scriptstyle H(\alpha)} \\ \mathbf{M}' & \xrightarrow{\varphi'} & Z(\mathbf{A}') & \to & H(\mathbf{A}') \end{array}$$

commutative, then there exists an isomorphism of differential Γ-*algebras* $\omega\colon \mathbf{A}\langle\mathbf{M};\varphi\rangle\to\mathbf{A}'\langle\mathbf{M}';\varphi'\rangle$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{A} & \hookrightarrow & \mathbf{A}\langle\mathbf{M};\varphi\rangle \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\omega} \\ \mathbf{A}' & \hookrightarrow & \mathbf{A}'\langle\mathbf{M}';\varphi'\rangle \end{array}$$

Proof. From the commutativity of the diagram (3) it follows that for each $m \in M$ the element

$$T'_m = a\varphi(m) - \varphi'\psi(m) \tag{4}$$

is a boundary in A'. Let a set $\{x_i\}$ be a homogeneous base of the R-module M. We choose arbitrarily homogeneous elements $t'_{x_i} \in A'$ such that $T'_{x_i} = d't'_{x_i}$. Then for an element $m = \sum \lambda_i x_i$ we define $t'_m = \sum \lambda_i t'_{x_i}$. By (4) we have for each $m \in M$

$$a\varphi(m) - \varphi'\psi(m) = d(t'_m) \tag{5}$$

and the mapping $m \mapsto t'_m$ is R-linear. We define $\overline{\omega}\colon M \to A'\langle M'; \varphi'\rangle$ by putting $\overline{\omega}(m) = t'_m \otimes 1 + 1 \otimes \psi(m)$. From (3.8) it follows that $\overline{\omega}$ has a unique extension (also denoted by $\overline{\omega}$) to the homomorphism $\overline{\omega}\colon \Gamma\langle M\rangle \to A'\langle M'; \varphi'\rangle$. We define ω by $\omega(a \otimes x) = \alpha(a)\overline{\omega}(x)$ for $a \in A$, $x \in \Gamma\langle M\rangle$. It can easily be proved by using the formula (5) that ω is a homomorphism of differential Γ-algebras.

Now write $\beta = \alpha^{-1}$, $\eta = \psi^{-1}$ and define for each $m' \in M'$ $t_{m'} = -\beta(t'_{\eta(m')})$. By applying the differential d to the last equality we obtain

$$\beta\varphi'(m') - \varphi\eta(m') = d(t_{m'})\,. \tag{6}$$

Similarly as in the case of ω we define a homomorphism $\overline{\vartheta}\colon \Gamma\langle M'\rangle \to A\langle M; \varphi\rangle$ such that $\overline{\vartheta}(m') = t_{m'} \otimes 1 + 1 \otimes \eta(m')$ for $m' \in M'$. Further, let $\vartheta\colon A'\langle M'; \varphi'\rangle \to A\langle M; \varphi\rangle$ be a map such that $\vartheta(a' \otimes x') = \beta(a') \cdot \overline{\vartheta}(x')$. ϑ is in fact the homomorphism of differential Γ-algebras and we shall prove that ω and ϑ are inverse to each other. By the universal property of the functor Γ it is sufficient to prove $\vartheta\omega(a \otimes m) = a \otimes m$ for $m \in M$. Immediately from the definition we have

$$\begin{aligned}
\vartheta\omega(a \otimes m) &= \vartheta\big((\alpha(a) \otimes 1)(t'_m \otimes 1 + 1 \otimes \psi(m))\big)\\
&= \vartheta\big(\alpha(a)t'_m \otimes 1 + \alpha(a) \otimes \psi(m)\big)\\
&= \beta\big(\alpha(a)t'_m\big) \otimes 1 + \big(\beta\alpha(a) \otimes 1\big)\big(t_{\psi(m)} \otimes 1 + 1 \otimes \eta\psi(m)\big)\\
&= a\beta(t'_m) \otimes 1 + at_{\psi m} \otimes 1 + a \otimes m\\
&= a\big(\beta(t'_m) + t_{\psi(m)}\big) \otimes 1 + a \otimes m = a \otimes m\,.
\end{aligned}$$

A similar computation shows that $\omega\vartheta = \mathrm{Id}$.

We recall that if A is an R-algebra, then $I' = \bigoplus_{i \geq 1} A_i$.

(3.16) THEOREM. *Let A, A' be differential Γ-algebras over A and let $\alpha\colon A \to A'$ be an isomorphism of differential Γ-algebras. Suppose that $H_n(A)$ (and so $H_n(A')$) is annihilated by I' as an A-module. Let M, M' be free R-modules from R-Bimod. If $\varphi\colon M \to Z_n(A)$, $\varphi'\colon M' \to Z_n(A')$ are such R-homomorphisms of degree $(0, -1)$ that the composed maps $\widetilde{\varphi}\colon A \otimes M \to Z_n(A) \to H_n(A)$, $\widetilde{\varphi}'\colon A \otimes M' \to Z_n(A') \to H_n(A')$ are minimal epimorphisms in the category A-Mod, then there exists an isomorphism $\omega\colon A\langle M; \varphi\rangle \to A'\langle M'; \varphi'\rangle$ of differential Γ-algebras such that the diagram*

$$\begin{array}{ccc}
A & \hookrightarrow & A\langle M; \varphi\rangle\\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\omega}\\
A' & \hookrightarrow & A'\langle M'; \varphi'\rangle
\end{array}$$

is commutative.

