One dimensional locally setwise homogeneous continua

by

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1. Introduction. In this paper we obtain a classification of the one dimensional locally setwise homogeneous (l-s-h) continua without "boundary". It is shown that the simple closed curve, Sierpiński curve, universal (Menger) curve, and "copies" of the Sierpiński curve on the compact 2-manifolds without boundary are the only such continua.

L-s-h continua were first introduced in [2], where it is shown that their groups of homeomorphisms are not zero dimensional. Other such continua include the compact manifolds, with or without boundary and of all dimensions, and the Hilbert cube. The notions of *near basis* and *strongly* l-s-h (s-l-s-h) continua are used in the proofs of the theorems of this paper. These were first defined in [3], where it is shown that every s-l-s-h continuum has a near basis. It is not known whether every l-s-h continuum is also s-l-s-h.

See Section 2 for definitions of the above terms.

2. Preliminaries. All spaces are separable metric. $H(X)$ (or H) denotes the group of all homeomorphisms of X onto itself. If $U \subseteq X$ and $h \in H$, h is *supported* on U means $h(x) = x$ for $x \in C(U)$, the complement of U. If $\mathfrak{D}$ is a collection of sets, $\mathfrak{D}^*$ denotes the union of the members of the collection. X^0 denotes the *interior* of X. A double arrow denotes an *onto* function.

DEFINITION 2.1. A continuum X is called *locally setwise homogeneous* (l-s-h) iff there exist a dense subset A of X and a basis $\mathfrak{B}$ of connected open subsets of X such that for any $B \in \mathfrak{B}$ and $a, b \in A \cap B$, there is a homeomorphism $h \in H$, h supported on B, such that $h(a) = b$. $\{X, A, \mathfrak{B}, H\}$ is called a l-s-h *structure* for X. Note that an l-s-h continuum is necessarily locally connected.

DEFINITION 2.2. A continuum is called *strongly locally setwise homogeneous* (s-l-s-h) iff there exists an l-s-h structure $\{X, A, \mathfrak{B}, H\}$ for X such

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that for each $B \in \mathcal{B}$ and $x \in A \cap B$, there is a neighborhood U of x with $\overline{U} \subseteq B$, satisfying the following property: For each open subset V of B, there is a homeomorphism $h \in H$ such that

(1) h is supported on B, and

(2) $h(\overline{U}) \subseteq V$.

DEFINITION 2.3. Let $\{X, A, \mathcal{B}, H\}$ be a l-s-h structure, and let $B \in \mathcal{B}$, $B \neq \emptyset$. Let $\mathcal{E} = \{h(B)|\ h \in H\}$. $\mathcal{E}$ is called a *near basis* for X iff every open set U of X contains the closure of an element of $\mathcal{E}$. $\mathcal{E}$ is said to be *generated* by B.

DEFINITION 2.4. Let X be a l-s-h continuum, and let $\mathcal{E}$ be a near basis for X. Let $M(X) = \bigcup \{E \in \mathcal{E}\}$. Then $M(X)$ is called the *core* of X, and $X - M(X)$ is called the *boundary* of X (BdX).

We note that if X is a compact manifold with boundary, then the core of X is the interior of the manifold, and BdX is the boundary of the manifold in the usual sense. (See [3].)

DEFINITION 2.5. The *Sierpiński* (*universal plane*) *curve* is any continuum homeomorphic to the following: A closed disk D in E^2 minus the interiors of a countable dense set of pairwise disjoint circles in D^0 whose diameters have limit 0, and no one of which is a subset of the interior of another. It has been characterized [8] as the only one dimensional, locally connected, plane continuum, with no local cut points.

The *Menger* (*universal*) *curve* is obtained in the following way: Let F_1, F_2, and F_3 be three faces of a cube in E^3 such that no two of these are opposite each other. Punch out, to the opposite side, the interior of the square which is the middle ninth of each of these. Then punch out, to the opposite sides, the middle ninths of the remaining eight squares on each of the three faces. Continue the process. The resulting continuum is the standard construction of the Menger curve. It has been characterized [1] as the only one dimensional locally connected continuum, with no local cut points, such that no open subset is imbeddable in the plane.

