**References**

[1] A. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. 66 (1970), pp. 185–193.
[2] L. Gillman and M. Jerison, *Rings of continuous functions*, 1960.

*Reçu par le Rédaction le 17. 11. 1970*

# Remarks on open-closed mappings

by

**J. Chaber** (Warszawa)

I this paper we shall investigate the invariance and the inverse invariance of some topological properties under open-closed mappings. In the first section we shall prove a generalization of theorems proved by N. Dykes [4] and A. V. Arhangel'skiĭ [2] which shows a connection between the classes of open-closed and open-perfect mappings. In the second section we shall consider inverse images of complete spaces (1) under open-closed, open, and closed mappings. The last section is devoted to an investigation of invariance and inverse invariance of axioms of separation. In particular, we generalize (and simplify) an example of M. Henriksen and J. R. Isbell from [12] to show that an inverse image of a normal, complete space under an open-perfect mapping need not be completely regular.

We shall use the terminology and notation from [5]. By a *mapping* we always mean a continuous function. Perfect mappings will be assumed to be defined on Hausdorff spaces.

We shall often use the following simple criteria for the openness and closedness of mappings: A mapping $f\colon X \to Y$ is open iff the image of each element of a base of $X$ is open in $Y$; $f$ is closed iff $f(X) = Y$ and for each point $y$ in $Y$ and an open set $U$ in $X$ which contains $f^{-1}(y)$ there exists a neighbourhood $V$ of $y$ in $Y$ such that $f^{-1}(V) \subset U$.

**1. Invariance of real-compactness and strong paracompactness.** In this section the following notion will be widely applied.

Definition 1.1. A space $Y$ will be called a *c-space* iff for each non-isolated point $y \in Y$ and each sequence $\{U_n\}_{n=1}^{\infty}$ of neighbourhoods of $y$ there exists a set $\{y_n\}_{n=1}^{\infty}$ which is not closed and is such that $y_n \in U_n$ for every $n$.

---

(1) By a *complete space* we always mean a space which is complete in the sense of Čech. A space with the topology induced by complete uniformity will be called a *uniformly complete space* (such spaces are also called complete in the sense of Dieudonné).

THEOREM 1.2. *Let $f\colon X\to Y$ be an open-closed mapping of a uniformly complete space $X$ onto a $c$-space $Y$. If $y$ is a non-isolated point of $Y$, then $f^{-1}(y)$ is compact.*

Proof. Let $y$ be a non-isolated point of $Y$ and assume that $f^{-1}(y)$ is not compact. Therefore we can pick a point $x$ of $\beta X\setminus X$ in the closure of $f^{-1}(y)$ in $\beta X$ and, by the uniform completeness of $X$, we can find a paracompact space $P$ such that $X\subset P\subset \beta X\setminus\{x\}$ [9]. Thus there exists a locally finite open covering $\mathfrak{U}=\{U_a\}_{a\in A}$ of $X$ such that the closure of $U_a$ in $\beta X$ does not contain $x$ for any $a\in A$. Since $x\in\overline{f^{-1}(y)}\setminus f^{-1}(y)$, the set of elements of $\mathfrak{U}$ which intersect $f^{-1}(y)$ is infinite. Let us choose a countable subset $\{U_{a_n}\}_{n=1}^{\infty}$ consisting of different elements of $\mathfrak{U}$ all intersecting $f^{-1}(y)$. From the assumption that $Y$ is a $c$-space it follows that there exists a set $N=\{y_n\}_{n=1}^{\infty}$, where $y_n\in f(U_{a_n})$ for every $n$, which is not closed. Let us put $M=\{x_n\}_{n=1}^{\infty}$, where $x_n$ is an arbitrary point of the set $f^{-1}(y)\cap U_{a_n}$. From the local finitness of $\mathfrak{U}$ it follows that $M$ is closed. On the other hand, $N=f(M)$ is not closed, which contradicts the closedness of $f$.

A similar theorem is proved in [2], 4.1 (the space $Y$ being assumed to be a $k$-space) and in [4], 3.3 (the space $Y$ being assumed to be a $q$-space (²)). As follows from the next two lemmas, Theorem 1.2 is more general than those two statements.

LEMMA 1.3. *If the space $Y$ is the image under an open-closed mapping $f$ of a regular space $X$, then $Y$ is regular.*

Proof. As a closed image of a $T_1$-space, $Y$ is a $T_1$-space. Let $y\in Y$ and let $C$ be a closed set in $Y$ not containing $y$. Take any point $x\in f^{-1}(y)$ and disjoint open sets $U,V\subset X$ such that $x\in U$ and $f^{-1}(C)\subset V$. The open sets $f(U)$ and $Y\setminus f(X\setminus V)$ contain, respectively, $y$ and $C$ and are disjoint.