Proof. Since $A \otimes M$, $A \otimes M'$ are free graded A-modules and $\widetilde{\varphi}\colon A \otimes M \to H_n(A)$, $\widetilde{\varphi}'\colon A \otimes M' \to H_n(A')$ are minimal epimorphisms, then it follows from (1.12) that there exists an isomorphism $\psi\colon A \otimes M \to A \otimes M'$ of graded A-modules such that the diagram

$$\begin{array}{ccc}
A \otimes M & \xrightarrow{\widetilde{\varphi}} & H_n(A)\\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle H_n(\alpha)}\\
A \otimes M' & \xrightarrow{\widetilde{\varphi}'} & H_n(A')
\end{array} \tag{7}$$

is commutative. Now consider a functor $T\colon A$-Bimod $\to R$-Bimod defined by $T(Y) = Y/I'Y$. Since $T(A \otimes M) \simeq M$, $T(A \otimes M') \simeq M'$ and $H_n(A)$, $H_n(A')$ are annihilated by I', then by applying the functor T to the diagram (7) we obtain the commutative diagram

$$\begin{array}{ccccc}
M & \xrightarrow{\varphi} & Z_n(A) & \longrightarrow & H_n(A)\\
{\scriptstyle T(\psi)}\downarrow & & & & \downarrow{\scriptstyle H_n(\alpha)}\\
M' & \xrightarrow{\varphi'} & Z_n(A') & \longrightarrow & H_n(A')
\end{array}$$

Lemma (3.15) implies the existence of an isomorphism with the required properties.

§ 4. Tate resolutions. Let A be an R-algebra. We recall that $I = \mathfrak{m} \oplus \bigoplus_{i \geq 1} A_i$ and $k = A/I$. Notice that a differential Γ-algebra A over A furnishes us with a left differential complex in the category A-Mod

$$\ldots \to A_{*,n+1} \xrightarrow{d_A} A_{*,n} \xrightarrow{d_A} \ldots \xrightarrow{d_A} A_{*,0} \to 0\,.$$

(4.1) THEOREM. *There exists a differential Γ-algebra X over A which is a free resolution of the graded A-module k.*

Proof. We shall obtain X as the union of an ascending chain of differential Γ-algebras $F_0X \subset F_1X \subset \ldots$ We define F_0X to be the trivial Γ-algebra A, i.e. $(F_0X)_{*,0} = A$, $(F_0X)_{*,p} = 0$ for $p > 0$, $d = 0$. To define F_1X we first choose a free bigraded R-module $M_{*,1}$ and a homomorphism $\varphi_1\colon M_{*,1} \to I$ of bigraded R-modules of degree $(0, -1)$ such that the homomorphism $A\otimes M_{*,1}\to I$, $a\otimes m\mapsto a\varphi_1(m)$, is a minimal epimorphism in the category A-Mod. For any homogeneous element $x\in M_{*,1}$ we have $\omega(x)=1$ of course. We define $F_1X=F_0X\langle M_{*,1};\varphi_1\rangle$ (see (3.12), (3.13)). Clearly, $H_0(F_1X)\simeq k$. Now let $M_{*,2}$ be a free bigraded R-module and let $\varphi_2\colon M_{*,2}\to Z_1(F_1X)$ be such a mapping of bigraded R-modules of degree $(0,-1)$ that the composed A-homomorphism $A\otimes M_{*,2}\to Z_1(F_1X)\to H_1(F_1X)$ is a minimal epimorphism in A-Mod. We define $F_2X=F_1X\langle M_{*,2};\varphi_2\rangle$. Clearly, $(F_2X)_{*,0}=(F_1X)_{*,0}$, $(F_2X)_{*,1}=(F_1X)_{*,1}$ and consequently $H_0(F_2X)\simeq k$, $H_1(F_2X)=0$. Proceeding in this way, we define inductively for $q>0$

$$F_{q+1}X=F_qX\langle M_{*,q+1};\varphi_{q+1}\rangle\,,$$

where $M_{*,q+1}$ is a free bigraded R-module, $\omega(x)=q+1$ for homogeneous $x\in M_{*,q+1}$, and $\varphi_{q+1}\colon M_{*,q+1}\to Z_q(F_qX)$ is such an R-homomorphism of bigraded R-modules of degree $(0,-1)$ that the A-homomorphism $A\otimes M_{*,q+1}\to Z_q(F_qX)\to H_q(F_qX)$ is a minimal epimorphism in A-Mod. Observe that it is always possible to choose a map φ_{q+1} with the above-mentioned properties. If the elements of the set $\{\xi_i\}_{i\in\Lambda}$, $\xi_i\in Z_q(F_qX)$, are representatives of a minimal set of generators of the A-module $H_q(F_qX)$, then we define $M_{*,q+1}$ to be the free R-module with the homogeneous base $\{x_i\}_{i\in\Lambda}$, $\partial(x_i)=\partial(\xi_i)$, $\omega(x_i)=q+1$, and we put $\varphi_{q+1}(x_i)=\xi_i$.

We define $X=\bigcup_{q\geqslant 0}F_qX$. It follows immediately from the construction that

$$H_0(F_qX)\simeq k\,,\qquad H_i(F_qX)=0\quad\text{ for }\quad 0<i<q,\ q\geqslant 1\,,$$

and

$$X_{*,q}=(F_qX)_{*,q}=(F_{q+1}X)_{*,q}=\ldots$$

This proves that X is acyclic and is therefore a free resolution of the graded A-module k.

(4.2) Definition. The differential Γ-algebra X constructed as in the proof of Theorem (4.1) is called a *Tate resolution of the R-algebra A*.

(4.3) Remark. By the same method as in the proof of Theorem (4.1) a free resolution of an arbitrary cyclic A-module can be built. More generally this construction leads to the notion of an acyclic closure of the bigraded Γ-algebra over A (see [5]).

(4.4) Remark. If $A=R$ is a local ring with trivial grading, then a Tate resolution of the ring R was first constructed by Tate in [11].

