Non-standard analysis and homology

by

M. C. McCord (Lexington, Ky)

Introduction. The spirit of Vietoris homology and Alexander–Spanier cohomology is to use chains and cochains based on "small" simplexes ($(n+1)$-tuples of points) in a space. The situation for Čech homology and cohomology is similar, except that the simplexes are $(n+1)$-tuples of members of open covers of the space. Although one thinks intuitively of "small" simplexes, one must resort technically to various limiting processes. Except for the case of Alexander–Spanier cohomology, the groups are not even computed from a single chain or cochain complex.

The object of this paper is to show how the methods of non-standard analysis, due to Abraham Robinson [3], can be used to define a homology theory based on chains of infinitesimal simplexes. The homology groups are computed from a single chain complex. All of the Eilenberg–Steenrod axioms [1] (*including exactness*) are verified. The proofs are given in such a way that one automatically gets an associated cohomology theory by composing the chain complex functor with $\mathrm{Hom}(\cdot, G)$. The proofs are geometrically intuitive and technically easy. In the case of excision, for example, one does not have a subdivision problem as with singular homology, because the chains are already "infinitely fine".

The relationship with Čech homology has not yet been determined (see § 6). However the theory has at least one property characteristic of Čech theories: for compact Hausdorff spaces with a finite number of components, the 0-dimensional group detects components, not path-components (see § 5).

We follow essentially the set-theoretic version of non-standard analysis of M. Machover and J. Hirschfeld [2] (cf. also Robinson and Zakon [4]), but in the next section we offer a slightly different exposition of the basic set-up. Hopefully the paper will be readable by topologists having no particular familiarity with non-standard analysis.

1. Preliminaries. We work within Zermelo–Fraenkel set theory. If X is a set, let $\bigcup X$ denote the union of all elements of X and let $\boldsymbol{P}X$ denote the set of all subsets of X. By a *universe* let us mean a non-empty set $\mathfrak{U}$ satisfying the following condition:

(1.1) *If* $X \in \mathfrak{U}$ *and* $Y \in \mathfrak{U}$, *then* $X \subset \mathfrak{U}$, $\bigcup X \in \mathfrak{U}$, $\boldsymbol{P}X \in \mathfrak{U}$, *and* $\{X, Y\} \in \mathfrak{U}$.

Given any set A we can construct a universe (in fact the smallest universe) $\mathfrak{U}$ such that $A \in \mathfrak{U}$, as follows. Let $A_0 = A$ and for $n \geqslant 0$ let $A_{n+1} = A_n \cup (\bigcup A_n) \cup (\boldsymbol{P}A_n)$. Then let $\mathfrak{U} = \bigcup_{n=0}^{\infty} A_n$. In the remainder of this paper we assume given a fixed universe $\mathfrak{U}$ such that $\boldsymbol{N} \in \mathfrak{U}$.

An *enlargement* of $\mathfrak{U}$ consists of a set ${}^*\mathfrak{U}$ containing $\mathfrak{U}$ and a relation ${}^*\in$ on ${}^*\mathfrak{U}$ satisfying the two conditions (1.2) and (1.3) below. Construct a first-order language; $\mathfrak{L}$ having symbols for the usual connectives, quantifiers, variables, and punctuation; $\mathfrak{L}$ shall have $\mathfrak{U}$ as its set of constants and shall have two binary relation symbols $\dot{=}$ and $\dot{\in}$. Interpreting $\dot{=}$ as equality and $\dot{\in}$ as ordinary set membership, and understanding quantification to be over $\mathfrak{U}$, one can define in the usual way the notion that a *statement* S *of* $\mathfrak{L}$ *holds in* $\mathfrak{U}$. Interpreting $\dot{=}$ as equality, replacing $\dot{\in}$ by ${}^*\in$, and understanding quantification to be over ${}^*\mathfrak{U}$, one has the notion that a *statement* S *of* $\mathfrak{L}$ *holds in* ${}^*\mathfrak{U}$. We then assume

(1.2) *If S is a statement of $\mathfrak{L}$ that holds in $\mathfrak{U}$, then S holds in* ${}^*\mathfrak{U}$.

If $A \in {}^*\mathfrak{U}$, the *scope* $\hat{A}$ of A is defined as the set of all x in ${}^*\mathfrak{U}$ such that $x \,{}^*\!\in A$. One concludes from (1.2) that if $A \in \mathfrak{U}$, then $A \subset \hat{A}$.