LEMMA 1.4. (a) *If a regular space $Y$ is a $q$-space, then it is also a $c$-space.*

(b) *If $Y$ is a $k$-space, then it is also a $c$-space.*

Proof. (a) Let $y$ be a non-isolated point of $Y$. From the definition of a $q$-space it follows that there exists a sequence $\{V_n\}_{n=1}^{\infty}$ of neighbourhoods of $y$ such that each set $\{y_n\}_{n=1}^{\infty}$, where $y_n\in V_n$ and $y_n\neq y_m$ for $n\neq m$, has an accumulation point. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of neighbourhoods of $y$. Using the fact that $Y$ is regular, one can easily define a sequence

(²) A space $X$ is called a *$q$-space* iff for each point $x\in X$ there exists a sequence of its neighbourhoods $\{V_n\}_{n=1}^{\infty}$ such that each set $\{y_n\}_{n=1}^{\infty}$, where $y_n\in V_n$ and $y_n\neq y_m$ for $n\neq m$, has an accumulation point [14].



$\{G_n\}_{n=1}^{\infty}$ of open sets and a sequence $\{y_n\}_{n=1}^{\infty}$ of elements of $Y$ which satisfy the following conditions:

(1) $\quad G_1=U_1\cap V_1\quad$ and $\quad G_n\cap U_{n+1}\cap V_{n+1}\supset\overline{G_{n+1}}\supset G_{n+1}\ni y$,

(2) $\quad y_n\in G_n\setminus\overline{G_{n+1}}$.

From (1) and (2) it follows that $y_n\in V_n$ and $y_n\neq y_m$ for $n\neq m$; therefore $\{y_n\}_{n=1}^{\infty}$ has an accumulation point which, as follows from (2), does not belong to $\{y_n\}_{n=1}^{\infty}$. Then the set $\{y_n\}_{n=1}^{\infty}$ is not closed and, as $y_n\in U_n$ for every $n$, it follows that $Y$ is a $c$-space.

(b) Let $y$ be an accumulation point of $Y$. There exists a compact set $F\subset Y$ such that $y$ is an accumulation point of $F$. As a compact space, $F$ is a regular $q$-space. Hence, by virtue of (a), the point $y$ satisfies the condition from Definition 1.1 in the subspace $F$ and *a fortiori* in the space $Y$.

We shall now give two examples to show that implications (a) and (b) of Lemma 1.4 are independent and that their converses are false.

The space obtained from the Euclidean plane by matching a line to a point is a normal $k$-space which is not a $q$-space. The following example of a normal $q$-space which is not a $k$-space is due to R. Engelking.

EXAMPLE 1.5. Let $X$ be the space of all ordinals not greater than $\omega_2$ that are not cofinal with $\omega_1$. The space $X$ is normal ([5], Problem 3.C). From the construction it follows that $X$ is countably compact, and hence $X$ is a $q$-space. The set $X\setminus\{\omega_2\}$ is not closed, but its intersection with every compact subset of $X$ is closed because the compact subsets of $X$ are of power not greater than $\aleph_0$. Thus $X$ is not a $k$-space.

Using our Theorem 1.2, one can easily generalize Theorem 4.4 from [2] and Corollary 4.4 from [4] (see also [16] and [13], 6.6). Namely, we have

THEOREM 1.6. *If $f\colon X\to Y$ is an open-closed mapping of a strongly paracompact (real-compact) space $X$ (³) onto a $c$-space $Y$, then the space $Y$ is strongly paracompact (real-compact).*

At last let us observe that, by virtue of Lemma 1.4, the class of $c$-spaces contains all spaces satisfying the first axiom of countability, countably compact regular spaces and $p$-spaces in the sense of Arhangel'skiĭ (⁴). In particular, locally compact spaces and complete spaces are $c$-spaces ([1], 2.3).

(³) Real-compact spaces and paracompact spaces are uniformly complete, see [5], p. 336 and Problem 8.0.

(⁴) A completely regular space $X$ is called a *$p$-space* iff in some (or, equivalently, in each) of its compactifications $rX$ there exists a sequence $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ of coverings of $X$ open in $rX$ and such that for each $x$ in $X$ we have $\bigcap_{n=1}^{\infty}\mathrm{St}(x,\mathfrak{U}_n)\subset X$ [1]. Clearly, every $p$-space is a $q$-space [14], and it can be verified that it is also a $k$-space ([1], 2.7).

**2. Spaces complete in the sense of Čech.** All spaces considered in this section are assumed to be completely regular. In the next section we shall show that this assumption is necessary.

To begin with, we shall prove a lemma.