(4.5) Remark. Suppose that R is a field. Under this assumption the class of algebras for which Tate resolutions exist may be enlarged. If in the definition of an R-algebra we do not assume that homogeneous components are finitely generated, then all the results of § 1 will still be valid. This allows us to build Tate resolutions for such generalized R-algebras. The appropriate results of the present paper suitably modified are valid in this more general situation.

A priori a Tate resolution depends on the choice of the modules $M_{*,n}$ and the mappings φ_n. But we shall prove the following

(4.6) Theorem. *Any two Tate resolutions of the R-algebra A are isomorphic as differential Γ-algebras.*

Proof. Suppose that we have two Tate resolutions X, X', determined by the modules $M_{*,n}$, $M'_{*,n}$ and the homomorphisms φ_n, φ'_n, respectively. We shall construct isomorphisms $\omega_n\colon F_nX\to F_nX'$, $n=0,1,\ldots$, such that the diagrams

$$\begin{array}{ccc} F_nX & \hookrightarrow & F_{n+1}X\\ {\scriptstyle\omega_n}\downarrow & & \downarrow{\scriptstyle\omega_{n+1}}\\ F_nX' & \hookrightarrow & F_{n+1}X' \end{array}$$

are commutative. By passing to the direct limit we shall obtain the required isomorphism $\omega\colon X\to X'$.

Since $F_0X=F_0X'=A$, we put $\omega_0=\mathrm{Id}$. Next we construct the isomorphism ω_1. Since the mappings $A\otimes M_{*,1}\to I$, $a\otimes m\mapsto a\cdot\varphi_1(m)$, $A\otimes M'_{*,1}\to I$, $a\otimes m'\mapsto a\varphi'_1(m')$, are minimal epimorphisms, there exists an isomorphism $\overline{\psi}\colon A\otimes M_{*,1}\to A\otimes M'_{*,1}$ making the diagram

$$\begin{array}{ccc} A\otimes M_{*,1} & \longrightarrow & I\\ {\scriptstyle\overline{\psi}}\downarrow & & \|{\scriptstyle\mathrm{Id}}\\ A\otimes M'_{*,1} & \longrightarrow & I \end{array}$$

commutative. We define the homomorphism $\psi\colon\Gamma\langle M_{*,1}\rangle\to A\otimes\Gamma\langle M'_{*,1}\rangle$ by putting $\psi(m)=\overline{\psi}(m)$ for $m\in M_{*,1}$ and extending it to the whole algebra $\Gamma\langle M_{*,1}\rangle$ in a unique manner by the universal property of the functor Γ (see (3.8)). Finally we define $\omega_1\colon A\otimes\Gamma\langle M_{*,1}\rangle\to A\otimes\Gamma\langle M'_{*,1}\rangle$, $\omega_1(a\otimes x)=(a\otimes 1)\psi(x)$, $a\in A$, $x\in\Gamma\langle M_{*,1}\rangle$. It is easy to verify that ω_1 is an isomorphism of differential Γ-algebras, the inverse being a map defined similarly by means of the homomorphism $\overline{\psi}^{-1}$.

Now let $n\geqslant 1$. Observe that $H_n(F_nX)$ is annihilated by the ideal I. Indeed, any element from I is a boundary in F_nX for $n\geqslant 1$. Since all the assumptions of Theorem (3.16) are satisfied for the algebras F_nX, F_nX' and the isomorphism ω_n, we obtain the required isomorphism ω_{n+1}.

By Theorem (4.6) we may speak of *the* Tate resolution. In (1.10) we defined the notion of a minimal free resolution in the category A-Mod. Minimal resolutions are important because they determine completely the homology of an appropriate object, i.e. if F is a minimal free resolution of $M\in\operatorname{ob}A$-Mod, then we have $\operatorname{Tor}^A(M,k)=F\otimes k$. Fortunately the Tate resolution possesses this important property. We have

(4.7) THEOREM. *The Tate resolution of the R-algebra A is a minimal free resolution of the residue class field k, i.e. the associated differential complex*

$$\ldots \longrightarrow X_{*,p} \xrightarrow{d_X} X_{*,p-1} \xrightarrow{d_X} \ldots \xrightarrow{d_X} X_{*,0} \longrightarrow 0$$

is minimal in the category A-Mod.

In the case of a local ring (i.e. $A = R$) several proofs of Theorem (4.7) are known (see [4], [8], [10]). The theorem in its full generality can be proved by using Gulliksen's idea of applying a special class of mappings, so called derivations. But the notion of derivation should be adapted to the bigraded case in the following way:

(4.8) DEFINITION. Let $\boldsymbol{A}$ be a differential Γ-algebra over A. A homogeneous mapping $J\colon \boldsymbol{A} \to \boldsymbol{A}$ is called a *derivation of the algebra* $\boldsymbol{A}$ if

(i) J is an A-homomorphism,

(ii) $d_{\boldsymbol{A}} J = J d_{\boldsymbol{A}}$,

(iii) $J(xy) = (-1)^{\partial(J)\partial(y)+\omega(J)\omega(y)} J(x)\cdot y + x\cdot J(y)$

for homogeneous $x, y \in \boldsymbol{A}$. The pair $(\partial(J), \omega(J))$ denotes the degree of J.

The proof of Theorem (4.7) can be based on Gulliksen's beautiful paper [4] by using this notion of derivation.

§ 5. $\boldsymbol{X} = \boldsymbol{F}_1\boldsymbol{X}$. The aim of this section is to give the full characterization of those R-algebras for which the Tate resolution is attained at the first step of the construction presented in § 4.