If $x, y \in {}^*\mathfrak{U}$, the **unordered pair* ${}^*\{x, y\}$ is defined to be the unique element z of ${}^*\mathfrak{U}$ such that $\hat{z} = \{x, y\}$. The existence (and uniqueness) of z follows from (1.1) and (1.2). We define the **ordered pair* ${}^*(x, y)$ to be ${}^*\{x, {}^*\{x, y\}\}$. This corresponds to $x\,{}^*\mathrm{pr}\,y$ in [2].

A relation R is called *concurrent* if for any finite subset $\{a_1, \ldots, a_n\}$ of the left domain of R there exists an object b such that $(a_i, b) \in R$ for each $i = 1, \ldots, n$. The second assumption on enlargements is

(1.3) *For each concurrent relation* $R \in \mathfrak{U}$ *there exists an element* b *of* ${}^*\mathfrak{U}$ *such that* ${}^*(a, b) \,{}^*\!\in R$ *for all* a *(in* $\mathfrak{U}$*) in the left domain of* R.

In the remainder of the paper we assume given a fixed enlargement $({}^*\mathfrak{U}, {}^*\in)$ of $\mathfrak{U}$.

If X and Y are sets, let $F(X, Y)$ denote the set of all functions from X to Y. If $X, Y \in \mathfrak{U}$, then by (1.1), $F(X, Y) \in \mathfrak{U}$; each $f \,{}^*\!\in F(X, Y)$ gives rise to an element ${}^*f$ of $F(\hat{X}, \hat{Y})$ as follows. If $x \in \hat{X}$, let ${}^*f(x)$ be the unique element y of $\hat{Y}$ such that ${}^*(x, y) \,{}^*\!\in f$. The existence (and uniqueness) of y follows from (1.2). The evaluation of ${}^*f$ at x corresponds to the operation ${}^*\mathrm{ap}$ in [2]. When there is no danger confusion we write $f(x)$ instead of ${}^*f(x)$.

If $X, Y \in \mathfrak{U}$, then it follows from (1.2) that the correspondence $(x, y) \mapsto {}^*(x, y)$ gives a bijection $\hat{X} \times \hat{Y} \to (X \times Y)\hat{}$. We shall make the practice of identifying these two sets under this correspondence, and in fact shall do a similar thing for iterated Cartesian products.

We understand that every *group* $(G, \cdot)$ shall be such that $G \in \mathfrak{U}$ (hence by (1.1) such that $\cdot \in \mathfrak{U}$). The set $\hat{G}$ will be given the group operation induced by that on G.

Similarly we understand that every *space* $(X, \mathfrak{J})$ shall have $X \in \mathfrak{U}$ (hence $\mathfrak{J} \in \mathfrak{U}$). If $a \in X$, then the *monad* of a is $\mu(a) = \mu_X(a) = \bigcap \{\hat{V}: a \in V \in \mathfrak{J}\} \subset \hat{X}$.

2. Definition of the homology theory. Let G be a fixed abelian group, which we will use for coefficients. If X is a set, let GX denote the abelian group of all functions $u\colon X \to G$ such that $u(x) = 0$ for all but finitely many x in X. If $g \in G$ and $x \in X$, then gx denotes the element of GX whose value at x is g and whose value elsewhere is 0. If $f\colon X \to Y$ is a function, there is a unique homomorphism $Gf\colon GX \to GY$ such that $(Gf)(gx) = g(fx)$ for all $g \in G$, $x \in X$. In this way, $G(\cdot)$ is a functor from sets to abelian groups.

Now we define a functor $C(\cdot) = C(\cdot\,; G)$ from sets to chain complexes by letting $C_n(X) = GX^{n+1}$ for $n \geqslant 0$ and $C_n(X) = 0$ for $n < 0$. The differential $d\colon C_n(X) \to C_{n-1}(X)$ is given as usual by

$$d\big(g(x_0, \ldots, x_n)\big) = \sum_{i=0}^{n} (-1)^i g(x_0, \ldots, \hat{x}_i, \ldots, x_n) \quad \text{for} \quad n > 0 .$$

If $f\colon X \to Y$ is a function, then $C(f)$ is given by $C_n(f) = Gf^{n+1}$ for $n \geqslant 0$.