LEMMA 2.1. *If $f\colon X\to Y$ is a closed mapping of a space $X$ onto a complete space $Y$ which has a closed countable covering $\{C_i\}_{i=1}^{\infty}$ such that all sets $f^{-1}(C_i)$ are complete, then $X$ is complete.*

Proof. Let $F\colon \beta X\to\beta Y$ be the extension of $f$ over $\beta X$. Each set ${}^{-1}(C_i)$ is closed in $X$ and is a $G_\delta$-set in $\overline{f^{-1}(C_i)}$ (in this proof closures will always be taken in the Čech-Stone compactifications) and therefore $X=\bigcup_{i=1}^{\infty} f^{-1}(C_i)$ is a $G_\delta$-set in $Z=\bigcup_{i=1}^{\infty}\overline{f^{-1}(C_i)}$. The inverse image $F^{-1}(Y)$ of the complete space $Y$ under the mapping $F$ is a $G_\delta$-set in $\beta X$. Then to prove our lemma it suffices to show that $F^{-1}(Y)\subset Z$. We shall prove that for $i=1,2,\ldots$ we have

$$F^{-1}(C_i)\subset\overline{f^{-1}(C_i)}\,.$$

Let us assume that for some $i$ there exists a point $x\in F^{-1}(C_i)\setminus\overline{f^{-1}(C_i)}$. Let $U$ be a neighbourhood of $x$ in $\beta X$ such that $\overline{U}\cap f^{-1}(C_i)=\emptyset$. The point $F(x)$ belongs to $C_i\subset Y$ and, on the other hand, we have

$$F(x)\in Y\cap F(\overline{\overline{U}\cap X})=Y\cap\overline{F(\overline{U}\cap X)}=f(\overline{U}\cap X)\subset Y\setminus C_i\,,$$

and the lemma is proved.

From the fact that an inverse image of a complete space under a perfect mapping is complete ([12], 2.2, 2.7), one can easily deduce, using Theorem 1.2 and Lemma 2.1, the following theorem.

THEOREM 2.2. *Let $f\colon X\to Y$ be an open-closed mapping of a uniformly complete space $X$ onto a complete space $Y$ such that the inverses of points are complete. If the derived set $Y^d$ of $Y$ is a $G_\delta$-set in $Y$, then $X$ is complete.*

We shall show that none of the assumptions in Theorem 2.2 can be omitted.

EXAMPLE 2.3. Let us put $Y=W\times I$ and $X=Y\times W\setminus[(V\times I)\times\{\omega_1\}]$, where $W$ denotes the space of all ordinals not greater than $\omega_1$, $V=W\setminus\{\omega_1\}$, and $I$ is the closed interval $[0,1]$. It is easy to check that the projection $f\colon X\to Y$ is open and closed (cf. the proof in [11] of the fact that the projection parallel to a countably compact axis onto a first countable space is closed). The inverses of points are locally compact, $Y$ is a compact space and $Y^d=Y$. The space $X$ is not complete because it is not a $G_\delta$-set in its compactification $Y\times W$. This is due to the fact that $X$ is not uniformly complete.



EXAMPLE 2.4. Let $\omega D$ be the Aleksandrov compactification of an uncountable discrete space $D$ and $\omega N$ the Aleksandrov compactification of the countable discrete space $N$; let $X=\omega N\times\omega D\setminus[(\omega N\setminus N)\times D]$. The projection $f\colon X\to Y=\omega D$ is both open and closed, the inverses of points under $f$ are locally compact and $Y$ is a compact space. The space $X$ is a Lindelöf space and therefore it is uniformly complete, but $X$ is not complete because it is not a $G_\delta$-set in its compactification $\omega N\times\omega D$. This is due to the fact that the derived set $Y^d$ is not a $G_\delta$-set in $Y$.

A similar example was defined by Filippov in [7] for a different purpose.

We shall now show that the assumption of openness of $f$ in Theorem 2.2 cannot be omitted. This is a consequence of the following theorem (to obtain the corresponding counter-example we take $Y=I$).

THEOREM 2.5. *In order that every inverse image of a space $Y$ under a closed mapping with complete inverses of points be complete it is necessary and sufficient that $Y$ be a complete space which can be expressed as a countable union of discrete closed subspaces. Moreover, if $Y$ is a paracompact (Lindelöf) space, we can restrict ourselves to paracompact (Lindelöf) inverse images of $Y$.*

Proof. From Lemma 2.1 it follows that our condition is sufficient; we shall show that it is also necessary.

Let $rY$ be an arbitrary compactification of the space $Y$ and let $\omega N$ be the Aleksandrov compactification of the set of integers with discrete topology.