Let A be an R-algebra and let $\{\xi_1, \ldots, \xi_n\}$ be the set of homogeneous elements contained in the ideal I. Denote by $\boldsymbol{M}$ the free bigraded R-module with base $S_1, \ldots, S_n$, $\partial(S_i) = \partial(\xi_i)$, $\omega(S_i) = 1$. Let $\varphi\colon \boldsymbol{M} \to I$ be such a map of R-modules that $\varphi(S_i) = \xi_i$ and denote by $\boldsymbol{Y}$ the differential Γ-algebra obtained from A by adjoining the module $\boldsymbol{M}$ by means of the map φ, i.e. $\boldsymbol{Y} = A\langle \boldsymbol{M}; \varphi\rangle = A\langle S_1, \ldots, S_n; \varphi(S_i) = \xi_i\rangle$. We shall prove at first

(5.1) PROPOSITION. *The following conditions are equivalent:*

(i) *the set $\{\xi_1, \ldots, \xi_n\}$ is a normal set in A,*

(ii) *$\boldsymbol{Y}$ is a free resolution of the graded A-module $A/(\xi_1, \ldots, \xi_n)$,*

(iii) $H_1(\boldsymbol{Y}) = 0$.

Proof. The implication (ii) $\Rightarrow$ (iii) is obvious. We shall prove the implication (iii) $\Rightarrow$ (i) by induction on the number of elements in the set $\{\xi_1, \ldots, \xi_n\}$.

Let $n = 1$; if $\partial(\xi_1)$ is even and $x\xi_1 = 0$, $x \in A$, then we have $d(x \otimes S_1) = x\xi_1 \otimes 1 = 0$ and by assumption $x \otimes S_1$ is a boundary in $\boldsymbol{Y}$. But $B_1(\boldsymbol{Y}) = 0$, so that $x \otimes S_1 = 0$ and consequently $x = 0$. Now let $\partial(\xi_1)$ be odd and let $x\xi_1 = 0$, $x \in A$. Similarly as above $x \otimes S_1$ is a one-dimensional cycle in $\boldsymbol{Y}$. From $H_1(\boldsymbol{Y}) = 0$ it follows that $x \otimes S_1$ is a boundary, i.e. there exists $y \in A$ such that $x \otimes S_1 = d(y \otimes \gamma_2(S_1))$. This implies $x = y\xi_1$, and ξ_1 is a normal sequence in A.

Now let $n > 1$ and assume that the implication (iii) $\Rightarrow$ (i) is true for $p < n$. By (3.14) we may write $\boldsymbol{Y} = \boldsymbol{Y}'\langle S_n; \varphi(S_n) = \xi_n\rangle$ where $\boldsymbol{Y}' = A\langle S_1, \ldots, S_{n-1}; \varphi(S_i) = \xi_i\rangle$. Consider two cases:

1) $\partial(\xi_n)$ is odd.

We have an exact sequence $0 \to \boldsymbol{Y}' \xrightarrow{\sigma} \boldsymbol{Y} \xrightarrow{\tau} \boldsymbol{Y} \to 0$, $\sigma(y') = y' \otimes 1$, $\tau(y' \otimes \gamma_q(S_n)) = y' \otimes \gamma_{q-1}(S_n)$, $y' \in \boldsymbol{Y}'$, which induces the long homology sequence

$$(1)\qquad \begin{gathered} \ldots \to H_p(\boldsymbol{Y}') \to H_p(\boldsymbol{Y}) \to H_{p-1}(\boldsymbol{Y}) \to \ldots \\ \ldots \to H_1(\boldsymbol{Y}) \xrightarrow{\Delta} H_1(\boldsymbol{Y}') \to H_1(\boldsymbol{Y}) \to H_0(\boldsymbol{Y}) \xrightarrow{\Delta} H_0(\boldsymbol{Y}') \to H_0(\boldsymbol{Y}) \to 0 . \end{gathered}$$

From the exactness of (1) and from the assumption of $H_1(\boldsymbol{Y}) = 0$ it follows that $H_1(\boldsymbol{Y}') = 0$, and so, by the induction hypothesis, the sequence $\xi_1, \ldots, \xi_{n-1}$ is normal. Since $H_0(\boldsymbol{Y}) = A/(\xi_1, \ldots, \xi_n)$, $H_0(\boldsymbol{Y}') = A/(\xi_1, \ldots, \xi_{n-1})$ we obtain from the homology sequence the short exact sequence

$$0 \to A/(\xi_1, \ldots, \xi_n) \xrightarrow{\Delta} A/(\xi_1, \ldots, \xi_{n-1}) \to A/(\xi_1, \ldots, \xi_n) \to 0 ,$$

the map Δ being multiplication by ξ_n. The injectivity of Δ means simply that ξ_n is a non-zero divisor $\bmod(\xi_1, \ldots, \xi_{n-1})$ and consequently $\xi_1, \ldots, \xi_n$ form a normal sequence in A.

2) $\partial(\xi_n)$ is even.

In this case we have an exact sequence $0 \to \boldsymbol{Y}' \xrightarrow{\sigma} \boldsymbol{Y} \xrightarrow{\tau} \boldsymbol{Y}' \to 0$, $\sigma(y') = y' \otimes 1$, $\tau(y' \otimes 1) = 0$, $\tau(y' \otimes S_n) = y'$, $y' \in \boldsymbol{Y}'$. This short exact sequence induces the long exact homology sequence

$$(2)\qquad \begin{gathered} \ldots \longrightarrow H_p(\boldsymbol{Y}') \xrightarrow{\sigma_*} H_p(\boldsymbol{Y}) \xrightarrow{\tau_*} H_{p-1}(\boldsymbol{Y}') \xrightarrow{\Delta} H_{p-1}(\boldsymbol{Y}') \longrightarrow \ldots \\ \ldots \longrightarrow H_1(\boldsymbol{Y}') \xrightarrow{\Delta} H_1(\boldsymbol{Y}') \xrightarrow{\sigma_*} H_1(\boldsymbol{Y}) \xrightarrow{\tau_*} H_0(\boldsymbol{Y}') \xrightarrow{\Delta} H_0(\boldsymbol{Y}') \xrightarrow{\sigma_*} H_0(\boldsymbol{Y}) \longrightarrow 0 . \end{gathered}$$

