Countable dense homogeneous spaces

by

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1. Introduction. This paper is devoted to the study of topological spaces which are countable dense homogeneous (defined below). A first countable connected space with this property is homogeneous and a metric continuum with this property is necessarily decomposable, not irreducible between any pair of points and has no cut points. Although manifolds are countable dense homogeneous, this property does not characterize manifolds as the universal curve also has this property. A proof that the universal curve is countable dense homogeneous is obtained here as a result of a theorem which shows that many spaces are countable dense homogeneous. The paper concludes with statements of some open problems.

I thank Professor Ben Fitzpatrick for suggesting the problem of deciding whether countable dense homogeneity characterizes manifolds and both Professors Fitzpatrick and Ralph Ford for stimulating conversations which were held while I was working on this problem.

2. Basic properties. A topological space X is defined to be countable dense homogeneous if it is separable and for any two countable dense subsets M and N of X, there is a homeomorphism h of X onto X such that $h(M)$ is N.

THEOREM 1. *A countable dense homogeneous connected space X satisfying the first axiom of countability is homogeneous.*

Proof. Let x be any point of X. Denote as M the set of all points y of X such that there is a homeomorphism h of X onto X taking x to y. Suppose first that neither M nor $X-M$ is dense in X. Then there is a countable dense subset $A \cup B$ of X such that A is a subset of $\mathrm{int}(\mathrm{cl}(M)) \cap M$ and B is a subset of $X-\mathrm{cl}(M)$. Because X is connected, there is a point z in $\mathrm{bd}(X-\mathrm{cl}(M))$. Then $\{z\} \cup A \cup B$ is countable and dense and some homeomorphism h of X onto X sends $\{z\} \cup A \cup B$ onto $A \cup B$. But clearly $h(A)$ is a subset of A and $h(B)$ is a subset of B and $h(z)$

could not be in either A or B. So either M is dense or $X-M$ is dense. But if M is dense, M must be all of X since if z is any point of X one could take a countable dense subset M_0 of M onto $M_0 \cup \{z\}$ and z would be in M. Similarly, if $X-M$ is dense, then M is empty. But M contains x, so M is all of X and X is homogeneous.

Remarks. One needs the assumption that X is connected in Theorem 1 because non-connected spaces such as the union of a 2-sphere and a disjoint simple closed curve are countable dense homogeneous but not homogeneous. The hypothesis that X satisfies the first axiom of countability is used only to ensure that every dense subset has a countable dense subset. There are separable spaces which do not have this property. However, I do not know whether there is a countable dense homogeneous space with a dense subset which is not itself separable.

As an immediate consequence of Theorem 1 and the well known theorem that every continuum has at least two non-cut points, one has that a countable dense homogeneous continuum has no cut points.

THEOREM 2. *A countable dense homogeneous metric continuum X is not irreducible between any two of its points.*

Proof. Let p be any point of X. It is well known that the composant $X(p)$ of X generated by p is dense in X. (See [5], pp. 208–209, for information about composants.) Let N be a countable dense subset of $X(p)$ containing p. If q is any point of X not in $X(p)$, then there is a homeomorphism of X onto X taking N onto $N \cup \{q\}$. But X is irreducible between two points of $N \cup \{q\}$, and not irreducible between any two points of N, which is absurd.

Because a non-degenerate indecomposable continuum is irreducible between two of its points, ([5], p. 213), a countable dense homogeneous metric continuum cannot be indecomposable. It is known [3] that the pseudo-arc is homogeneous. Therefore, the class of countable dense homogeneous metric continua is a proper subclass of the class of homogeneous metric continua.

3. A condition implying countable dense homogeneity. Define a topological space X to be *strongly locally homogeneous* if for any point x of X and open set U containing x there is an open set V containing x such that if y is in V then there is a homeomorphism h of X onto X such that $h(x)$ is y and $h(z)=z$ if z is not in U. Locally euclidean spaces are the most obvious examples of strongly locally homogeneous spaces.