DEFINITION 2.6. Let $\{C_i\}_{i=1}^{\infty}$ be a countable collection of pairwise disjoint simple closed curves in a compact two-manifold, M, such that C_i bounds a disk D_i in M, diam $D_i \to 0$, and, for $j \neq i$, $C_j \not\subseteq D_i$. We also require that $\bigcup_{i=1}^{\infty} D_i$ be dense in M. Let $X = M - \bigcup_{i=1}^{\infty} D_i^0$. Any continuum X obtained in this way is called a *copy of the Sierpiński curve on a compact two-manifold*.

THEOREM 2.1. *Every* s-l-s-h *continuum has a near basis.*

Proof. See Theorem 2.1 of [3].

THEOREM 2.2. *If X is a* s-l-s-h *continuum, $M(X)$ is a dense, open, connected subset of X, and is the union of the dense orbits of X.*

Proof. See Theorem 4.2 of [3].

THEOREM 2.3. *The Sierpiński curve, copies of it on a two-manifold, and the simple closed curve are* s-l-s-h *continua* (*without boundary*), *and therefore have near bases.*

Proof. See Section 5 of [3].

THEOREM 2.4. *The Menger curve has a near basis.*

Proof. Let M be the Menger curve in E^3, obtained by its standard construction, with $M \subseteq \prod_{i=1}^{3} I_i$, where $I_i = [0, 1]$. Let h be a continuous function of E^3 onto itself such that

(1) h is 1-1 except on the squares S, T,

$$S = \{(x, y, z) \in E^3|\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, z = 0\},$$
$$T = \{(x, y, z) \in E^3|\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, z = 1\},$$

(2) $h(S) = (0, 0, 0) = p$, $h(T) = (0, 0, 1) = q$, and

(3) $h(x, y, z) = (x', y', z)$ for all points of E^3.

Let $M' = h(M)$. Then M' is a locally connected continuum with no local cut points, and no open set is imbeddable in E^2, since each open subset contains a homeomorphic image of an open subset of M. Thus, by the characterization of the Menger curve, M' and M are homeomorphic.

We now show that there exists a neighborhood U of p in M' such that U generates a near basis for M'. Let $\{P_i\}_{i=-\infty}^{\infty}$ be a sequence of points of $[0, 1]$ such that $\lim_{i \to -\infty} p_i = 1$, $\lim_{i \to \infty} p_i = 0$, $p_i < p_{i+1}$, and p_i is the midpoint of a deleted interval of the Cantor set on $[0, 1]$. Let $F_i = \{(x, y, z) \in M|\ z_i = p_i\}$. Then $h(F_i)$ separates $h(M)$ into two disjoint sets. Let $G_i = \{(x, y, z) \in h(M)|\ p_{i-1} \leqslant z \leqslant p_i\}$. Then there exists a homeomorphism g: $M' \twoheadrightarrow M'$ such that $g(G_i) = G_{i+1}$, all i. Further g^k is also a homeomorphism of M' onto itself taking G_i onto G_{i+k}. Now let $U = \text{Int}(\bigcup_{i=1}^{\infty} G_i \cup \{p\})$ in M'. Then U is a neighborhood of p in M'. Furthermore U generates a basis $\mathcal{B}$ for M', for if V is any neighborhood of p, there exists an integer n such that $g^n(U) \subseteq V$. Thus, by the homogeneity of M', U generates a basis, $\mathcal{B} = \{h(U)|\ h \in H\}$. Further, it follows from a statement in the proof of Theorem 16 of [1], that $\{M', M', \mathcal{B}, H\}$ is an l-s-h structure for M'. Then $\mathcal{B}$ is also a near basis for M', and the theorem follows.

3. Main theorem. In this section we obtain the main theorem of the paper. We first prove some lemmas.

LEMMA 3.1. *Let X be a* l-s-h *continuum. If X contains an open subset which is homeomorphic to the interior of an arc, then X is an arc or simple closed curve.*

Proof. Let O be an open subset of X such that $\bar{O}$ is an arc and O is the interior of $\bar{O}$. Let $\mathfrak{U}$ be a maximal chain of open subsets of X which are interiors of arcs and which contain O. Let U be the union of the elements of $\mathfrak{U}$. Then U is an open subset of X which is homeomorphic to the interior of an arc. Thus $\bar{U}-U$ has at most two points, and $\bar{U}$ is either an arc or simple closed curve.