It is easy to compute that Δ is just multiplication by $\pm\xi_n$. So by the exactness of the above sequence and from the assumption of $H_1(\boldsymbol{Y}) = 0$ it follows that $H_1(\boldsymbol{Y}') = \xi_n H_1(\boldsymbol{Y}')$ and from Nakayama Lemma we get $H_1(\boldsymbol{Y}') = 0$. From the induction hypothesis we infer that $\{\xi_1, \ldots, \xi_{n-1}\}$ is a normal set in A. The normality of the whole sequence $\xi_1, \ldots, \xi_n$ follows from the exactness of the sequence

$$0 \longrightarrow H_0(\boldsymbol{Y}') = A/(\xi_1, \ldots, \xi_{n-1}) \xrightarrow{\xi_n} H_0(\boldsymbol{Y}') = A/(\xi_1, \ldots, \xi_{n-1}) .$$

The implication (i) $\Rightarrow$ (ii) will also be proved by induction on n. Let $n = 1$. If $\partial(\xi_1)$ is even, then immediately from definition we get $H_i(\boldsymbol{Y}) = 0$ for $i \geqslant 1$. Now let $\partial(\xi_1)$ be odd. If an element $y = x \otimes \gamma_i(S_1)$ is an i-dimensional cycle, then $x\xi_1 = 0$ and from the normality of ξ_1 it follows that there exists an element $x' \in A$ with $x = x'\xi_1$. This implies that $y = x \otimes \gamma_i(S_1) = d\big(x' \otimes \gamma_{i+1}(S_1)\big)$ is a boundary, i.e. $H_i(Y) = 0$ for $i \geqslant 1$.

Now suppose that the implication (i) ⇒ (ii) is true for $p < n$. We write as above $Y = Y'\langle S_n; \varphi(S_n) = \xi_n\rangle$, $Y' = A\langle S_1, \ldots, S_{n-1}; \varphi(S_i) = \xi_i\rangle$. If $\partial(\xi_n)$ is even then from the induction hypothesis it follows that $H_p(Y') = 0$ for $p \geqslant 1$, and from the sequence (2) we obtain $H_p(Y) = 0$ for $p > 1$. Since $H_0(Y') = A/(\xi_1, \ldots, \xi_{n-1})$ and because the sequence $\xi_1, \ldots, \xi_n$ is normal, the short sequence $0 \to H_0(Y') \xrightarrow{\xi_n} H_0(Y') \to H_0(Y) \to 0$ is exact. But this means that $H_1(Y) = 0$ and $H_0(Y) = A/(\xi_1, \ldots, \xi_n)$. Now assume that $\partial(\xi_n)$ is odd. From the induction hypothesis and from the exactness of the long homology sequence (1) we obtain $H_i(Y) \simeq H_{i-1}(Y)$ for $i > 1$. To finish the proof we have to show that $H_1(Y) = 0$. Consider again the exact sequence (1). Since $\{\xi_1, \ldots, \xi_n\}$ is a normal set, we know that the connecting homomorphism $\Delta\colon H_0(Y) = A/(\xi_1, \ldots, \xi_n) \to H_0(Y') = A/(\xi_1, \ldots, \xi_{n-1})$ is injective (because Δ is just multiplication by ξ_n). So the exactness of (1) gives us $H_1(Y) = 0$.

From (2.8), Theorem (2.6), Proposition (5.1) and by easy direct limit arguments we obtain

(5.2) Theorem. *Let A be an R-algebra. The following conditions are equivalent*:

(i) *the ring R is a regular local ring and A is isomorphic with the tensor product of an exterior algebra ΛM and a symmetric algebra SN, $A \simeq \Lambda M \otimes SN$, where M and N are free R-modules,*

(ii) *any minimal set of generators of the ideal I is a normal set in A,*

(iii) *there exists a minimal normal set of generators of the ideal I,*

(iv) *if X is the Tate resolution of A, then $X = F_1X$,*

(v) *if X is the Tate resolution of A, then $H_1(F_1X) = 0$.*

§ 6. Application to the Poincaré series.

Let A be an R-algebra which is finitely generated as an algebra over R. From this assumption it follows that each homogeneous component F_i of a minimal free resolution F of a graded A-module k is a free A-module of finite rank. Since any two minimal resolutions are isomorphic (see (1.12)), the numbers $b_i = \operatorname{rank}_A F_i$ are independent of the choice of a particular minimal resolution. The number b_i will be called the *i-th Betti number of the R-algebra A* and the formal power series $\mathfrak{P}(A) = \sum_{i=0}^{\infty} b_i t^i$ will be called the *Poincaré series of A*. In this section we shall apply the theory of the Tate resolutions to the computation of the Poincaré series.