Next, we get a functor $\hat{C}(\cdot)$ from sets in $\mathfrak{U}$ to chain complexes by letting $\hat{C}_n(X) = \big(C_n(X)\big)\hat{}$. The differential in $\hat{C}(X)$ is induced by that in $C(X)$ (see the preceding section for induced functions). The fact that we indeed get a differential follows immediately from (1.2). Similarly, if $f\colon X \to Y$ is a function (in $\mathfrak{U}$), the induced chain map $\hat{C}(f)$ is given by $\hat{C}_n(f) = {}^*\big(C_n(f)\big)$.

Using $\hat{C}(\cdot)$, we define a functor $\bar{C}(\cdot)$ from spaces (in $\mathfrak{U}$) to chain complexes. If X is a space, an "n-simplex" $(x_0, \ldots, x_n) \in \hat{X}^{n+1}$ will be called *infinitesimal* if $\{x_0, \ldots, x_n\}$ is contained in the monad of some point of X. If $n \geqslant 0$, let $\bar{C}_n(X)$ be the subset of $\hat{C}_n(X)$ consisting of all u such that if $\sigma \in \hat{X}^{n+1}$ and $u(\sigma) \neq 0$ then σ is infinitesimal. Let $\bar{C}_n(X) = 0$ for $n < 0$. The following lemma allows us to consider $\bar{C}(X) = \big(\bar{C}_n(X)\big)$ as a sub chain complex of $\hat{C}(X)$ and to define induced chain maps by restriction.

2.1. Lemma. *If X is a space, then* (a) $\bar{C}_n(X)$ *is a subgroup of* $\hat{C}_n(X)$, *and* (b) *the differential of* $\hat{C}(X)$ *maps* $\bar{C}_n(X)$ *into* $\bar{C}_{n-1}(X)$. (c) *If* $f\colon X \to Y$ *is a continuous map, then* $\hat{C}_n(f)$ *maps* $\bar{C}_n(X)$ *into* $\bar{C}_n(Y)$.

Proof. We prove these three statements by informal application of (1.2).

(a) We have the following true statement in $\mathfrak{U}$: "If $u, v \in C_n(X)$, $\sigma \in X^{n+1}$, and $(u \pm v)(\sigma) \neq 0$, then $u(\sigma) \neq 0$ or $v(\sigma) \neq 0$". Then from (1.2) we get the statement: "If $u, v \in \hat{C}_n(X)$, $\sigma \in \hat{X}^{n+1}$, and $(u \pm v)(\sigma) \neq 0$, then $u(\sigma) \neq 0$ or $v(\sigma) \neq 0$". From this we conclude immediately that if $u, v \in \bar{C}_n(X)$ then $u \pm v \in \bar{C}_n(X)$.

(b) By a similar "transfer" we get the statement: "If $u \in \hat{C}_n(X)$, $(x_0, \ldots, x_{n-1}) \in \hat{X}^n$, and $(du)(x_0, \ldots, x_{n-1}) \neq 0$, then there exists $(y_0, \ldots, y_n) \in \hat{X}^{n+1}$ such that $u(y_0, \ldots, y_n) \neq 0$ and $\{x_0, \ldots, x_{n-1}\} \subset \{y_0, \ldots, y_n\}$". From this it follows that if $u \in \bar{C}_n(X)$ then $du \in \bar{C}_{n-1}(X)$.

(c) Suppose $u \in \bar{C}_n(X)$, $(y_0, \ldots, y_n) \in \hat{Y}^{n+1}$, and $\hat{C}_n(f)(u)(y_0, \ldots, y_n) \neq 0$. By transfer, we see that there must exist $(x_0, \ldots, x_n) \in \hat{X}^{n+1}$ such that $u(x_0, \ldots, x_n) \neq 0$ and $(y_0, \ldots, y_n) = (fx_0, \ldots, fx_n)$. Since $u(x_0, \ldots, x_n) \neq 0$, there exists $a \in X$ such that $\{x_0, \ldots, x_n\} \subset \mu(a)$. But since f is continuous at a, we have $f(\mu(a)) \subset \mu(f(a))$ (see [3] or [2]). Hence $\{y_0, \ldots, y_n\} \subset \mu(f(a))$. This completes the proof.