We show that $\bar{U}=X$. Suppose there exists a point $x\in X-\bar{U}$. Let $p\in\bar{U}\cap C$, where C is a minimal arc in X from x to $\bar{U}$. Since each point of U is a point of order two in X, C contains no point of U. Now let $W\in\mathfrak{B}$ be a neighborhood of p such that p separates W. Then there are points $a,b\in A$ such that $a\in W\cap U$ and $b\in W-\bar{U}$, and a homeomorphism h supported on W such that $h(a)=b$. Let $\widehat{pq}$ be the largest arc containing a and contained in $W\cap\bar{U}$, with endpoint p. Then $\widehat{aq}$ is an arc. Therefore $h(\widehat{aq})$ is also an arc. Since h is a homeomorphism of X onto itself, $h\big((\mathrm{Int}(\widehat{aq})\big)$ is open in X. But then $[h(\widehat{aq})\cup\bar{U}]$ is an arc whose interior is an open subset of X and properly contains U. Therefore U was not maximal. This is a contradiction. Thus $X-\bar{U}=\emptyset$ and $\bar{U}=X$.

LEMMA 3.2. *Let X be a* l-s-h *continuum, p a local cut* (1) *point of X. Then p is of order* (2) *two in X.*

Proof. Let $\{X,A,\mathfrak{B},H\}$ be a l-s-h structure for X, and let $B\in\mathfrak{B}$ such that $p\in B$ and p is a cut point of B. We first show that p is not isolated. Let $B-\{p\}=M\cup N$ where M and N are mutually separate. We may further assume that $M\cap \mathrm{Bd}B\neq\emptyset\neq N\cap\mathrm{Bd}B$. Then there are points $a\in A\cap M$ and $b\in A\cap N$ such that the components of these points, respectively, meet $\mathrm{Bd}B$, and there exists a homeomorphism h supported on B such that $h(a)=b$. Let q be a point of $M\cap\mathrm{Bd}B\cap$ (component of a in M). Then M contains an arc S from q to a. Now $h(S)$ contains p. Thus S must contain a cut point of B different from p. Clearly, for each $\varepsilon>0$, we may choose B so that $\mathrm{diam}B<\varepsilon$. Thus p is not isolated; that is, there exists a sequence of cut points of B converging to p from M. By a similar argument, there exists a sequence of cut points of B converging to p from N.

Now let x and y be cut points of B in M and N, respectively, chosen as in the above paragraph. Let D be the closure of the component of p in $B-\{x,y\}$. Then D^0 is a connected open set containing p with a two point boundary in X. Since D can be made as small as we please, p is of order two in X.

(1) p is a *local cut point* of X means that there exists a neighborhood V of p such that $V-\{p\}$ has at least two components.

(2) p is of *order two* in X means that there exist arbitrarily small neighborhoods of p with two-point boundaries in X.

LEMMA 3.3. *Let X be a* l-s-h *continuum. Then X contains no local cut points, unless it is an arc or simple closed curve.*

Proof. (3) If X is not an arc or simple closed curve, by Lemma 3.1, we may assume that X contains no free arc — that is, an open set which is homeomorphic to the interior of an arc. We suppose, by way of contradiction, that X contains a local cut point p. By Lemma 3.2, p is of order two in X. Thus, as in Lemma 3.2, there exists a connected open set D containing p such that $\mathrm{Bd}D$ consists of exactly two points, say a and b, each a local cut point of order two. Since D is a connected open set, $\bar{D}$ is l-s-h.

We will show that $\bar{D}$ is not a dendron and contains no simple closed curves, a contradiction. Therefore, X contains no local cut points, and the theorem follows.

Since X cannot contain a free arc, $\bar{D}$ is not an arc. Thus, if $\bar{D}$ is a dendron, it must contain (a dense set of) branch points. But then $\bar{D}$ is not l-s-h. Therefore $\bar{D}$ cannot be a dendron, and must contain a simple closed curve. But if $\bar{D}$ contains a simple closed curve, it must contain a non-trivial cyclic element E (p. 312 of [5]). Let A' be any arc from a to b in $\bar{D}$, and let t be the first point of A', in the order from a to b, that lies in E. It follows (Theorems 4,11 on pp. 308, 316 of [5]) that t is a cut point of $\bar{D}$, and therefore t is a local cut point of X. But t is of order greater than 2. This is a contradiction, and thus $\bar{D}$ cannot contain a simple closed curve. Hence, $\bar{D}$ is not a dendron, and $\bar{D}$ contains no simple closed curve.