Let X be the Tate resolution of the R-algebra A. We recall that there exists an increasing filtration $F_0X \subset F_1X \subset \ldots$ in X and $F_{q+1}X = F_qX\langle M_{*,q+1}; \varphi_{q+1}\rangle$, $q \geqslant 0$. For the bigraded R-module M write as in (3.6) $M^+ = \bigoplus_{p+q\,\text{even}} M_{p,q}$, $M^- = \bigoplus_{p+q\,\text{odd}} M_{p,q}$. Further, write $n_p = \operatorname{rank}_R M^-_{*,p}$, $m_p = \operatorname{rank}_R M^+_{*,p}$, $p = 1, 2, \ldots$ From the minimality of the Tate resolution (see (4.7)) it follows

(6.1) Corollary. *The Poincaré series of a finitely generated R-algebra A has the form*

$$\mathfrak{P}(A) = \frac{(1+t)^{n_1}(1+t^2)^{n_2}(1+t^3)^{n_3}\ldots}{(1-t)^{m_1}(1-t^2)^{m_2}(1-t^3)^{m_3}\ldots}.$$

If every homogeneous element of the algebra A has an even degree, then $M^+_{*,p} = 0$ for p odd and $M^-_{*,p} = 0$ for p even. Thus we have

(6.2) Corollary. *The Poincaré series of a finitely generated R-algebra A which has only homogeneous elements of even degrees has the form*

$$\mathfrak{P}(A) = \prod_{p=0}^{\infty} \frac{(1+t^{2p+1})^{n_{2p+1}}}{(1-t^{2p+2})^{m_{2p+2}}}.$$

In particular, if $A = R$ then the Poincaré series of a local ring R has the above form (see [4], [10]).

(6.3) Corollary. *If the algebra A is not isomorphic with a polynomial algebra over a regular local ring, then the sequence of Betti numbers of A is non-decreasing, $b_0 \leqslant b_1 \leqslant b_2 \leqslant \ldots$*

Proof. If the algebra A contains a homogeneous element of odd degree, then $M^+_{*,1} \neq 0$ and $m_1 > 0$. Thus $\sum b_i \cdot t^i = \mathfrak{P}(A) = \big(1/(1-t)\big)\sum c_i t^i = \big(\sum t^i\big)\big(\sum c_i t^i\big)$, $c_i \geqslant 0$, and the corollary is proved. Suppose now that all homogeneous elements in A have even degrees. If A is not a polynomial algebra over a regular local ring, then by Theorem (5.2) we have $H_1(F_1X) \neq 0$, where X is the Tate resolution of A. But by (6.2) this means that $m_2 > 0$ and similarly as above $\sum b_i t^i = \mathfrak{P}(A) = (1/1-t^2)\sum c_i t^i$.

From Theorem (5.2) it follows

(6.4) Corollary. *The Poincaré series of an R-algebra A has the form*

$$\mathfrak{P}(A) = \frac{(1+t)^{n_1}}{(1-t)^{m_1}}$$

if and only if R is a regular local ring and A is isomorphic with the tensor product of an exterior algebra ΛM and a symmetric algebra SN, $A \simeq \Lambda M \otimes SN$, where M and N are free R-modules.

Now as an application of the Tate resolutions we shall prove the following change of rings theorem

(6.5) THEOREM. *Let A be a finitely generated R-algebra and let x be a non-zero divisor in A. Write $\bar{A} = A/xA$. For $x \notin I^2$ we have*

$$\mathcal{T}(A) = (1+t)\mathcal{T}(\bar{A}) \quad \textit{if} \quad \partial(x) \quad \textit{is even},$$

$$\mathcal{T}(A) = \frac{1}{1-t}\mathcal{T}(\bar{A}) \quad \textit{if} \quad \partial(x) \quad \textit{is odd}.$$

For $x \in I^2$ we have

$$\mathcal{T}(A) = (1-t^2)\mathcal{T}(\bar{A}) \quad \textit{if} \quad \partial(x) \quad \textit{is even},$$

$$\mathcal{T}(A) = \frac{1}{1+t^2}\mathcal{T}(\bar{A}) \quad \textit{if} \quad \partial(x) \quad \textit{is odd and } \operatorname{char} k = 0 .$$

For the proof of the theorem we shall need two lemmas.

(6.6) LEMMA. *Let A be an R-algebra and let x be a non-zero divisor in A. Assume that $\boldsymbol{A}$ is a differential Γ-algebra over A and $\underline{x} \in B_0(\boldsymbol{A})$. Therefore there exists an element $S \in \boldsymbol{A}_{*,1}$ such that $dS = x$. Write $\bar{\boldsymbol{A}} = \boldsymbol{A}/x\boldsymbol{A}$, $\bar{S} = S + xA \in Z_1(\bar{\boldsymbol{A}})$, $\sigma = \bar{S} + B(\bar{\boldsymbol{A}}) \in H_1(\bar{\boldsymbol{A}})$. If $f\colon \boldsymbol{A} \to \bar{\boldsymbol{A}}$ is a natural mapping and $f_*\colon H(\boldsymbol{A}) \to H(\bar{\boldsymbol{A}})$ a mapping induced by f, then $H(\bar{\boldsymbol{A}})$ is isomorphic with $\big(f_*H(\boldsymbol{A})\big)\langle Rw\rangle$ where Rw is a free R-module on one generator w. The isomorphism is given by the correspondence $\sigma \leftrightarrow w$.*

Proof. If $\partial(x)$ is even, then the arguments given in [11], Theorem 3, for local rings work without changes in this more general situation.