We shall define a topology in the set

$$Z=\omega N\times rY\setminus[(\omega N\setminus\{1\})\times(rY\setminus Y)]\,,$$

taking as neighbourhoods of all points of the form $(n,y)\in Z\setminus(\{1\}\times rY)\subset Z$ the sets

$$(U\setminus\{1\})\times\{y\}\,,$$

where $U$ is a neighbourhood of $n$ in $\omega N$. For all points of the form $(1,y)\in\{1\}\times rY\subset Z$ we shall take as neighbourhoods the sets

$$[\omega N\times(V\cap Y\setminus\{y\})]\cup(\{1\}\times V)\,,$$

where $V$ is a neighbourhood of $y$ in $rY$. Let us take $X=N\times Y$ with the topology of a subspace of $Z$. It is easy to check that $Z$ is a compactification of $X$. Moreover, if the space $Y$ is a paracompact (Lindelöf) space, then the space $X$ is also a paracompact (Lindelöf) space. The projection $f\colon X\to Y$ is closed and the inverses of points under $f$ are locally compact.

If the space $X$ is complete, then the space $Z\setminus X$ is $\sigma$-compact, and hence its discrete subspace $\omega N\setminus N\times Y$ is the union of a countable family $\{F_i\}_{i=1}^{\infty}$ of closed subsets of the subspace $\omega N\times Y$ of $Z$. The projection

$f'\colon \omega N\times Y\to Y$ of the subspace $\omega N\times Y$ of $Z$ onto $Y$ is closed. As the restriction $f'|F_i\colon F_i\to Y$ is closed and one-to-one, all $f'(F_i)$ are closed and discrete subsets of $Y$. Hence $\{f'(F_i)\}_{i=1}^{\infty}$ is a countable family consisting of discrete subspaces which cover $Y$, and each $f'(F_i)$ is closed in $Y$.

The assumption of openness of $f$ in Theorem 2.2, however, can be replaced by the assumption that $X$ is metrizable. Namely, we have the following theorem:

THEOREM 2.6. *Let $f\colon X\to Y$ be a closed mapping of a metrizable space $X$ onto a complete space $Y$.*

(a) *If the inverses of points are complete, then $X$ is complete.*

(b) *If there exists a metric $\varrho_X$ on $X$ such that $f^{-1}(y)$ is complete in this metric for all $y\in Y$ and $\varrho_Y$ is a complete metric on $Y$* [5], *then $\varrho=\varrho_X+\varrho_Y\circ(f\times f)$ is a complete metric on $X$.*

The part (a) of this theorem was also announced in [17].

Proof. (a) Let $\widetilde{X}$ be a complete metric space containing $X$ as a dense subset ([5], 4.3.10). It is sufficient to prove that $X$ is a $G_\delta$-set in $\widetilde{X}$ ([5], 4.3.9 and 4.3.11). Let us put $X_1=\bigcup_{y\in Y}\operatorname{Int}f^{-1}(y)=\bigoplus_{y\in Y}\operatorname{Int}f^{-1}(y)$ and $X_2=\bigcup_{y\in Y^d}\operatorname{Fr}f^{-1}(y)$. The space $X_1$ is complete as the sum of complete spaces and hence is a $G_\delta$-set of $\widetilde{X}$. The space $X_2$ is complete as the inverse image of the complete space $Y^d$ under the perfect mapping $f|X_2$ and hence is also a $G_\delta$-set in $\widetilde{X}$. Since $X$ is the union of $X_1$ and $X_2$, it follows that it is a $G_\delta$-set in $\widetilde{X}$, and therefore $X$ is a complete space.

(b) It is easy to check that metrics $\varrho$ and $\varrho_X$ are equivalent, and by virtue of the Cantor theorem ([5], 4.3.4) one can easily prove that the metric space $(X,\varrho)$ is complete.

In the case where the mapping $f$ is open Theorem 2.2 does not hold even if one assumes that both $X$ and $Y$ are metric.

EXAMPLE 2.7. Let $X$ be the space obtained from the square by removing from its lower base any set homeomorphic to the set of irrational numbers, and let $f$ be the projection of $X$ onto its upper base $Y$. The space $Y$ is compact, the space $X$ is not complete, the mapping $f$ is open and has locally compact inverses of points.

In Example 2.7 the inverses of points are metrizable and complete, but there exists no metric on $X$ such that all inverses of points are complete in this metric.

[5] From the assumptions of theorem it follows that $Y$ is a $q$-space. Therefore $\operatorname{Fr}f^{-1}(y)$ is compact for every $y\in Y$ [14] and hence $Y$ is metrizable in a complete manner ([5], Problem 4.U. and 4.3.11).



THEOREM 2.8. *Let $f\colon X\to Y$ be an open mapping of a metric space $(X,\varrho)$ onto a space $Y$ metrizable in a complete manner. If the inverses of points are all complete with respect to $\varrho$, then $X$ is metrizable in a complete manner.*

The proof of Theorem 2.8 and Example 2.7 are due to R. Engelking.