The theorem follows.

LEMMA 3.4. *Let X be a locally connected continuum in E^2 (or S^2) obtained by removing the interiors of countably many pairwise disjoint disks, D_n. Then $\lim_{n\to\infty}\mathrm{diam}(D_n)=0$.*

Proof. Suppose, by way of contradiction, that $\lim_{n\to\infty}\mathrm{diam}(D_n)=\varepsilon>0$. By Theorem 59 of p. 24 of [7], we may assume, without loss of generality, that $\{D_n\}_{n=1}^{\infty}$ converges sequentially to a limiting set C. It is easy to see that C is a one dimensional continuum and $\mathrm{diam}(C)>0$. Let $0<\delta<\varepsilon/3$, and let $\mathfrak{U}$: $U_1,U_2,\ldots,U_k$ be a minimal, finite, one dimensional, δ-cover of C by connected open sets in X, and let $\mathfrak{V}$: $V_1,V_2,\ldots,V_k$ be open sets in E^2 (or S^2) such that $E^2\cap V_i=U_i$. We may assume that $\mathfrak{V}$ is also a one dimensional cover. (By a *one dimensional cover*, we mean a cover whose nerve is one dimensional.) Let N be an integer such that for $n\geqslant N$, $D_n\subseteq\bigcup_{i=1}^{k}V_i$, and D_n meets each V_i.

(3) I am indebted to G. T. Whyburn for a letter in which he sent me a much simpler proof of Lemma 4.4 of [2]. His proof made use of cyclic element theory, and his ideas are used in the proof of Lemma 3.3 here.

We show that some V_i disconnects both D_N and $\mathrm{Bd}\,D_N$. Let C_N be the boundary simple closed curve of D_N. Since $\mathfrak{V}$ is a one dimensional cover containing at least three elements, there exists j, $1 \leqslant j \leqslant k$, such that V_j separates $\mathfrak{V}^*$. But $\mathfrak{V}$ is also a one dimensional minimal cover of both D_N and C_N. Thus V_j separates both D_N and C_N.

Let K be a component of $V_j \cap D_N$ which separates D_N. Then K also separates C_N. Thus $K \cap C_N = A \cup B$, where A and B are mutually separate. Therefore $K \cap D_N^0$ separates A from B, and it follows that U_j is not connected, since $U_j \cap (K \cap D_N^0) = \emptyset$. This is a contradiction.

Remark. We note that the above lemma is implicitly assumed in the proofs of Lemma 1 and Theorem 4 of [8].

LEMMA 3.5. *Let X be a one dimensional, locally connected, metric continuum with no local cut points, such that some open subset U of X is imbeddable in E^2. Then there is an open set W such that $\overline{W} \subseteq U$ and $\overline{W}$ is homeomorphic to a Sierpiński curve.*

Proof. Let $h\colon U \to E^2$ and $g\colon E^2 \twoheadrightarrow S^2 - \{p\}$, where p is the north pole of S^2, be homeomorphisms. Let V be open and connected in U such that $\overline{V} \subseteq U$ and $\overline{V}$ is a Peano Continuum. Then $gh(\overline{V}) \subseteq S^2 - \{p\}$. Let D' be the complementary domain of $gh(\overline{V})$ in S^2 that contains p. Let $\mathfrak{D} = \{D'' \mid D''$ is a complementary domain of $gh(\overline{V})$ in S^2 and $\overline{D''} \cap \overline{D'} \neq \emptyset\}$. Let $D = D' \cup \mathfrak{D}^*$. Clearly, $\overline{D}$ is compact and connected.

We show that $\overline{D}$ doesn't separate S^2. Suppose $\overline{D}$ separates S^2. Then $S^2 - \overline{D} = A \cup B$, where A and B are mutually separate. We may assume that $gh(\overline{V}) \subseteq \overline{A}$, and it follows that $B \cap gh(\overline{V}) = \emptyset$. Therefore B is a subset of the union of the complementary domains of $gh(\overline{V})$. Now $\overline{D} \cup B'$ is connected. Therefore, for any component B' of B, $\overline{D} \cup B'$ is connected, and thus B' must be a subset of $\overline{D}$. Thus $B = \emptyset$ and this is a contradiction. Therefore $\overline{D}$ doesn't separate S^2.