Thus we have the functor $\bar{C}(\cdot)$. For *single* spaces the homology groups we are after are given by

$$H_n(X; G) = H_n(\bar{C}(X)) .$$

For pairs of spaces we do the customary thing, but we must be a little careful. Let (X, A) be a pair of spaces and let $i\colon A \to X$ be the inclusion map. For each n, $C_n(i)\colon C_n(A) \to C_n(X)$ is injective. By transfer $\hat{C}_n(i)$ is injective, and by restriction $\bar{C}_n(i)$ is injective. This allows us to identify $\bar{C}(A)$ with a sub chain complex of $\bar{C}(X)$ and we define

$$H_n(X, A; G) = H_n(\bar{C}(X)/\bar{C}(A)) ,$$

with the obvious definition of induced homomorphisms. Since $\bar{C}(\emptyset) \subset \hat{C}(\emptyset) = \hat{0} = 0$, we can identify $H_n(X, \emptyset; G)$ with $H_n(X; G)$.

For each pair (X, A) of spaces we get, by the above, a short exact sequence of chain complexes

$$0 \to \bar{C}(A) \to \bar{C}(X) \to \bar{C}(X)/\bar{C}(A) \to 0 ,$$

hence a homology exact sequence with an (obviously natural) homology boundary operator. It is easy to see that this short exact sequence splits if A is closed.

3. The homotopy axiom. Under a standard construction of the set $\boldsymbol{R}$ of real numbers, it follows from (1.1) that $\boldsymbol{R} \in \mathfrak{U}$, hence also $I \in \mathfrak{U}$, where $I = [0, 1]$. The homotopy axiom is a consequence of the following proposition.

3.1. Proposition. *If $f_0, f_1\colon (X, A) \to (Y, B)$ are homotopic maps, then the chain maps from $\bar{C}(X)/\bar{C}(A)$ to $\bar{C}(Y)/\bar{C}(B)$ induced by f_0 and f_1 are chain homotopic.*

Proof. Let $h\colon (X, A) \times I \to (Y, B)$ be a homotopy from f_0 to f_1. For each positive integer m and each integer n, define a morphism

$$\big(C_n(X), C_n(A)\big) \xrightarrow{D_n^{(m)}} \big(C_{n+1}(Y), C_{n+1}(B)\big)$$

by letting $D_n^{(m)} = 0$ if $n < 0$ and for $n \geqslant 0$ letting $D_n^{(m)}$ be the unique homomorphism whose value on a generator $g(x_0, \ldots, x_n)$ of $C_n(X)$ is

$$\sum_{j=1}^{m} \sum_{i=0}^{n} (-1)^i g\Big(h\Big(x_0, \frac{j-1}{m}\Big), \ldots, h\Big(x_i, \frac{j-1}{m}\Big), h\Big(x_i, \frac{j}{m}\Big), \ldots, h\Big(x_n, \frac{j}{m}\Big)\Big) .$$

It is straightforward to calculate that $D^{(m)} = (D_n^{(m)})$ is a chain homotopy from $C(f_0)$ to $C(f_1)$. Now let m be an *infinite* positive integer (an element of $\hat{P} - P$ where P is the set of positive integers). By transfer, we get a chain homotopy $D^{(m)} = (D_n^{(m)})$ from $\hat{C}(f_0)$ to $\hat{C}(f_1)$ with the following property:

If $n \geqslant 0$, $u \in \hat{C}_n(X) [u \in \hat{C}_n(A)]$, $\tau \in \hat{Y}^{n+2}$, and $(D_n^{(m)} u)(\tau) \neq 0$, then there exist $(x_0, \ldots, x_n)$ in $\hat{X}^{n+1}$ [in $\hat{A}^{n+1}$], $t, t' \in \hat{I}$, and $i \in \{0, \ldots, n\}$ such that $u(x_0, \ldots, x_n) \neq 0$, $|t - t'| = 1/m$, and $\tau = \big(h(x_0, t), \ldots, h(x_i, t), h(x_i, t'), \ldots, h(x_n, t')\big)$.

It follows easily then that $D_n^{(m)}$ maps $\big(\bar{C}_n(X), \bar{C}_n(A)\big)$ into $\big(\bar{C}_{n+1}(Y), \bar{C}_{n+1}(B)\big)$. Passing to quotients, we get the required chain homotopy.

3.2. Remark. This proposition (and its proof) can be generalized by replacing the triple $(I, 0, 1)$ by any triple (K, a, b) where K is a connected compact Hausdorff space and $\{a, b\} \subset K$. For the extra ingredient see the proof of 5.2.