Proof. Let us denote by $(\widetilde{X},\widetilde{\varrho})$ a space metrizable in a complete manner, containing isometrically $(X,\varrho)$ as a dense subset, and such that there exists an extension $F\colon \widetilde{X}\to Y$ of $f\colon X\to Y$ ([5], 4.3.10, Problem 4.J, and 4.3.9). We shall prove that the set

$$G_n=\{x\in\widetilde{X}\colon\ \widetilde{\varrho}\big(x,f^{-1}(F(x))\big)<1/n\}$$

is open for every $n$. Let $x$ be an arbitrary point of $G_n$ and take $\varepsilon>0$ such that

$$\widetilde{\varrho}\big(x,f^{-1}(F(x))\big)<1/n-\varepsilon\,.$$

The open set $U=f\big(B(x,1/n-\varepsilon)\cap X\big)$ is a neighbourhood of $F(x)$ in $Y$. It is easy to check that the neighbourhood $V=B(x,\varepsilon)\cap F^{-1}(U)$ of $x$ in $\widetilde{X}$ is contained in $G_n$. Hence $G_n$ is open. Since the inverses of points under $f$ are complete in the metric $\widetilde{\varrho}$, it follows that

$$X=\bigcap_{n=1}^{\infty}G_n\,,$$

and that $X$ is metrizable in a complete manner.

Let us finish this section with a few remarks and problems. First of all, let us observe that by a modification of Examples 2.3, 2.4 and 2.7 the following can be obtained:

THEOREM 2.9. *A necessary condition for a complete space $Y$ that each of its inverse images under an open-closed mapping with complete inverses of points be complete is that every closed subset of $Y$ be a $G_\delta$-set. A similar condition for open mappings is that every subset of $Y$ be a $G_\delta$-set.*

The condition for open mappings is not sufficient. There exists an open finite-to-one mapping from a countable non-metrizable space $X$ onto a compact space $Y$ (take a point $x_0$ in $\beta N\backslash N$, a sequence $x_1, x_2, \ldots$ converging to $x_0$ and map the space $X=N\cup\{x_0,x_1,\ldots\}$ onto $Y=\{0,1,1/2,\ldots\}$ by assigning $0$ to $x_0$ and $1/n$ to $n$ and $x_n$, ([6], Example 10)). The space $X$ is paracompact and, by virtue of a theorem of Arhangel'skiĭ [3], it cannot be even a $p$-space. We do not know whether the condition for open-closed mappings is sufficient. This condition is satisfied when $Y$ is metrizable. We do not know whether every inverse image of a complete metrizable space under an open-closed mapping which has complete (complete and metrizable) inverses of points is complete.

Theorems 2.5 and 2.9 can be generalized to $G(\mathfrak{m})$-spaces [6]. In particular, we infer that a necessary and sufficient condition that each of the inverse images of a space $Y$ under an open-closed (open or closed) mapping with locally compact inverses of points be locally compact is that $Y$ be a discrete space.

It is known that an open image of a complete space is complete if it is paracompact [15], and it need not be complete in general (every locally complete space is an open image of a complete space, and there exist ([5], Problem 5.P) locally complete non-complete spaces). Examples are also known which show that a closed image of a complete space need not be complete. From a theorem of N. Dykes ([4], 3.3) and from the fact that a perfect image of a complete space is complete ([12], 2.2 and 2.7) it follows that a closed image of a complete and uniformly complete space is complete if it is a $q$-space. We do not know whether an open-closed image of a complete space is complete.

**3. Axioms of separation.** We shall first investigate the invariance of separation axioms.

EXAMPLE 3.1 ([6], Example 3). Let us consider the set of integers $Z$ with a topology in which the only open sets are intervals of the form $(-\infty, l)$. Let us define the equivalence relation $T$ on $Z$, taking $kTl$ if and only if $k-l \not\equiv 0 (\mathrm{mod} 2)$. This relation generates an open-closed mapping which does not preserve the $T_0$ axiom.

It is easy to check that a closed (open) image of a $T_1$-space is (need not be) a $T_1$-space. However, as has been observed by Mr. K. Alster, an open-closed image of a $T_2$-space need not be a $T_2$-space.

