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# Concerning first countable spaces*


by

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By a development for a topological space $S$ (all spaces are to be $T_1$) is meant a sequence $G_1, G_2, \ldots$ of open covers of $S$ such that for each point $p$ of $S$ and each open set $D$ containing $p$, there is a positive integer $n$ such that each element of $G_n$ containing $p$ is contained in $D$. The statement that the development $G = (G_1, G_2, \ldots)$ for the space $S$ satisfies Axiom C at the point $p$ of $S$ means that if $D$ is an open set containing $p$ then there exists a positive integer $n$ such that each element of $G_n$ intersecting an element of $G_n$ which contains $p$ is contained in $D$. A regular developable space is a Moore space. A Moore space $S$ is metrizable if and only if it has a development which satisfies Axiom C at each point of $S$ [14]. There has been considerable work done in [18], [6], [7], [15], and [17] concerning Axiom C and the existence of dense metrizable subspaces in Moore spaces.

In part I the author establishes necessary and sufficient conditions under which first countable spaces have dense developable and dense metrizable subspaces. The statement that the subset $M$ of the first countable space $S$ is developable ($C$-developable) in $S$ means there is a sequence $G_1, G_2, \ldots$ of open covers of $S$ such that (1) for each point $x$ of $M$ and open set $D$ containing $x$ there is a positive integer $n$ such that each element of $G_n$ containing $x$ (each element of $G_n$ intersecting an element of $G_n$ containing $x$) is contained in $D$ and (2) for each point $p$ of $S$ there exists a non-increasing sequence $g_1(p), g_2(p), \ldots$ of open sets in $S$ forming a local base at $p$ such that for each positive integer $i$, $g_i(p)$ is an element of $G_i$.

A first countable stratifiable space (a Nagata space [4]) is a $\sigma$-space (a regular space with a $\sigma$-locally finite network) [10]. A first countable


* The work on this paper was done under the direction of Ben Fitzpatrick, Jr. at Auburn University. The author would like to acknowledge the many helpful suggestions of the referee. Also, the author has recently learned that the fact that each Nagata space has a dense metrizable subspace has been independently discovered by U. J. Christian at the University of Houston.

$\sigma$-space is a first countable semi-stratifiable space (equivalently a semi-metric space) [5]. The following theorems are established: (i) Each semi-metric space has a dense developable subspace. (ii) Each first countable $\sigma$-space $S$ has a dense subset which is developable in $S$ (hence a Moore space). (iii) Each Nagata space $S$ has a dense subset which is $C$-developable in $S$ (hence metrizable).

In part II a question raised by Heath in [9] is answered. In [8] Heath gave an example of a paracompact semi-metric space which is not a Nagata space. In [9] he asked for a necessary and sufficient condition for a semi-metric space to be a Nagata space. Lutzer in [12] gave an answer to this question with the concept of $k$-semi-stratifiable spaces. He showed that a first countable semi-stratifiable space is stratifiable if and only if it is $k$-semi-stratifiable. However, each $k$-semi-stratifiable space is a semi-stratifiable space. Borges in [2] defined $w\Delta$-spaces which have proved extremely useful ([2], [3], and [11]) in exploring the relationships between $M$-spaces, semi-stratifiable spaces, and Moore spaces. The author defines a more general class of spaces called $w\delta$-spaces and proves that a paracompact semi-metric space is a Nagata space if and only if it is a $w\delta$-space.

I. Notation. (i) The letters $i$, $j$, $m$, $n$ will be used exlusively to denote variables with integer values. (ii) $\mathrm{st}(x, G)$, where $x$ is a point and $G$ is a collection of point sets, will denote $\{g \text{ in } G \mid x \text{ in } g\}^*$. (iii) If $M$ is a subset of the space $S$, $\mathrm{CL}(M)$ will denote the closure of $M$ in $S$.

DEFINITIONS. (i) A collection $U$ of subsets of the space $S$ is *discrete* provided that for each point $p$ in $S$ there exists an open set containing $p$ which intersects at most one element of $U$. (ii) A subset $K$ of the space $S$ is *discrete* provided that no point of $S$ is a limit point of $K$. (iii) A subset $K$ (subspace $K$) of the space $S$ is *$\sigma$-discrete* provided that $K$ is the union of countably many point sets each of which has no limit point in $S$ (in $K$).