We note further that $\overline{D} \neq S^2$. For let W_1 be open in V and W_2 be open in W_1 such that $\overline{W}_2 \subseteq W_1$. Since X contains no local cut points, $\overline{W}_2$ is not a dendron, and therefore $\overline{W}_2$ contains a simple closed curve, C. Now $gh(C)$ is a simple closed curve in $gh(\overline{V}) \subseteq S^2$, and therefore bounds a disk T in S^2 which misses p. This disk is two dimensional and therefore not a subset of $gh(\overline{V})$. Thus there exists a complementary domain O of $gh(\overline{V})$ such that $\overline{O} \subseteq T$. Now, since $\overline{V}$ is a locally connected, metric continuum, so is $gh(\overline{V})$. Further, since V is connected and contains no (local) cut points, no point of $\overline{V}$ is a cut point of $\overline{V}$. Thus $gh(\overline{V})$ is a locally connected, plane continuum with no cut points. Then by Theorem 9, p. 212 of [6], each complementary domain of $gh(\overline{V})$ is a simple closed curve. Thus $\mathrm{Bd}\,\overline{O}$ is a simple closed curve and $\mathrm{Bd}\,\overline{O} \subseteq gh(\overline{V})$.

To show that $\overline{D} \neq S^2$, we will show that $\overline{D} \cap \mathrm{Bd}\,\overline{O} = \emptyset$. We first note that $\overline{D'} \cap \mathrm{Bd}\,\overline{O} = \emptyset$. For if not, there exists a point $t \in \overline{D'} \cap \mathrm{Bd}\,\overline{O}$.

Then t is accessible from each of the complementary domains D' and O, and therefore, by Theorem 6, p. 308 of [10], t is a local cut point of $gh(\overline{V})$. But $t \in gh(V)$ also, and V contains no local cut points. This is a contradiction. We finally show that $\overline{D} \cap \mathrm{Bd}\,\overline{O} = \emptyset$. Suppose $q \in \overline{D} \cap \mathrm{Bd}\,\overline{O}$. Then q is not an element of any complementary bounding simple closed curve in $\mathfrak{D}$, by the above argument. Thus q is a limit point of a sequence of complementary bounding simple closed curves in $\mathfrak{D}$. By Lemma 3.4, this sequence of simple closed curves has diameters with limit 0. Thus some subsequence converges to q, and by the construction of $\mathfrak{D}$, q must be on $\mathrm{Bd}\,\overline{D'}$. But again, by the above argument, this cannot happen. Thus it follows that $\overline{D} \cap \mathrm{Bd}\,O = \emptyset$, and $\overline{D} \neq S^2$.

Thus $\overline{D}$ is a continuum which doesn't separate S^2. By Theorem 15, p. 363 of [7], there exists a continuous function $f\colon S^2 \twoheadrightarrow S^2$ such that f is 1-1 off $\overline{D}$ and $f(\overline{D}) = p$. Now $fgh(\overline{V})$ is a continuum in S^2 (containing p). It follows from the previous paragraphs that $fgh(\overline{V})$ contains many complementary domains with simple closed curve boundaries. Further, no two of these complementary domain boundaries can intersect, for if y is such a point of intersection then $y \neq p$ and y is accessible from both domains, and therefore is a local cut point of $fgh(V)$ by Theorem 6 of [10]. It follows that $(fgh)^{-1}(y)$ is a local cut point of X and this is a contradiction.

We have now shown that $fgh(\overline{V})$ is a continuum in S^2 which contains complementary domains, no two of whose closures meet. Further the diameters of these complementary domains have limit 0, by Lemma 3.4. Thus by Theorem 15, page 363 of [7], there exists a continuous function $\varphi\colon S^2 \twoheadrightarrow S^2$ such that $\varphi(p) = p$ and each of whose non-degenerate inverse sets is the closure of exactly one complementary domain of $fgh(\overline{V})$. Let $F = \{z \in S^2 \mid \varphi^{-1}(z)$ is non-degenerate$\}$. Then F is countable and, since $fgh(\overline{V})$ is one-dimensional, F is also dense in S^2. Since F is countable, there exists a sequence of simple closed curves $\{K_i\}_{i=1}^{\infty}$ in $S^2 - F$ which bound open cells whose intersection is p. Then $\{\varphi^{-1}(K_i)\}_{i=1}^{\infty}$ is a sequence of simple closed curves closing down on $\varphi^{-1}(p) = p$ in S^2, and each of these misses the union of the closures of the complementary domains of $fgh(\overline{V})$.