EXAMPLE 3.2. Let us denote by $Q$ the set of rationals and by $R$ the set of real numbers and let $T$ be the equivalence relation in $R$ given by the formula $tTs$ if and only if $t-s \in Q$. Let us put

$$X = (Q\times\{-1\}) \cup R \cup (Q\times\{1\}), \qquad Y = \{-1\} \cup R/T \cup \{1\}.$$

All points $t \in R \subset X$ are open in $X$. A base at the point $(p, -1) \in X$ consists of all the sets of the form

$$W = \{(p, -1)\} \cup [(p-\varepsilon, p)\setminus S],$$

where $\varepsilon$ is a positive number and $S$ is a countable subset of $R$. Similarly, a base at the point $(q, 1) \in X$ consists of all the sets of the form

$$W = \{(q, 1)\} \cup [(q, q+\varepsilon)\setminus S],$$

where $\varepsilon$ is a positive number and $S$ is a countable subset of $R$. The space $X$ with this topology is a $T_2$-space. We shall define a function $f$ as follows:

$$f(x) = \begin{cases} -1 & \text{for} \quad x \in Q\times\{-1\}, \\ [x]_T & \text{for} \quad x \in R, \\ 1 & \text{for} \quad x \in Q\times\{1\}. \end{cases}$$

The function $f$ generates in $Y$ the quotient topology. Applying the fact that every open subset of $Y$ which contains $-1$ or $1$ has a countable complement, it is easy to check that $f$ is open-closed and that $Y$ is not a $T_2$-space.

We have proved above that $T_3$ is an invariant of open-closed mappings (Lemma 1.3); we have also

THEOREM 3.3. *If a space $Y$ is the image under an open-closed mapping $f$ of a completely regular space $X$, then $Y$ is completely regular.*

Proof. The space $Y$ is a $T_1$-space as a closed image of the $T_1$-space $X$. Let $y \in Y$ and let $C$ be a closed subset of $Y$ not containing $y$. Take any point $x \in f^{-1}(y)$. Since $X$ is completely regular, there exists a continuous function $h\colon X\to I$ such that $h(x) = 1$ and $h(f^{-1}(C)) \subset \{0\}$. It is now easy to show that the function $h^s\colon Y\to I$ defined by the formula

$$h^s(y) = \sup\{h(x)\colon x \in f^{-1}(y)\}$$

is continuous ([10], 3.4) and separates the point $y$ from the set $C$.

The Examples are known showing that neither regularity nor complete regularity is preserved by open closed mappings (see e.g. [6], Examples 4 and 5).

For the sake of completeness let us recall that normality is preserved by closed mappings ([5], Exercise 1.5.E).

We shall now investigate the inverse invariance of axioms of separation under open-perfect mappings. It is known that regularity is an inverse invariant of perfect mappings [12, 4.2]. Before considering complete regularity and normality we shall recall an example, given in [12], 4.2, of a perfect mapping $f$ from a regular but not completely regular space $X$ onto a completely regular space $Y$. We shall describe this example here in a simplified form.

EXAMPLE 3.4. Let us consider the Cartesian square $A = W\times W$ of the set $W$ of all ordinals not greater than $\omega_1$ with the order topology, and let us put $B = A\setminus\{(\omega_1, \omega_1)\}$. The set of pairs of the form $(\alpha, \omega_1) \in A$ will be called the *left edge* of $A$ and will be denoted by $E_-$. The set of pairs of the form $(\omega_1, \alpha) \in A$ will be called the *right edge* of $A$ and will be denoted by $E_+$. Let us notice that for each continuous function $h\colon B\to R$ there exists an $\alpha_0 < \omega_1$ such that

(1) $$h((\alpha, \omega_1)) = h((\omega_1, \alpha)) \quad \text{for} \quad \alpha > \alpha_0.$$

[6] Let $\mathfrak{m}$ be a cardinal number. A completely regular space $X$ is called a $G(\mathfrak{m})$-*space* iff in some (or, equivalently, in each) compactification of $X$ there exists a family of power $\mathfrak{m}$ of open sets such that $X$ is the intersection of that family [8]. In this notation locally compact spaces are $G(0)$-spaces and complete spaces are $G(\aleph_0)$-spaces.

This follows from the fact that none of the sets $E_- \cap B$ and $E_+ \cap B$ can be separated from the diagonal $D = \{(\alpha, \alpha): \alpha < \omega_1\} \subset B$ ([5], Problem 3.0) and from the fact that each continuous function defined on $V = W \setminus \{\omega_1\}$ is constant beyond an initial interval ([5], p. 131).

Let us define $K^n$, for each positive integer $n$, as the space obtained by identification in the sum $\bigoplus_{k=1}^{n} A^k$, where each $A^k$ is homeomorphic to $A$, of the right edge of $A^k$ with the left edge of $A^{k+1}$ (each point of the form $(\omega_1, \alpha) \in A^k$ is identified with the point $(\alpha, \omega_1) \in A^{k+1}$, $k = 1, \ldots, n-1$) and let $\varphi_n\colon \bigoplus_{k=1}^{n} A^k \to K^n$ be the quotient mapping generated by this identification. Let us put

$$L^n = \varphi_n\Big(\bigoplus_{k=1}^{n} B^k\Big),$$

where $B^k$ is the copy of $B$ in $A^k$. For each $0 \leqslant k \leqslant n$ we define an open subset $U_k^n \subset L^n$, putting

$$U_k^n = \begin{cases} \operatorname{Int}\varphi_n(B^{k+1}) & \text{for} \quad k = 0, \\ \operatorname{Int}\varphi_n(B^k \cup B^{k+1}) & \text{for} \quad k = 1, \ldots, n-1, \\ \operatorname{Int}\varphi_n(B^k) & \text{for} \quad k = n. \end{cases}$$