Remark 1.1. Consider the space $X$ such that the points of $X$ are the points of the real line and define a basis $B$ for $X$ such that (1) if $x$ is irrational then $\{x\}$ is in $B$ and (2) if $x$ is rational and $i$ is a positive integer then the segment (with respect to the topology of the real line) $(x-1/i, x+1/i)$ is in $B$. $X$ is a regular first countable space such that $I = \{x \text{ in } X \mid x \text{ is irrational}\}$ is a dense metrizable subspace of $X$ but no dense subset of $X$ is developable in $X$. In [17] the author gave an example of a Moore space $S$ which has a dense metrizable subset but for which there exists no development for $S$ satisfying Axiom C at each point of a dense subset. It follows that this space $S$ has a dense metrizable subset which is developable in $S$ but there exists no dense subset of $S$ which is $C$-developable in $S$.



LEMMA 1.2. *If $K$ is a discrete subset of the first countable space $S$ then $K$ is developable in $S$.*

Proof. For each point $p$ of $S$, let $g_1(p), g_2(p), \ldots$ be a non-increasing sequence of open sets in $S$ which forms a local base at $p$ and which is such that $g_1(p)$ contains no point of $K$ other than (possibly) $p$. It is easy to verify that $G_1, G_2, \ldots$, where for each $i$, $G_i = \{g_i(p) \mid p \text{ in } S\}$, is a sequence of open covers of $S$ with the desired properties.

LEMMA 1.3. *If $K$ is a subset of the regular first countable space $S$ and there exists a discrete collection $U$ of open sets in $S$ such that $U$ covers $K$ and each element of $U$ contains only one point of $K$, then $K$ is $C$-developable in $S$.*

Proof. For each points $p$ of $S$, let $g_1(p), g_2(p), \ldots$ be a non-increasing sequence of open sets in $S$ which forms a local base at $p$. Consider the following defining process: (1) For each point $p$ in $S$, let $g_p^1 = g_i(p)$ for some $i \geq 1$ such that: if $p$ is in $K$, $\mathrm{CL}(g_i(p))$ is contained in an element of $U$; if $p$ is in $S-K$ and $p$ is in $\mathrm{CL}(g_x^1)$ for some $x$ in $K$, $g_i(p)$ is contained in an element of $U$ and $g_i(p) \cap K = \emptyset$; if $p$ is not in $\mathrm{CL}(g_x^1)$ for some $x$ in $K$, $g_i(p) \cap \{g_x^1 \mid x \text{ in } K\}^* = \emptyset$. (2) For each point $p$ in $S$, let $g_p^2 = g_i(p)$ for some $i \geq 2$ such that: if $p$ is in $K$, $\mathrm{CL}(g_i(p)) \subset g_p^1$; if $p$ is in $S-K$ and $p$ is in $\mathrm{CL}(g_x^2)$ for some $x$ in $K$, $g_i(p) \subset g_x^1$ and $g_i(p) \subset g_p^1$; if $p$ is not in $\mathrm{CL}(g_x^2)$ for some $x$ in $K$, $g_i(p) \subset g_p^1$ and $g_i(p) \cap \{g_x^2 \mid x \text{ in } K\}^* = \emptyset$. Continue this process such that if $n > 2$ and $p$ is a point of $S$, then $g_p^n = g_i(p)$ for some $i \geq n$ such that: if $p$ is in $K$, $\mathrm{CL}(g_i(p)) \subset g_p^{n-1}$; if $p$ is in $S-K$ and $p$ is in $\mathrm{CL}(g_x^n)$ for some $x$ in $K$, $g_i(p) \subset g_x^{n-1}$ and $g_i(p) \subset g_p^{n-1}$; if $p$ is not in $\mathrm{CL}(g_x^n)$ for some $x$ in $K$, $g_i(p) \subset g_p^{n-1}$ and $g_i(p) \cap \{g_x^n \mid x \text{ in } K\}^* = \emptyset$.

For each $i$, let $G_i = \{g_p^i \mid p \text{ in } S\}$. From the construction it follows that for each point $p$ in $S$, $g_p^1, g_p^2, \ldots$ forms a non-increasing local base at $p$. Now suppose that $p$ is a point of $K$ and $D$ is an open set in $S$ containing $p$. There exists an $n$ such that $g_n(p) \subset D$. Consider $G_{n+1}$. If $p$ is in $g_x^{n+1}$ for some $x$ in $S$, then $x$ is $p$. And if $g_p^{n+1} \cap g_x^{n+1} \neq \emptyset$ for some $x$ in $S$, then it follows that $g_x^{n+1} \subset g_p^n$. Thus the sequence of open covers $G_1, G_2, \ldots$ has the desired properties.

