[9] — *Analytic topology*, Amer. Math. Soc. Colloquium Publication, Vol. 28, Amer. Math. Soc., Providence, R. I., 1942, reprinted 1955.
[10] — *Local separating points of continua*, Monatsh. für Math. und Phys. 36 (1929), pp. 305–314.

UNIVERSITY OF FLORIDA
Gainesville, Florida

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Some results on fixed points — IV

by

R. Kannan (Lafayette, Ind.)

The notions of normal structure and diminishing orbital diameters have been used by Belluce and Kirk ([1], [2], [3]) to study the existence of fixed points of nonexpansive mappings. In this paper we fix our attention on mappings of another type defined as follows: Let (E, ϱ) be a metric space and let T be a mapping of E into itself such that

$$\varrho(Tx, Ty) \leqslant \tfrac{1}{2}\{\varrho(x, Tx) + \varrho(y, Ty)\} \qquad x, y \in E.$$

Mappings T of the above type will be referred to as having property A over E in this paper. Such mappings have been discussed in [5], [6], [7] and [8], dealing with fixed point and other allied problems. The author would like to mention in this connection that the referee of the present paper has suggested the name "semi-nonexpanding mappings" for mappings of this type.

Here we obtain some fixed point theorems for mappings having property A by using certain additional hypotheses. Then we compare the notions of diminishing orbital diameters [1], normal structure ([3], [4]) and property B (defined below). If $a \in E$ then the sequence of iterates of a by T will be written as $\{T^n a\}$ or $\{a_n\}$, $T^0 a = a$.

Before going into the theorems, we recollect some known definitions.

Definition 1 ([1]). A mapping T of a metric space (E, ϱ) into itself is called *nonexpansive* if

$$\varrho(Tx, Ty) \leqslant \varrho(x, y) \qquad \text{for each } x, y \in E.$$

Definition 2 ([1]). For a subset A of a metric space (E, ϱ) let $\delta(A) = \sup \varrho(x, y)$, $x, y \in A$, denote the diameter of A and let T be a mapping of E into itself. Let $\{T^n x\}$ denote the sequence of iterates of $x \in E$ and let $O(T^r x) = \{T^r x, T^{r+1} x, \ldots\}$, $r = 0, 1, \ldots$, $T^0 x = x$. If $r(x) = \lim_n \delta(O(T^n x)) < \delta(O(x))$ at a point $x \in A$ where $\delta(O(x)) > 0$, then we say that T has a *diminishing orbital diameter at* x. If T has a diminishing orbital diameter for every $x \in E$, then we say that T has a *diminishing orbital diameters over* E.

DEFINITION 3 ([3], [4]). Let A be a bounded subset of a Banach space E. A point $a \in A$ is said to be a *non-diametral point of* A if $\sup\{\|x-a\|, x \in A\} < \delta(A)$. A bounded convex subset K of E is said to have *normal structure* if for each convex subset H of K which contains more than one point there exists an $x \in H$ which is a non-diametral point of H.

We now introduce the following definition.

DEFINITION 4. Let (E, ϱ) be a metric space and let T be a mapping of E into itself. Then T is said to have the *property B on* $G \subset E$ if for every closed subset F of G which contains more than one element and is mapped into itself by T there exists an $x \in F$ such that $\varrho(x, Tx) < \sup_{y \in F} \varrho(y, Ty)$.

THEOREM 1. *Let (E, ϱ) be a compact metric space and let T be a mapping of E into itself having properties A and B over E. Then if T be such that, for any non-empty subset F of E mapped into itself by T, $F' \subset (TF)'$ (the dash standing for the derived set), then T has a unique fixed point in E (TF standing for the set of points which are the transforms of the points of F by T).*

Proof. A partial ordering is introduced in the space of $E(K)$ of sets $K_\alpha \subset E$ which are non-empty, closed and invariant relative to T in the following manner: $K_{\alpha_1} \lesssim K_{\alpha_2}$, $K_{\alpha_1} \supset K_{\alpha_2}$, $K_{\alpha_1} \neq K_{\alpha_2}$. Using the Kuratowski–Zorn lemma, we can get a set K which is minimal with respect to being non-empty, closed and invariant relative to T.