Now assume that $\partial(x)$ is odd. Since x is a non-zero divisor, we have an exact sequence of complexes $0 \to \bar{\boldsymbol{A}} \xrightarrow{\varphi} \boldsymbol{A} \xrightarrow{\psi} \bar{\boldsymbol{A}} \to 0$, $\varphi(y + x\boldsymbol{A}) = xy$, $\psi(y) = y + x\boldsymbol{A}$, $y \in \boldsymbol{A}$. This exact sequence induces an exact homology triangle

$$\begin{array}{ccc} H(\bar{\boldsymbol{A}}) & \xrightarrow{\varphi_*} & H(\boldsymbol{A}) \\ & {}_{\Delta}\nwarrow \quad \swarrow_{\psi_*} & \\ & H(\bar{\boldsymbol{A}}) & \end{array}$$

where Δ is a connecting homomorphism of degree -1. First we shall prove

$$\text{(i)} \qquad \begin{aligned} \Delta\big(\gamma_i(\sigma)\big) &= \gamma_{i-1}(\sigma), \\ \Delta(\tau\cdot\tau') &= \Delta(\tau)\cdot\tau' + (-1)^{\partial(\tau)+\omega(\tau)}\tau\cdot\Delta(\tau'). \end{aligned}$$

Write $\bar{y} = y + x\boldsymbol{A}$. From the definition of Δ it follows immediately that the homology class of the cycle $\bar{y}$ is sent by Δ to the homology class of $\bar{z}$ where $dy = xz$. Since $d\gamma_i(S) = x\gamma_{i-1}(S)$, we have the first formula in (i). The second equality in (i) follows from the product formula for the differential d.

To prove the lemma it suffices to show that the arbitrary homogeneous element $\tau \in H_n(\bar{\boldsymbol{A}})$ can be written uniquely in the form

$$\text{(ii)} \qquad \tau = \psi_*(a_0) + \sigma\psi_*(a_1) + \ldots + \gamma_n(\sigma)\psi_*(a_n), \qquad a_i \in H_{n-1}(\boldsymbol{A}).$$

This will be proved by induction on n. If $\tau \in H_0(\bar{\boldsymbol{A}})$, then (ii) is obvious. Now if $\tau \in H_n(\bar{\boldsymbol{A}})$, then $\Delta(\tau) \in H_{n-1}(\bar{\boldsymbol{A}})$ and by the induction hypothesis we have

$$\Delta(\tau) = \psi_*(a_1) + \sigma\psi_*(a_2) + \ldots + \gamma_{n-1}(\sigma)\psi_*(a_n).$$

Consider the element

$$\tau' = \tau - \sigma\psi_*(a_1) - \ldots - \gamma_n(\sigma)\psi_*(a_n).$$

Since $\Delta\psi_* = 0$, we obtain from (i) $\Delta\big(\gamma_p(\sigma)\psi_*(a)\big) = \gamma_{p-1}(\sigma)\psi_*(a)$. Thus we have $\Delta(\tau') = \Delta(\tau) - \psi_*(a_1) - \sigma\psi_*(a_2) - \ldots - \gamma_{n-1}(\sigma)\psi_*(a_n) = 0$. The exactness of the homology triangle implies that $\tau' = \psi_*(a_0)$, and consequently we have the formula (ii). To prove the uniqueness observe that for τ satisfying (ii) we have $\Delta^n(\tau) = \psi_*(a_n)$. So if $\tau = 0$, we obtain successively $\psi_*(a_n) = 0$, $\psi_*(a_{n-1}) = 0, \ldots, \psi_*(a_0) = 0$.

Now we state the following lemma, proved in [11] for local rings:

(6.7) LEMMA. *Let $\boldsymbol{A}$ be a differential Γ-algebra over A. Assume that s is a cycle in $\boldsymbol{A}$ and let $\boldsymbol{B} = \boldsymbol{A}\langle S; dS = s\rangle$. If the homology class σ of s is a non-zero divisor in $H(\boldsymbol{A})$, then the inclusion map $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ induces in the homology a surjection with kernel $\sigma H(\boldsymbol{A})$, i.e. $H(\boldsymbol{B}) \simeq H(\boldsymbol{A})/\sigma H(\boldsymbol{A})$.*

Proof of Theorem (6.5). Consider first the case $x \notin I^2$. This assumption implies that x is a member of some minimal set of generators of the ideal I (see (1.7)). But then in the construction of the Tate resolution $\boldsymbol{X}$ we can choose the base $\{S_i\}$ of the module $\boldsymbol{M}_{*,1}$ such that for $S = S_1$ we have $dS = x$. If $\boldsymbol{X}$ is the Tate resolution of A, then write $\bar{\boldsymbol{X}} = \boldsymbol{X}/x\boldsymbol{X}$. By Lemma (6.6) we have $H(\bar{\boldsymbol{X}}) = k\langle\sigma\rangle$, where σ is the homology class of the cycle $\bar{S}$.

Now suppose that $\partial(x)$ is odd. We define a differential complex $\boldsymbol{V}$ by the exactness of the following sequence of complexes

$$\text{(iii)} \qquad 0 \to \bar{\boldsymbol{X}} \xrightarrow{\alpha} \bar{\boldsymbol{X}} \xrightarrow{\beta} \boldsymbol{V} \to 0,$$

where α is determined by the condition $\alpha\big(\bar{y}\gamma_i(\bar{S})\big) = \bar{y}\gamma_{i+1}(\bar{S})$, $\bar{y} \in \bar{\boldsymbol{X}}$. We shall prove that $\boldsymbol{V}$ furnishes us with a free minimal resolution of a graded $\bar{A}$-module k. From (ii) we get the exact homology sequence

$$\ldots \longrightarrow H_{n+1}(\boldsymbol{V}) \xrightarrow{\Delta} H_n(\bar{\boldsymbol{X}}) \xrightarrow{\alpha_*} H_{n+1}(\bar{\boldsymbol{X}}) \xrightarrow{\beta_*} H_n(\boldsymbol{V}) \xrightarrow{\Delta} H_{n-1}(\bar{\boldsymbol{X}}) \longrightarrow \ldots$$