Let us observe that

$$L^n = \bigcup_{k=0}^{n} U_k^n, \qquad n = 1, 2, \ldots \tag{2}$$

In the set

$$X = X_1 \cup I,$$

where $X_1 = \bigoplus_{n=1}^{\infty} L^n$ and $I$ is the closed interval $[-1, 1]$, we introduce a topology in the following way: the set $X_1$, considered as the sum of all $L^n$, is open in $X$; for an arbitrary point $t \in I$ as a base at the point $t$ in $X$ we take the family $\{H_l^m\}_{m,l=1}^{\infty}$, where

$$H_l^m = \bigcup_{n \geqslant m} \bigcup_{s \in G_l} U_{E(ns)}^n \cup G_l,$$

and $\{G_l\}_{l=1}^{\infty}$ is a base of neighbourhoods of $t$ in $I$, and the function $E$ assigns to every real number $x$ the greatest integer not greater than $x$.

It is easy to check that $X$ is a $T_2$-space. From the condition (1) it follows that each continuous function on $X$ is constant on $I \subset X$ and hence $X$ is not completely regular.

Let $f$ be the quotient mapping identifying the set $I \subset X$ to a point which will be denoted in the sequel by $w$,

$$f\colon X = X_1 \cup I \to X_1 \cup \{w\} = Y.$$



Obviously $f$ is perfect. To prove that $Y$ is completely regular we need only to show that any neighbourhood $H$ of the point $w$ in $Y$ contains almost all sets $L^n$. Let us consider the set $f^{-1}(H)$. From the compactness of $I$ it follows that there exists an integer $m$ such that

$$I \subset I \cup \bigcup_{n \geqslant m} \bigcup_{s \in I} U_{E(ns)}^n \subset f^{-1}(H).$$

Let us take an arbitrary $n \geqslant m$. For each $0 \leqslant k \leqslant n$ we have $k/n \in I$ and

$$U_k^n = U_{E(nk/n)}^n \subset f^{-1}(H).$$

Therefore, by virtue of (2), we have $L^n \subset f^{-1}(H)$.

The space $X$ is regular as the inverse image of the regular space $Y$ under the perfect mapping $f$.

In [12] the set $X \setminus X_1$ was defined in a different way, which made the notation and the proof of complete regularity of $Y$ more complicated.

We shall now describe an example of an open-perfect mapping $g$ from the space $X$ defined in the previous example onto a normal space $Z$. In this example we shall use the notation introduced in Example 3.4.

EXAMPLE 3.5. The mapping $g$ is the composition of the mapping $f$ defined above and of the quotient mapping $f'$ defined on $Y$ which is generated by the identification of all pairs $(\alpha, \beta)$ and $(\gamma, \delta)$ such that

$$\min(\alpha, \beta) = \min(\gamma, \delta)$$

and both $(\alpha, \beta)$ and $(\gamma, \delta)$ belong to the same set $L^n$ for a certain $n = 1, 2, \ldots$

It is easy to see that $Z = f'(Y) = \bigoplus_{n=1}^{\infty} V^n \cup \{w\}$, where, for each $n$, $V^n = g(L^n)$ is homeomorphic to the space $V$ of all ordinals smaller than $\omega_1$, all $V^n$ are open in $Z$, and the neighbourhoods of the point $w$ contain almost all $V^n$.

The restriction $g_n = g|L^n\colon L^n \to g(L^n)$ of the mapping $g$ is the restriction of the mapping $\tilde{g}_n\colon K^n \to W$ to the set $L^n = \tilde{g}_n^{-1}(V)$. Hence $g_n$ is perfect, because $\tilde{g}_n$ is perfect as a mapping defined on the compact space $K^n$.

Since each $g_n$ is perfect and each neighbourhood of the set $g^{-1}(w) = I$ in $X$ contains almost all sets $L^n$, it follows that $g$ is perfect.

To prove that $g$ is open it suffices to notice that all $g_n$ are open and the image of every set open in $X$ which intersects $I$ is open in $Z$.

The space $Z$ is normal because it can be embedded in a space of ordinals.

Finally, let us observe that the space $Z$ is complete. It is easy to check that it could not be locally compact.

I am gratefull to doc. R. Engelking for suggesting the problems and for some valuable remarks.

Added in proof. A generalization of Theorem 2.8 is contained in H. H. Wicke and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order*, Duke Math. Journ. 34 (1967), pp. 255–272. One can prove that the theorem remains valid if $Y$ is complete.