4. Excision. Recall that for any pair (X, A) of spaces we have agreed to identify $\bar{C}_n(A)$ with a subgroup of $\bar{C}_n(X)$ under the monomorphism $\bar{C}_n(i)$ induced by the inclusion $i\colon A \to X$. Let us also identify $\hat{C}_n(A)$ with a subgroup of $\hat{C}_n(X)$ under the monomorphism $\hat{C}_n(i)$. The excision axiom follows from

4.1. Proposition. *Suppose X is a space, X_1 and X_2 are subspaces, X_1 is closed, and $X = \operatorname{int} X_1 \cup \operatorname{int} X_2$. Then the chain map*

$$\bar{C}(X_1)/\bar{C}(X_1 \cap X_2) \to \bar{C}(X)/\bar{C}(X_2)$$

induced by inclusion is an isomorphism.

Proof. Let $X_0 = X_1 \cap X_2$. Considering all the groups $\hat{C}_n(X_0)$, $\hat{C}_n(X_1)$, $\hat{C}_n(X_2)$ as subgroups of $\hat{C}_n(X)$, we need only show two things: (a) $\bar{C}_n(X_1) \cap \bar{C}_n(X_2) \subset \bar{C}_n(X_0)$, and (b) $\bar{C}_n(X) \subset \bar{C}_n(X_1) + \bar{C}_n(X_2)$.

Proof of (a). Suppose $u \in \bar{C}_n(X_1) \cap \bar{C}_n(X_2)$. By transfer, we see then that $u \in \hat{C}_n(X_0)$. To show further that $u \in \bar{C}_n(X_0)$, suppose $u(x_0, \ldots, x_n)$

$\neq 0$, where $\{x_0, \ldots, x_n\} \subset \hat{X}_0$. Since $u \in \bar{C}_n(X_2)$, there exists $b \in X_2$ such that $\{x_0, \ldots, x_n\} \subset \mu(b)$. It suffices to show that $b \in X_1$. But if $b \in X - X_1$, then since X_1 is closed, $\mu(b) \subset (X - X_1)\hat{} = \hat{X} - \hat{X}_1$, so that $\{x_0, \ldots, x_n\} \subset \hat{X} - \hat{X}_1$, which is a contradiction.

Proof of (b). Suppose $v \in \bar{C}_n(X)$. By transfer, we get the existence of an element u of $\hat{C}_n(X_1)$ such that $u(x_0, \ldots, x_n) = v(x_0, \ldots, x_n)$ for all $(x_0, \ldots, x_n)$ in $\hat{X}_1^{n+1}$. Choose such a u. First we claim that $u \in \bar{C}_n(X_1)$. For suppose $\{x_0, \ldots, x_n\} \subset \hat{X}_1$ and $u(x_0, \ldots, x_n) \neq 0$. Then $v(x_0, \ldots, x_n) \neq 0$, so there exists $a \in X$ such that $\{x_0, \ldots, x_n\} \subset \mu(a)$. If a were not in X_1 then we would have $\{x_0, \ldots, x_n\} \subset \mu(a) \subset (X - X_1)\hat{} \subset \hat{X} - \hat{X}_1$, a contradiction. Thus we have shown that $u \in \bar{C}_n(X_1)$.

For the remainder, it suffices to show that $v - u \in \bar{C}_n(X_2)$. So suppose that $\{x_0, \ldots, x_n\} \subset \hat{X}$ and $(v-u)(x_0, \ldots, x_n) \neq 0$. Since u and v agree on $\hat{X}_1$, some x_i must be in $\hat{X} - \hat{X}_1$, and $v(x_0, \ldots, x_n) \neq 0$. Since $v \in \bar{C}_n(X)$, there exists $a \in X$ such that $\{x_0, \ldots, x_n\} \subset \mu(a)$. Now $a \in \operatorname{int} X_2$; otherwise, by the hypothesis of the proposition, $a \in \operatorname{int} X_1$, and we would have $\{x_0, \ldots, x_n\} \subset \mu(a) \subset (\operatorname{int} X_1)\hat{} \subset \hat{X}_1$. Thus $\{x_0, \ldots, x_n\} \subset \mu(a) \subset (\operatorname{int} X_2)\hat{} \subset \hat{X}_2$, and the proof is complete.