From $H(\bar{\boldsymbol{X}}) = k\langle\sigma\rangle$ and $\alpha_*\big(\gamma_i(\sigma)\big) = \gamma_{i+1}(\sigma)$ we infer that α_* is an isomorphism. But then from the exactness of the above sequence we get $H_i(\boldsymbol{V}) = 0$ for $i > 0$ and $H_0(\boldsymbol{V}) = H_0(\bar{\boldsymbol{X}}) \simeq k$. The minimality of $\boldsymbol{V}$ follows from the minimality of the Tate resolution $\boldsymbol{X}$. From (iii) we obtain the exact sequence of graded $\bar{A}$-modules $0 \to \bar{\boldsymbol{X}}_{*,p} \to \bar{\boldsymbol{X}}_{*,p+1} \to \boldsymbol{V}_{*,p+1} \to 0$ for arbitrary $p \geqslant 0$. Since $\operatorname{rank}_{\bar{A}} \bar{\boldsymbol{X}}_{*,p} = \operatorname{rank}_A \boldsymbol{X}_{*,p}$ and $\bar{b}_p = \operatorname{rank}_{\bar{A}} \boldsymbol{V}_{*,p}$ is the pth Betti number of $\bar{A}$, we have $\bar{b}_{p+1} = b_{p+1} - b_p$ for $p \geqslant 0$, $\bar{b}_0 = b_0$. Consequently $\mathfrak{I}(\bar{A}) = (1-t)\mathfrak{I}(A)$.

Now consider the case when $\partial(x)$ is even. Write $W = \bar{X}/\bar{S}\bar{X}$. Since $S^2 = 0$, we have an exact sequence of complexes

$$0 \to W \xrightarrow{\alpha} \bar{X} \xrightarrow{\beta} W \to 0\,, \quad \alpha(y + \bar{S}\bar{X}) = (-1)^{\omega(y)}\bar{S}y\,, \quad \beta(y) = y + \bar{S}\bar{X}\,,$$

$y \in \bar{X}$, and an appropriate exact homology sequence

$$\text{(iv)} \quad \ldots \longrightarrow H_{n+1}(\bar{X}) \xrightarrow{\beta_*} H_{n+1}(W) \xrightarrow{\Delta} H_{n-1}(W) \xrightarrow{\alpha_*} H_n(\bar{X}) \longrightarrow \ldots$$
$$\ldots \longrightarrow H_3(\bar{X}) \xrightarrow{\beta_*} H_3(W) \xrightarrow{\Delta} H_1(W) \xrightarrow{\alpha_*} H_2(\bar{X})$$
$$\xrightarrow{\beta_*} H_2(W) \xrightarrow{\Delta} H_0(W) \xrightarrow{\alpha_*} H_1(\bar{X}) \xrightarrow{\beta_*} H_1(W) \longrightarrow 0\,.$$

Since $H_0(W) \simeq k$, $H_1(\bar{X}) = k\sigma$ and $\alpha(1 + \bar{S}\bar{X}) = \bar{S}$, we have $\alpha_*(1) = \sigma$. This implies that α_* is an isomorphism and from (iv) we obtain $H_1(W) = 0$. Further, from the exactness of (iv) we know that $H_2(W) \xrightarrow{\Delta} H_0(W)$ is a zero homomorphism and $H_2(\bar{X}) \xrightarrow{\beta_*} H_2(W)$ is an epimorphism. But $H_2(\bar{X}) = 0$, so that $H_2(W) = 0$. Now assume that $n > 1$ and $H_{n-1}(W) = H_n(W) = 0$. From $H_m(\bar{X}) = 0$ for $m > 1$ and by the exactness of (iv) we get $H_{n+1}(W) = 0$. Since $H_1(W) = H_2(W) = 0$ and $H_0(W) \simeq k$, we have proved by induction that $H(W) = k$. Obviously W is a minimal resolution of k. From the exact sequence $0 \to W_{*,p} \to \bar{X}_{*,p+1} \to W_{*,p+1} \to 0$, $p \geqslant 0$, we obtain the required relation $\mathfrak{I}(A) = (1+t)\mathfrak{I}(\bar{A})$.

Now let $x \in I^2$. From this assumption it follows that in the Tate resolution X of A there exists such a homogeneous element $S \in X_{*,1}$ that $dS = x$ and $S \in IX$. If, as above, $\bar{X} = X/xX$, we know by (6.6) that $H(\bar{X}) = k\langle\sigma\rangle$. If $\partial(x)$ is even, then from Lemma (6.7) we infer that $W = \bar{X}\langle T; dT = \bar{S}\rangle$ is a free resolution of the graded $\bar{A}$-module k. Observe that W is a minimal resolution because $dT = \bar{S} \in \bar{I}\bar{X}$ and that $\omega(T) = 2$. This gives us the required formula $\mathfrak{I}(A) = (1-t^2)\mathfrak{I}(\bar{A})$. Suppose that $\partial(x)$ is odd. If $\operatorname{char} k = 0$, then $H(\bar{X}) = k\langle\sigma\rangle$ is isomorphic with the ring of polynomials of the variable σ. Thus σ is a non-zero divisor in $k\langle\sigma\rangle$ and by (6.7) we infer that the algebra $W = \bar{X}\langle T;\ dT = \bar{S}\rangle$ is a minimal resolution of k over $\bar{A}$. Hence $\mathfrak{I}(\bar{A}) = (1+t^2)\mathfrak{I}(A)$.

From Theorem (6.5) we obtain

(6.8) Corollary. *If $A = R[x_1, \ldots, x_n]$ is a polynomial algebra in variables $x_1, \ldots, x_n$, then*

$$\mathfrak{I}(A) = (1+t)^n\mathfrak{I}(R)\,.$$

(6.9) Corollary. *If $A = \Lambda(Rx_1 \oplus \ldots \oplus Rx_n)$ is the exterior algebra of the free R-module of rank n, then*

$$\mathfrak{I}(A) = \frac{1}{(1-t)^n}\mathfrak{I}(R)\,.$$

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