## References

[1] A. V. Arhangel'skiĭ, *A class of spaces which contains all metric and all locally compact spaces*, Mat. Sb. (N.S.) 67 (1965), pp. 55–88 (Russian).
[2] — *On closed mappings, bicompact sets and a certain problem of P. S. Alexandrov*, Mat. Sb. (N. S.) 69 (1966), pp. 13–34 (Russian).
[3] — *A theorem on the metrizability of the inverse image of a metric space under an open-closed finite-to-one mapping: Example and unsolved problems*, Dokl. Akad. Nauk SSSR 170 (1966), pp. 759–762 (Russian).
[4] N. Dykes, *Mappings and realcompact spaces*, Pacific Journ. of Math. 31 (1969), pp. 347–358.
[5] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[6] — *O zagadnieniach topologii ogólnej związanych z badaniem przekształceń*, Wiadom. Mat., 12 (1971), pp. 257–284.
[7] V. V. Filippov, *Feathery paracompacta*, Dokl. Akad. Nauk SSSR 178 (1968), pp. 555–558.
[8] Z. Frolík, *Generalizations of the $G_\delta$-property of complete metric spaces*, Czech. Math. Journ. 10 (1960), pp. 359–379.
[9] — *On approximation and uniform approximation of spaces*, Proc. Japan Acad. 37 (1961), pp. 530–532.
[10] — *Applications of complete families of continuous functions to the theory of Q-spaces*, Czech. Math. Journ. 11 (1961), pp. 115–133.
[11] S. Hanai, *Inverse images of closed mappings I*, Proc. Japan Acad. 37 (1961), pp. 298–301.
[12] M. Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Math. Journ. 25 (1958), pp. 83–106.
[13] T. Isiwata, *Mappings and spaces*, Pac. Journ. of Math. 20 (1967), pp. 455–480.
[14] E. Michael, *A note on closed maps and compact sets*, Israel Journ. of Math. 2 (1964), pp. 173–176.
[15] B. Pasynkov, *Open mappings*, Dokl. Akad. Nauk SSSR 175 (1966), pp. 292–295 (Russian).
[16] V. Ponomarev, *Proof of the invariance of the star finite property under open perfect mappings*, Bull. Acad. Pol. Sci. sér. math. 10 (1962), pp. 425–428 (Russian).
[17] F. G. Slaughter, *Some new results on inverse images of closed mappings*, Notices Amer. Math. Soc. 17 (1970), p. 842.

WYDZIAŁ MATEMATYKI I MECHANIKI UNIWERSYTETU WARSZAWSKIEGO
DEPARTMENT OF MATHEMATICS AND MECHANICS, WARSAW UNIVERSITY

*Reçu par la Rédaction le 21. 11. 1970*

# Tate resolutions for commutative graded algebras over a local ring

by

**Tadeusz Józefiak** (Bydgoszcz)

**Introduction.** Let $R$ be a commutative Noetherian ring with a unit element.

Tate has constructed in [11] for cyclic $R$-modules free resolutions with additional algebra structure and used them for the study of the functor $\mathrm{Tor}^R$. Only recently (see [4], [8], [10]) it has turned out that for a local ring $R$ and residue class field $k$ the Tate resolution has an important property: it is minimal. A minimal resolution $F$ determines completely the algebra $\mathrm{Tor}^R(k, k)$: we have $\mathrm{Tor}^R(k, k) \simeq F \otimes k$. These two properties: the algebra structure and minimality facilitate the investigation of the structure of the homology of the ring $R$.

The main purpose of the present paper is to build the theory of Tate resolutions for graded commutative algebras over a local ring $R$ (called $R$-algebras in this paper, cf. (1.1)).

From the existence of the Tate resolution of an $R$-algebra $A$ we obtain the following formula for the Poincaré series of $A$:

$$\mathcal{P}(A) = \frac{(1+t)^{n_1}(1+t^2)^{n_2}(1+t^3)^{n_3}\dots}{(1-t)^{m_1}(1-t^2)^{m_2}(1-t^3)^{m_3}\dots}.$$

The organization of the paper is as follows:

In § 1 we recall the definition of an $R$-algebra and some basic properties of the category of graded modules over such an $R$-algebra.

§ 2 contains the definition and properties of a normal sequence in an $R$-algebra. The main result of this section is a characterization of those $R$-algebras whose unique maximal homogeneous ideal is generated by a normal sequence.

In § 3 we define bigraded $\Gamma$-algebras and differential $\Gamma$-algebras. Furthermore we present the basic construction of the differential $\Gamma$-algebra $A\langle M; \varphi\rangle$ obtained from the differential $\Gamma$-algebra $A$ by the adjunction of the $R$-module $M$ by means of the map $\varphi$: $M \to Z(A)$.