If K contains only one element, then that element is a fixed point. If K contains more than one element, then, since T has the property B, there is an element x in K such that

$$\varrho(x, x_1) = r < \sup_{y \in K} \varrho(y, y_1).$$

Let $K_1 = \{z \in K\colon \varrho(z, z_1) \leqslant r\}$. Evidently K_1 is a non-empty proper subset of K. Further, if $z \in K_1$, then $z \in K$ and hence $z_1 \in K$. Now

$$\varrho(z_1, z_2) \leqslant \frac{\varrho(z, z_1)}{2} + \frac{\varrho(z_1, z_2)}{2}.$$

Therefore $\varrho(z_1, z_2) \leqslant \varrho(z, z_1) \leqslant r$. Hence $z_1 \in K_1$ i.e., K_1 is mapped into itself by T.

Also if y be a limit point of K_1, then by hypothesis $\exists p^{(n)} \in K_1$: $\{Tp^{(n)}\} \in TK_1$ such that, for any arbitrary $\varepsilon > 0$ $\exists N$ such that $\varrho(y, Tp^{(n)}) \leqslant \varepsilon$, $n > N$. Hence

$$\varrho(y, y_1) \leqslant \varrho(y, Tp^{(n)}) + \varrho(Tp^{(n)}, y_1)$$
$$\leqslant \varrho(y, Tp^{(n)}) + \frac{\varrho(p^{(n)}, Tp^{(n)})}{2} + \frac{\varrho(y, y_1)}{2}.$$

Therefore $\varrho(y, y_1) \leqslant 2\varrho(y, Tp^{(n)}) + r$, because $p^{(n)} \in K_1$. Hence $\varrho(y, y_1) \leqslant r$. This shows that $y \in K_1$, and consequently K_1 is closed.

Thus K_1, a proper subset of K, is non-empty, closed and mapped into itself by T, which contradicts the minimality of K. Hence K contains only one element, which is therefore a fixed point of T. The unicity of the fixed point follows from the fact that if $x = Tx$ and $y = Ty$ then

$$\varrho(x, y) = \varrho(Tx, Ty) \leqslant \tfrac{1}{2}\{\varrho(x, Tx) + \varrho(y, Ty)\} = 0 \quad \text{i.e.,} \quad x = y.$$

This proves the theorem.

We furnish below an example in support of Theorem 1.

EXAMPLE. Let $E = [0, 1/2] \cup (\{1/2 + 1/n\})$, n taking all positive integral values. T is defined by

$$Tx = \frac{1}{2} - x, \quad x \in \left[0, \frac{1}{2}\right] \quad \text{and} \quad Tx = \frac{1}{4} + \frac{1}{3n}, \quad x = \frac{1}{2} + \frac{1}{n}.$$

It can be seen that all the conditions of Theorem 1 are satisfied and 1/4 is the unique fixed point of T in E.

THEOREM 2. *Let T be a continuous mapping of a compact metric space (E, ϱ) into itself and let T have properties A and B over E. Then T has a unique fixed point in E.*

Proof. Let K be as in the previous theorem. If K contains more than one element, then, since T has property B over E, $\exists z \in K$ such that

$$\varrho(z, Tz) = r < \sup_{y \in K} \varrho(y, Ty).$$

Let $K_1 = \{t \in K\colon \varrho(t, Tt) \leqslant r\}$. Evidently K_1 is a non-empty proper subset of K. Also if $t \in K_1$, then

$$\varrho(Tt, T^2t) \leqslant \frac{\varrho(t, Tt)}{2} + \frac{\varrho(Tt, T^2t)}{2} \quad \text{by property } A.$$

Hence

$$\varrho(Tt, T^2t) \leqslant \varrho(t, Tt) \leqslant r.$$

So, K_1 is mapped into itself by T. Finally, if $p^{(n)} \in K_1$ be such that $p^{(n)} \to y$, then $y \in K$ and

$$\varrho(y, Ty) \leqslant \varrho(y, p^{(n)}) + \varrho(p^{(n)}, Tp^{(n)}) + \varrho(Tp^{(n)}, Ty).$$

The continuity of T implies $\varrho(y, Ty) \leqslant r$. So $y \in K_1$. Consequently K_1 is closed.

Thus K_1 is a non-empty closed subset of K which is mapped into itself by T. This contradicts the minimality of K. Hence K contains only one element which is a fixed point of T. The unicity follows as in the previous theorem. This proves the theorem.

We devote the rest of the paper to comparing the notions of property B, diminishing orbital diameters and normal structure.