THEOREM 3. *Suppose X is a locally compact separable metric space and X is strongly locally homogeneous. Then X is countable dense homogeneous.*

Proof. We may assume the metric d on X is such that each sphere $S(x, \varepsilon)$ of radius less than 1 has a compact closure, since X is locally compact. By assumption X is separable. Let M and N be two countable dense subsets of X.

A sequence $h_1, h_2, \ldots$ of homeomorphisms of X onto X will be constructed such that $h_1, h_2, \ldots$ converges uniformly to a homeomorphism h of X onto X taking M onto N.

The theorem is trivial if X is finite. Assume X is infinite. Denote the members of M as $p(1), p(2), \ldots$ in a sequence without repetitions. Denote the members of N as $q(1), q(2), \ldots$ in a sequence without repetitions.

Suppose α is a positive number less than 1/4. The homeomorphism h will be constructed so as to move no point more than α. Let $\varepsilon(1)$ and $\gamma(1)$ be positive numbers such that 1. $\varepsilon(1) < \alpha \cdot 2^{-1}$ and 2. if x and y are two points of $S(q(1), \gamma(1))$, then there is a homeomorphism of X onto X leaving every point of $X - S(q(1), \varepsilon(1)/2)$ fixed and sending x to y. Let $n(1)$ be the smallest positive integer j such $p(j)$ is in $S(q(1), \gamma(1))$. There is a homeomorphism f_1 of X onto X such that 1. $f_1(p(i))$ is in N if $i \leqslant n(1)$, 2. for some integer j such that $1 \leqslant j \leqslant n(1)$, $f_1(p(j))$ is $q(1)$, 3. if x is not in $\bigcup \{S(p(i), \varepsilon(1)/2)\colon i \leqslant n(1)\}$, then $f_1(x) = x$ and 4. if x is in X, then $d(x, f_1(x)) < \varepsilon(1)$. Denote as $K(1)$ the compact set $\mathrm{cl}\big(\bigcup \{S(p(i), \varepsilon(1)/2)\colon 1 \leqslant i \leqslant n(1)\}\big)$. The homeomorphism f_1 can be obtained in no more than $n(1)$ steps by first sending $p(n(1))$ to $q(1)$ and then sending the remaining $p(i)$ into N with functions leaving $q(1)$ fixed and moving points appropriately small distances.

Because $S(K(1), (1)/2)$ has a compact closure, there is a positive number $\delta(1)$ such that if x and y are any two points of X such that $d(x, y) \geqslant 2^{-1}$, then $d(f_1(x), f_1(y)) \geqslant \delta(1)$.

Let $r(1)$ be the smallest positive integer j such that $q(j)$ is not in $\{f_1(p(i))\colon i \leqslant n(1)\}$ Let $\varepsilon(2)$ and $\gamma(2)$ be positive numbers such that 1. $\varepsilon(2) < \alpha \cdot 2^{-2}$, 2. $\varepsilon(2) < \delta(1) \cdot 2^{-3}$, 3. $\varepsilon(2) < d\big(q(r(1)), \{f_1(p(i))\colon i \leqslant n(1)\}\big)$, and 4. if x and y are two points of $S\big(q(r(1)), \gamma(2)\big)$, then there is a homeomorphism of X onto X leaving $X - S\big(q(r(1)), \varepsilon(2)/2\big)$ fixed, and sending x to y. Let $n(2)$ be the smallest positive integer greater than $n(1)$ such that $f_1(p(n(2))$ is in $S\big(q(r(1)), \gamma(2)\big)$. There is a homeomorphism f_2 of X onto X such that 1. $f_2 f_1(p(i))$ is in N if $n(1) < i \leqslant n(2)$, 2. for some integer j such that $n(1) < j \leqslant n(2)$, $f_2 f_1(p(j))$ is $q(r(1))$, 3. if x is not in $\bigcup \{S(f_1(p(i)), \varepsilon(2)/2)\colon n(1) < i \leqslant n(2)\}$ then $f_2(x) = x$ and also $f_2 f_1(p(i)) = f_1(p(i))$ if $i \leqslant n(1)$ and 4. if x is in X, then $d(f_2(x), x) < \varepsilon(2)$. Denote as $K(2)$ the compact set $\mathrm{cl}\big(\bigcup \{S(f_1(p(i)), \varepsilon(1)/2)\colon n(1) < i \leqslant n(2)\}\big)$.