5. The dimension axiom. Behavior of H_0. For any space X we have the augmentation $\varepsilon\colon C_0(X) \to G$ given by $\varepsilon(\Sigma_i g_i(x_i)) = \Sigma_i g_i$. This induces homomorphisms from $\hat{C}_0(X)$ and $\bar{C}_0(X)$ into $\hat{G}$ which we still denote by ε. The following proposition shows that the dimension axiom holds for our theory and that its coefficient group is $\hat{G}$ (not G).

5.1. Proposition. *If X is a one-point space then the following sequence is exact:*

$$0 \longleftarrow \hat{G} \xleftarrow{\varepsilon} \bar{C}_0(X) \xleftarrow{d} \bar{C}_1(X) \xleftarrow{d} \ldots$$

Proof. The sequence

$$0 \longleftarrow G \xleftarrow{\varepsilon} C_0(X) \xleftarrow{d} C_1(X) \xleftarrow{d} \ldots$$

is exact, by a familiar calculation. Hence, by transfer, so is the sequence

$$0 \longleftarrow \hat{G} \xleftarrow{\varepsilon} \hat{C}_0(X) \xleftarrow{d} \hat{C}_1(X) \xleftarrow{d} \ldots$$

But $\hat{C}_n(X) = \bar{C}_n(X)$ for all n.

5.2. Proposition. *If X is a non-empty, connected, compact, Hausdorff space, then $H_0(X; G) \approx \hat{G}$.*

Proof. It suffices to show that the sequence

$$0 \longleftarrow \hat{G} \xleftarrow{\varepsilon} \bar{C}_0(X) \xleftarrow{d} \bar{C}_1(X)$$

is exact. The only non-trivial part is that $\ker \varepsilon \subset \operatorname{im} d$. Let $\mathcal{N}$ be the set of neighborhoods of the diagonal Δ in $X \times X$. We claim:

(5.3). *If $U \in \mathcal{N}$, $z \in C_0(X)$, and $\varepsilon(z) = 0$, then there exists $c \in C_1(X)$ such that $dc = z$ and such that c is U-fine in the sense that $c(x, y) \neq 0$ implies $(x, y) \in U$.*

Let $U \in \mathcal{N}$ be given, and choose a fixed point a of X. Since the set of U-fine 1-chains is a subgroup of $C_1(X)$, (5.3) is implied by the following statement: For each $b \in X$ and each $g \in G$ there exists a U-fine 1-chain c such that $dc = g(b) - g(a)$. But this is easy to see from the connectedness of X.

Let $\mu(\Delta) = \bigcap\{\hat{V}\colon V \in \mathcal{N}\}$. By [2], p.18 (cf. also [3], p. 90), there exists $U \in \hat{\mathcal{N}}$ such that $\hat{U} \subset \mu(\Delta)$. Fix such a U. Suppose now $z \in \bar{C}_0(X)$ and $\varepsilon(z) = 0$. Then by (5.3) and transfer, there exists $c \in \hat{C}_1(X)$ such that $dc = z$ and such that $c(x, y) \neq 0$ implies $(x, y) \in \hat{U} \subset \mu(\Delta)$. But it follows from Robinson's characterization of compactness ([3], p. 93) that every 1-simplex (x, y) in $\mu(\Delta)$ is infinitesimal, so that $c \in \bar{C}_1(X)$, and the proof is complete.

Remark. Every homology theory is finitely additive [1], so 5.2 allows us to determine $H_0(X; G)$ for any compact Hausdorff space with a finite number of components.

6. Questions. Since our homology groups have a definition that bears a resemblance to that of Vietoris homology (and Alexander–Spanier cohomology), and since the 0-dimensional group looks at components instead of path-components, one is led to ask:

(6.1). What is the relation of $H_n(\cdot\,; G)$ to Čech homology?

(6.2). What sort of inverse limit continuity does $H_n(\cdot\,; G)$ enjoy?