THEOREM 3. *Let (E, ϱ) be a metric space and let T be a mapping of E into itself having property A over E. Then if T has diminishing orbital diameters over E, T has the property B over E.*

Proof. Let F be a closed subset of E, mapped into itself by T, containing more than one element. If possible, let, for every element $x \in F$, $\varrho(x, x_1) = K = \sup_{y \in F} \varrho(y, y_1)$. K is evidently non-zero, for if $K = 0$ then F would contain more than one fixed point of T, which is not possible.

Now, for $x \in F$,

$$\varrho(x_r, x_s) \leqslant \frac{\varrho(x_{r-1}, x_r)}{2} + \frac{\varrho(x_{s-1}, x_s)}{2} = K, \quad r, s \geqslant 1.$$

Hence, for $r \geqslant 1$,

$$\delta(O(x_r)) = \delta(x_r, x_{r+1}, \ldots) = K$$

(because $\varrho(x_r, x_s) \leqslant K$ and $\varrho(x_r, x_{r+1}) = K$).

Hence, at $x_1 \in F$, T does not have a diminishing orbital diameter. This contradiction completes the proof.

THEOREM 4. *Let T be a continuous mapping of a metric space (E, ϱ) into itself and let T have properties A and B over E. Then T must have diminishing orbital diameters over E.*

Proof. If possible let $x \in E$ be a point at which $\varrho(x, x_1) > 0$ and T does not have diminishing orbital diameters at x.

Hence

$$\lim_n \delta(O(x_n)) = \delta(O(x)). \tag{1}$$

Also

$$\delta(O(x_n)) = \delta(x_n, x_{n+1}, \ldots).$$

Now

$$\varrho(x_r, x_s) \leqslant \frac{\varrho(x_{r-1}, x_r)}{2} + \frac{\varrho(x_{s-1}, x_s)}{2}.$$

Also

$$\varrho(x_{r-1}, x_r) \leqslant \frac{\varrho(x_{r-2}, x_{r-1})}{2} + \frac{\varrho(x_{r-1}, x_r)}{2}.$$

Hence

$$\begin{aligned}\varrho(x_{r-1}, x_r) &\leqslant \varrho(x_{r-2}, x_{r-1})\\ &\leqslant \varrho(x_{r-3}, x_{r-2})\\ &\ldots\ldots\ldots\ldots\\ &\leqslant \varrho(x, x_1).\end{aligned}$$

Therefore

$$\varrho(x_r, x_s) \leqslant \varrho(x, x_1).$$

So

$$\delta(O(x_n)) \leqslant \varrho(x_{n-1}, x_n) \leqslant \varrho(x, x_1).$$

Consequently $\delta(O(x_n)) = \varrho(x, x_1)$, for otherwise (1) would be contradicted. Hence

$$\varrho(x_{n-1}, x_n) = \varrho(x, x_1) \quad \text{for all } n \geqslant 1. \tag{2}$$

Let $y \in E$ be a limit point of $O(x)$. Then $\{x_n\} \supset \{x_{n_i}\} \to y \in E$. Therefore

$$|\varrho(y, y_1) - \varrho(x_{n_i}, x_{n_i+1})| \leqslant \varrho(y, x_{n_i}) + \varrho(y_1, x_{n_i+1}).$$

Since T is continuous and $\{x_{n_i}\} \to y$, we have

$$\varrho(y, y_1) = \lim_i \varrho(x_{n_i}, x_{n_i+1}).$$

So, from (2),

$$\varrho(y, y_1) = \varrho(x, x_1) = \varrho(x_{n-1}, x_n), \quad n \geqslant 1. \tag{3}$$

Now consider the set $\overline{O(x)}$, i.e., the closure of $O(x)$. It is non-empty, closed and contains more than one element. By virtue of the continuity of T it is mapped into itself. Hence, since T has property B over E, there exists an element $p \in \overline{O(x)}$ such that $\varrho(p, p_1) < \sup_{t \in \overline{O(x)}} \varrho(t, t_1)$, which however, is impossible by (3). The contradiction thus obtained proves the theorem.

THEOREM 5. *Let K be a bounded convex subset of a Banach space E and let K be mapped into itself by T. Suppose further that*

$$\|Tx - Ty\| \leqslant \tfrac{1}{2}\{\|x - Tx\| + \|y - Ty\|\}, \quad x, y \in K.$$

Then if K has normal structure, T has the following property: For every closed convex subset F of K mapped into itself by T and containing more than one element there exists an $x \in F$ such that $\|x - Tx\| < \sup_{y \in F} \|y - Ty\|$.