There is a positive number $\delta(2)$ such that if $d(x,y) \geqslant 2^{-2}$, then $d(f_2f_1(x),\ f_2f_1(y)) \geqslant \delta(2)$.

In the same way one can construct continuations of the sequences begun above and obtain a positive number sequence $\varepsilon(1), \varepsilon(2), \ldots$; a positive integer sequence $r(1), r(2), \ldots$; a positive integer sequence $n(1), n(2), \ldots$; a positive number sequence $\delta(1), \delta(2), \ldots$; a sequence of compact subsets of X, $K(1), K(2), \ldots$; a sequence of homeomorphisms of X onto $X, f_1, f_2, \ldots$; and a sequence of homeomorphisms of X onto $X, h_1, h_2, \ldots$ such that the following conditions are satisfied for each integer k greater than 2:

1. h_1 is f_1, h_2 is f_2f_1 and h_k is $f_k \ldots f_2f_1$.
2. $\varepsilon(k) < \alpha\cdot 2^{-k}$.
3. $\varepsilon(k) < \delta(j)\cdot 2^{-k-1}$ for $j < k$.
4. $r(k)$ is the smallest positive integer j such that $q(j)$ is not in $\{h_k(p(i))\colon\ i \leqslant n(k)\}$.
5. $q(r(k))$ is in $\{h_{k+1}(p(i))\colon n(k) < i \leqslant n(k+1)\}$, which is a subset of N.
6. If x is not in $K(k)$, then $f_k(x) = x$ and if $i \leqslant n(k-1)$, $h_k(p(i)) = h_{k-1}(p(i))$.
7. If x is in X, then $d(f_k(x), x) < \varepsilon(k)$.
8. If $d(x,y) \geqslant 2^{-k}$, then $d(h_k(x), h_k(y)) \geqslant \delta(k)$.

Because each f_j moves no point more than $\alpha\cdot 2^{-j}$, the sequence $h_1, h_2, \ldots$ converges uniformly to a continuous function h from X into X. We need to see that $h(X)$ is X, h has a continuous inverse, and $h(M) = N$. By properties 4, 5 and 6, each $q(j)$ is in some set $h_k(M)$ and is left fixed by h_m for $m > k$. So $h(M)$ contains N. By properties 5 and 6, the construction is clearly such that $h(M)$ is a subset of N. Therefore, h is one-to-one from M onto N.

Essentially the same argument can be used twice to show that h takes X onto X and that h is a closed function. Suppose x is in X. There is a sequence $q(t(1)), q(t(2)), \ldots$ converging to x such that $d(x, q(t(j))) < 1/4$ for each positive integer j. For each positive integer j there is only one integer $s(j)$ such that $h(p(s(j))) = q(t(j))$. Since h moves no point more than 1/4, $d(x, p(s(j))) < 1/2$. Because $S(x, 1/2)$ has a compact closure, some subsequence $p(s(m(1))), p(s(m(2))), \ldots$ converges to a point z of X. Since h is continuous, $h(z) = x$. Therefore, h takes X onto X.

Suppose that F is a closed subset of X and y is in $\mathrm{cl}(h(F))$. There is a sequence $x(1), x(2), \ldots$ in F such that $h(x(1)), h(x(2)), \ldots$ converges to y and $d(h(x(j)), y) < 1/4$ for each positive integer j. Since h moves no point more than 1/4, and $S(y, 1/2)$ has a compact closure, some subsequence $x(m(1)), x(m(2)), \ldots$ converges to a point z of F. Since h is continuous, $h(z) = y$ and y is in $h(F)$. Therefore, h is a closed function.