Let W' be the component of $\varphi^{-1}(K_1)$ in S^2 which doesn't contain p, and let $W = h^{-1}\, g^{-1}\, f^{-1}\,(W')$. Then W is open in X, $\overline{W} \subseteq V \subseteq U$, and $\overline{W}$ is a one dimensional locally connected, plane continuum with no local cut points, and therefore, by [8], a Sierpiński curve.

THEOREM 3.1. *The only one dimensional* 1-s-h *continua without boundary are the simple closed curve, Sierpiński curve, Menger curve, and copies of the Sierpiński curve on the compact two-manifolds without boundary.*

Proof. Let X be a one dimensional l-s-h continuum without boundary. If X contains local cut points, by Lemma 3.3, it is a simple closed curve. If X contains no local cut points, there are two cases.

Case 1. No open subset of X is imbeddable in E^2. Then X is a one dimensional locally connected continuum with no local cut points such that no open subset is imbeddable in E^2, and by [1], is homeomorphic to a Menger curve.

Case 2. Some open subset U is imbeddable in E^2. We first show that X is locally imbeddable in E^2. By Lemma 3.5, there exists an open set V such that $\overline{V} \subseteq U$ and $\overline{V}$ is a Sierpiński curve. Let $\mathcal{E}$ be a near basis for X and let $E \in \mathcal{E}$ such that $E \subseteq V$. Since X has no boundary, for each $x \in X$, there exists a homeomorphism $h_x\colon X \twoheadrightarrow X$ such that $h_x(x) \in E \subseteq V$. Thus $h_x^{-1}(V)$ is a neighborhood of x whose closure is $h_x^{-1}(\overline{V})$ and is therefore a Sierpiński curve. It follows that every point of X has a neighborhood whose closure is a Sierpiński curve. Hence X is locally imbeddable in E^2.

We note that a Sierpiński curve has exactly two orbits under its homeomorphism group. Thus the "complementary bounding" simple closed curves determined by a particular imbedding into the plane are independent of the imbedding, and are therefore uniquely determined by any such imbedding. Since X is compact, there exists a finite collection, $V_1, V_2, \ldots, V_n$, where V_i is an open subset of X, $\overline{V}_i$ is a Sierpiński curve, and $\bigcup_{i=1}^{n} V_i = X$. Thus there exists a well defined, unique, maximal, countable (because the collection is countable in each V_i) collection of "complementary bounding" simple closed curves, and as seen in the proof of Lemma 3.5, no two have a common point. We denote this collection by $\{C_i\}_{i=1}^{\infty}$, and note that, by Lemma 3.4, $\operatorname{diam} C_i \to 0$.

Our final aim is to "fill in" each of the C_i's with a disk bounded by it, in such a way that $\lim_{i\to\infty}(\operatorname{diam} D_i) = 0$ and thus insure that the resulting continuum will be a two-manifold without boundary. Then X is obtained from that two-manifold by removing the interiors of a countable dense set of pairwise disjoint simple closed curves, none contained in the interior of another, and whose diameters have limit 0. The theorem follows.

To this end, let $I^\infty = \prod_{i=1}^{\infty} I_i$, where $I_i = [-1/2i, 1/2i]$, be the Hilbert cube with metric $d(x, y) = [\sum_{i=1}^{\infty} d^2(x_i, y_i)]^{1/2}$. Since X is one dimensional, compact, and metric, by Theorem V.2 of [4], there exists a homeomorphism $\varphi\colon X \to I^3$. We think of I^3 as $I_1 \times I_2 \times I_3 \times \{(0, 0, 0, \ldots)\}$; that is, the product of the first three coordinate spaces of I^∞, imbedded as a hyperplane of I^∞. Let p_1 be a point in $I_1 \times I_2 \times I_3 \times (0, 1/2^4] \times \{(0, 0, 0, \ldots)\}$, where $[0, 1/2^4] \subseteq I_4$, and such that $d(p_1, C_1) < 1/2$. Let D_1 be the join of p_1 to C_1 in $\prod_{i=1}^{4} I_i \times \{(0, 0, 0, \ldots)\} \subseteq \prod_{i=1}^{\infty} I_i$. In general, let p_n be a point in $\prod_{i=1}^{n+2} I_i \times (0, 1/2^{n+3}] \times \{(0, 0, 0, \ldots)\}$, where $(0, 1/2^{n+3}] \subseteq I_{n+3}$, and such that $d(p_n, C_n) < 1/2^n$. Let D_n be the join of p_n to C_n in $\prod_{i=1}^{n+3} I_i \times \{(0, 0, 0, \ldots)\}$.