Proof. If possible, let the theorem be not true. Then there exists a closed convex subset F of K containing more than one element and mapped into itself by T and such that for every element $x \in F$

$$\|x - x_1\| = \sup_{y \in F} \|y - y_1\| = \beta \ (\neq 0) \text{ say}. \tag{4}$$

Now consider the set $T(F)$. If it contains only one element, then that element y is a fixed point of T and hence $\|y - y_1\| = 0$, which contradicts (4). Hence $T(F)$ contains more than one element.

Let $S = \mathrm{Co}[T(F)]$, i.e., the convex hull of $T(F)$, and let $x, y \in S$. Then there are three possibilites: 1) $x = Tx'$, $y = Ty'$, $x', y' \in F$,

2) $x = Tx'$, $y = \sum \alpha_i Ty'_i$, $x', y'_i \in F$ and $\sum \alpha_i = 1$, 3) $x = \sum \alpha_i Tx'_i$, $y = \sum \beta_i T\beta'_i$, $x'_i, y'_i \in F$, $\sum \alpha_i = \sum \beta_i = 1$.

It can be seen by virtue of

$$\|Tx - Ty\| \leqslant \frac{\|x - Tx\|}{2} + \frac{\|y - Ty\|}{2}, \quad x, y \in K,$$

that $\|x - y\| \leqslant \beta$. Hence $\delta(S) \leqslant \beta$.

Also for any element $x \in F$ [$x = Tx'$ or $\sum \alpha_i Tx'_i$, $x', x'_i \in F$, $\sum \alpha_i = 1$], $Tx \in \mathrm{Co}(TF)$ because T maps F into itself. Further, $\|x - Tx\| = \beta$ by (4). Hence $\sup_{y \in F} \|x - y\| = \beta$ for any $x \in S$. This contradicts the assumption of normal structure over K. Hence the theorem follows.

That the converse of the above theorem is not true may be seen from the following example.

EXAMPLE. Let m be the space of bounded sequences of numbers with the supremum norm ([9], p. 12) and let $K = \{x \in m: \|x\| \leqslant 2\}$. Clearly K is a bounded convex set in m. Now let F be the subset of K such that $F = [x_1, x_2, \ldots]$ where $x_K = \{0, 0, 0, \ldots, 1, 0, \ldots\}$ (1 in the Kth place). Evidently $\delta(F) = 1$. Also $\sup_{y \in F} \|x - y\| = 1$ for every $x \in F$. Hence K does not have normal structure. But the operator $T: K \to K$ defined by $Tx = x/3$, $x \in K$ is such that

$$\|Tx - Ty\| \leqslant \tfrac{1}{2}[\|x - Tx\| + \|y - Ty\|], \quad x, y \in K$$

and for every closed subset F' of K mapped into itself by T and containing more than one element there exists an $x \in F'$ such that $\|x - Tx\| < \sup_{y \in F'} \|y - Ty\|$.

Finally from Theorems 1 and 3 we obtain the following theorem.

THEOREM 6. *Let (E, ϱ) be a compact metric space and T be a mapping of E into itself having property A over E. Also let T have diminishing orbital diameters over E.*

Then if T is such that, for any non-empty subset F of E mapped into itself by T, $F' \subset (TF)'$ (the dash standing for the derived set), then T has a unique fixed point in E.

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References

[1] L. P. Belluce and W. A. Kirk, *Fixed point theorems for certain classes of nonexpansive mappings*, Proc. Amer. Math. Soc. 20 (1) 1969, pp. 141–146.

[2] — — *Some fixed point theorems in metric and Banach spaces*, Canad. Math. Bull. 12 (4) (1969), pp. 481–491.

[3] — — *Fixed point theorems for families of contraction mappings*, Pac. Jour. Math. 18 (2) (1966), pp. 213–217.

[4] M. S. Brodskii and D. P. Milman, *On the centre of a convex set*, Dokl. Akad. Nauk SSSR 59 (1948), pp. 837–840.

[5] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc. 60 (1968), pp. 71–76.

[6] — *Some results on fixed points — II*, Amer. Math. Month. 76 (4) (1969), pp. 405–408.

[7] — *On a class of operators in Banach spaces*, Communicated for publication.

[8] — *Some results on fixed points — III*, Fund. Math. 70 (1971), pp. 169–177.

[9] L. A. Liusternik and V. J. Sobolev, *Elements of functional analysis*, 1961.

ST. XAVIER'S COLLEGE
Calcutta, India

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