Next, we must see that h is one-to-one. Suppose x and y are two points of X. There is a positive integer k such that $2^{-k} < d(x,y)$. Then $d(h_k(x), h_k(y)) \geqslant \delta(k)$ by property 8. But for each positive integer j, f_{k+j} moves no point as much as $2^{-j-1}\cdot\delta(k)$ and so $d(h(x), h_k(x)) < \delta(k)/2$ and $d(h(y), h_k(y)) < \delta(k)/2$ and it cannot be that $h(x) = h(y)$.

Finally, we have that h is one-to-one, onto, continuous, closed and takes M onto N and the theorem is proved.

THEOREM 4. *The* 1*-dimensional universal curve in* E^3 *is countable dense homogeneous.*

Proof. R. D. Anderson showed ([2], p. 15) that this curve is strongly locally homogeneous.

THEOREM 5. *The following statements are equivalent for a* 1*-dimensional locally connected metric continuum* M: 1. M *is a simple closed or a universal curve.* 2. M *is homogeneous.* 3. M *is countable dense homogeneous.*

Proof. It was shown by Anderson [1], [2] that statement 1 is equivalent to statement 2. Theorem 1 shows that statement 3 implies statement 2. Theorems 3 and 4 show that statement 1 implies statement 3.

4. Open problems. The problems listed below are not yet solved, as far as I know. They indicate that there is still some very basic work to be done in the study of countable dense homogeneous spaces.

Is every countable dense homogeneous continuum necessarily locally connected? (This question was asked by Ben Fitzpatrick.)

Is every countable dense homogeneous continuum necessarily n-homogeneous for every positive integer n? (See [4] for a definition of n-homogeneous.)

Is the property of being countable dense homogeneous preserved in products? If so, this would provide one with examples of n-dimensional non-manifolds which are countable dense homogeneous continua for $n \geqslant 1$. It might be easier to resolve the question for particular products, such as the product of a finite number of simple closed curves and universal curves.

References

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Remarks on a paper by Bernstein

by

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A. Bernstein [1] introduced the concept of D-compactness. We note here how the concept can be expressed in a convenient equivalent form so that the analogue of the Tychonoff theorem becomes immediate. In addition, other kinds of compactness suggest themselves.

Let D be a non-principal ultrafilter on the set I of positive integers, and let βI be the Stone–Čech compactification of I regarded as a discrete space. Then $I \cup \{D\} \subset \beta I$. Then definition 3.1 on page 187 in [1] says that x is a D-limit of $\langle x_i \rangle$ precisely when the mapping $I \cup \{D\} \to X$ such that $f(i) = x_i$ and $f(D) = x$ is continuous. Thus definition 3.2 on page 188 in [1] is equivalent of the following:

X is *D-compact* if and only if every map $I \to X$ can be extended to a continuous map $I \cup \{D\} \to X$.

In general let $A \subset B$ be two topological spaces. Call a space X, *(A, B)-compact* if every continuous map from A into X can be extended to B. By considering projections, it is immediate that any product of (A, B)-compact spaces is (A, B)-compact. (A, B)-compactness is most interesting when A is completely regular and B is a subspace of the Stone–Čech compactification of A. In this case all compact spaces are (A, B)-compact.

Definition 3.3 on page 188 in [1] says X is *ultracompact* if and only if X is D-compact for every D. By an exercise 6H in [2], p.95, if a map from I into X can be extended to a continuous map $I \cup \{D\} \to B$ for every D, then it can be extended to βI. The converse is obvious. Thus ultracompactness can be expressed in the following form: X is *ultracompact* if and only if every map $I \to X$ can be extended to a continuous map $\beta I \to X$. This finally suggests the problem of studying $(A, \beta A)$-compactness where A is some well-known completely regular space other than I.