Let $M = \varphi(X) \cup \bigcup_{i=1}^{\infty} D_i$. M is clearly compact, connected, and metric. We show that M is a two-manifold. Let $x \in M$. If $x \in \bigcup_{i=1}^{\infty} D_i^0$, then x has an open two-cell neighborhood. If $x \in \varphi(X)$, then there exists an open set V containing x such that $\overline{V}$ is a Sierpiński curve in $\varphi(X)$; that is, $\overline{V}$ is the image of such a set in X. We note the proof of Lemma 3.5 shows that $\overline{V}$ may be chosen so that $\operatorname{Bd}\overline{V}$ is a simple closed curve not meeting $\bigcup_{i=1}^{\infty} C_i$. Thus $\overline{V}$ is a Sierpiński curve neighborhood of x. Let $\mathfrak{D} = \{D_i \mid \operatorname{Bd} D_i \subseteq V\}$. Then there exists a "natural" homeomorphism from $\overline{V} \cup \mathfrak{D}^*$ onto a two-cell. Thus x has an open two-cell neighborhood, and M is a two-manifold. Therefore, X is a copy of the Sierpiński curve on a compact two-manifold without boundary.

The theorem follows.

Remark. The author does not know examples of one dimensional l-s-h continua with non-empty boundary, except for an arc.

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Some results on fixed points — IV

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The notions of normal structure and diminishing orbital diameters have been used by Belluce and Kirk ([1], [2], [3]) to study the existence of fixed points of nonexpansive mappings. In this paper we fix our attention on mappings of another type defined as follows: Let (E, ϱ) be a metric space and let T be a mapping of E into itself such that

$$\varrho(Tx, Ty) \leqslant \tfrac{1}{2}\{\varrho(x, Tx)+\varrho(y, Ty)\} \qquad x, y \in E.$$

Mappings T of the above type will be referred to as having property A over E in this paper. Such mappings have been discussed in [5], [6], [7] and [8], dealing with fixed point and other allied problems. The author would like to mention in this connection that the referee of the present paper has suggested the name "semi-nonexpanding mappings" for mappings of this type.

Here we obtain some fixed point theorems for mappings having property A by using certain additional hypotheses. Then we compare the notions of diminishing orbital diameters [1], normal structure ([3], [4]) and property B (defined below). If $a \in E$ then the sequence of iterates of a by T will be written as $\{T^n a\}$ or $\{a_n\}$, $T^0 a = a$.

Before going into the theorems, we recollect some known definitions.

Definition 1 ([1]). A mapping T of a metric space (E, ϱ) into itself is called *nonexpansive* if

$$\varrho(Tx, Ty) \leqslant \varrho(x, y) \quad \text{ for each } x, y \in E.$$

Definition 2 ([1]). For a subset A of a metric space (E, ϱ) let $\delta(A) = \sup \varrho(x, y)$, $x, y \in A$, denote the diameter of A and let T be a mapping of E into itself. Let $\{T^n x\}$ denote the sequence of iterates of $x \in E$ and let $O(T^r x) = \{T^r x, T^{r+1}x, \ldots\}$, $r = 0, 1, \ldots$, $T^0 x = x$. If $r(x) = \lim_n \delta\big(O(T^n x)\big) < \delta\big(O(x)\big)$ at a point $x \in A$ where $\delta\big(O(x)\big) > 0$, then we say that T has a *diminishing orbital diameter at* x. If T has a diminishing orbital diameter for every $x \in E$, then we say that T has a *diminishing orbital diameters over* E.