In view of the fact that our theory does satisfy the exactness axiom, and in view of the continuity-versus-exactness problem ([1], p. 265) for homology, one might at first wonder how positive responses to (6.1) or (6.2) could be possible. But the counterexample given in [1] does not appear to work for our theory (for any G). It applies to a theory whose coefficient group $H_0(pt)$ is the integers $\mathbf{Z}$. It depends on the fact that the limit of the inverse system of (vertically written) exact sequences

$$\begin{array}{ccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbf{Z} & \xleftarrow{\times 3} & \mathbf{Z} & \xleftarrow{\times 3} & \mathbf{Z} & \xleftarrow{\times 3} & \ldots \\
\downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \\
\mathbf{Z} & \xleftarrow{\times 3} & \mathbf{Z} & \xleftarrow{\times 3} & \mathbf{Z} & \xleftarrow{\times 3} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbf{Z}_2 & \xleftarrow{\text{id}} & \mathbf{Z}_2 & \xleftarrow{\text{id}} & \mathbf{Z}_2 & \xleftarrow{\text{id}} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \ldots
\end{array}$$

is not exact. But the coefficient group $H_0(pt;\boldsymbol{Z})$ for our theory is $\hat{\boldsymbol{Z}}$; and it turns out, interestingly, that the limit of the system corresponding to (6.3) (with $\boldsymbol{Z}$ replaced by $\hat{\boldsymbol{Z}}$) is exact.

It is clear from §§ 2–5 that for each abelian group G we get a cohomology theory for closed pairs with

$$H^n(X,A;G) = H^n\big(\mathrm{Hom}\,(\bar{C}(X;\boldsymbol{Z})/\bar{C}(A;\boldsymbol{Z}),\,G)\big)\,.$$

(6.4). What is the relation of this theory to Čech cohomology?

References

[1] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[2] M. Machover and J. Hirschfeld, *Lectures on non-standard analysis*, Springer Lecture Notes in Mathematics 96 (1969).

[3] A. Robinson, *Non-Standard Analysis*, Amsterdam 1966.

[4] — and E. Zakon, *A set-theoretical characterization of enlargements*, Applications of Model Theory to Algebra, Analysis and Probability, New York 1969.

UNIVERSITY OF KENTUCKY
Lexington, Kentucky

Reçu par la Rédaction le 14. 4. 1970

The product of certain measurable spaces

by

Christopher Todd (Bolton, Conn.)

Let X and Y be topological spaces and let $S(X)$ and $S(Y)$ be the σ-rings of Baire sets of X and Y respectively, i.e. the σ-rings of subsets generated by the compact G_δ sets of X and Y. $S(X)\times S(Y)$, the cartesian product of $S(X)$ and $S(Y)$, is the σ-ring of subsets of the topological product $X\times Y$ generated by the family $G=\{E\times F|\ E\in S(X),F\in S(Y)\}$ of subsets of $X\times Y$. It is a well known fact that for locally compact Hausdorff spaces X and Y, $S(X)\times S(Y)=S(X\times Y)$. It is the purpose of this note to examine the corresponding situation for the σ-rings $Z(X)$ and $\mathrm{wb}(X)$ generated respectively by the zero sets and the closed G_δ sets of X. Following Berberian [1] we will call the elements of $\mathrm{wb}(X)$ the *weakly Baire sets of* X. In the following sufficient conditions on the product $X\times Y$ and on the spaces X and Y will be given to insure that $Z(X)\times Z(Y)=Z(X\times Y)$ and that $\mathrm{wb}(X)\times\mathrm{wb}(Y)=\mathrm{wb}(X\times Y)$. In the latter case the results give a partial answer to a question posed by Berberian ([1], p. 183).

Definitions and notation will be introduced as they become necessary. All topological spaces under consideration will be assumed to be completely regular and Hausdorff. The Stone–Čech compactification of such a topological space X will be denoted by βX.

A subset Z of a topological space X is said to be a *zero set of* X if there exists a continuous real valued function f on X such that $Z=Z(f)=\{x\in X|\ f(x)=0\}$. It is obvious that f may be so chosen such that $0\leqslant f(x)\leqslant 1$ for every x in X. A subset of X is called a *co-zero set* if it is the complement of a zero set of X. Since X itself is always a zero set (and a co-zero set), the σ-ring $Z(X)$ is in fact always a σ-algebra. It follows that $Z(X)$ is generated also by the co-zero sets of X. Similarly, since any topological space is a closed G_δ set we have the same situation for the σ-ring $\mathrm{wb}(X)$. Every compact G_δ set in a completely regular Hausdorff space X is a zero set and every zero set is a closed G_δ so that for these spaces we always have the relation $S(X)\subseteq Z(X)\subseteq\mathrm{wb}(X)$. If a topological space X is normal then every closed G_δ set is a zero set so that in this case $\mathrm{wb}(X)=Z(X)$.