LEMMA 1.4. *If $K = \bigcup_{i=1}^{\infty} K_i$ is a subset of the first countable space $S$ and for each $i$, $K_i$ is developable in $S$ ($C$-developable in $S$), then $K$ is developable in $S$ ($C$-developable in $S$).*

Proof. For each $i$, let $G_1^i, G_2^i, \ldots$ be the sequence of open covers of $S$ required for $K_i$ to be developable in $S$ ($C$-developable in $S$). For each $n$, let $G_n$ be an open cover of $S$ which refines $G_j^i$ where each of $i$ and $j$ is $\leq n$. It follows that $G_1, G_2, \ldots$ has the required properties for $K$ to be developable in $S$ ($C$-developable in $S$).

THEOREM 1.5. *In a first countable space $S$ the following are equivalent:*

(i) *$S$ has a dense $\sigma$-discrete subspace.*

(ii) *$S$ has a dense developable subspace.*

Proof. Suppose (i) is true. It follows from Lemma 1.2 and Lemma 1.4 that each $\sigma$-discrete first countable space is developable.

Suppose (ii) is true. It follows from Theorem 18 (Chapter I) of [13], that each developable space has a dense $\sigma$-discrete subset.

THEOREM 1.6. *In a first countable space $S$ the following are equivalent:*

(i) *$S$ has a dense $\sigma$-discrete subset.*

(ii) *$S$ has a dense subset which is developable in $S$.*

Proof. Suppose (i) is true. Then (ii) follows from Lemma 1.2 and Lemma 1.4.

Suppose (ii) is true. Let $G_1, G_2, \ldots$ be a sequence of open covers of $S$ as required for the subset $K$ to be developable in $S$. By an argument similar to the one given for Theorem 18 (Chapter I) of [13], it follows that for each $i$, there exists a subset $K_i$ of $K$ such that $K_i$ is discrete in $S$ and each point of $K$ is contained in an element of $G_i$ which intersects $K_i$. Thus $K' = \bigcup_{i=1}^{\infty} K_i$ has the desired properties.

Remark 1.7. A space $S$ is screenable provided that for each open cover $G$ of $S$ there is an open cover $H = \bigcup_{i=1}^{\infty} H_i$ of $S$ which refines $G$ such that for each $i$, $H_i$ is a collection of mutually exclusive open sets. A space $S$ has property $J$ provided that if $D$ is an open set in $S$ then there exists a sequence $d_1, d_2, \ldots$ of open sets in $S$ such that $D \subset \mathrm{CL}(\bigcup_{i=1}^{\infty} d_i)$ and for each $i$, $\mathrm{CL}(d_i) \subset D$. In [17] the author proved that in a Moore space $S$ there exists a development for $S$ which satisfies Axiom C at each point of a dense subset if and only if $S$ has property $J$ and a dense screenable subspace. Theorem 1.8 is a generalization of this result.

THEOREM 1.8. *In a regular first countable space $S$ the following are equivalent:*

(i) *$S$ has property $J$ and a dense metrizable subspace.*

(ii) *$S$ has property $J$ and a dense $\sigma$-discrete, screenable subspace.*

(iii) *$S$ has a dense subset which is $C$-developable in $S$.*

Proof. Suppose that (i) is true. Let $M$ be a dense metrizable subspace of $S$. Then $M$ is screenable and has a dense $\sigma$-discrete subset. Thus (ii) is true.

Suppose that (ii) is true. Let $K$ be a $\sigma$-discrete, screenable subspace of $S$. It follows that $K = \bigcup_{i=1}^{\infty} K_i$ where for each $i$, $K_i$ is discrete in $K$ and there is a collection $U_i$ of mutually exclusive open sets (with respect to $K$) covering $K_i$ such that each element of $U_i$ contains only one point of $K_i$. Since $K$ is dense in $S$, there exists such a collection $U'_i$ of open sets in $S$. For each point $p$ of $K_i$, let $g_1(p), g_2(p), \ldots$ be a non-increasing sequence of open sets in $S$ which forms a local base at $p$ such that $g_1(p)$ is contained in the element of $U'_i$ containing $p$. Note that for each $n$, $H_n = \{g_n(p) |\ p \text{ in } K_i\}$ is a collection of mutually exclusive open sets. Now for each $n$, there exists a sequence $d_1^n, d_2^n, \ldots$ of open sets in $S$ such that $H_n^* \subset \mathrm{CL}(\bigcup_{j=1}^{\infty} d_j^n)$ and for each $j$, $\mathrm{CL}(d_j^n) \subset H_n^*$. For each $n$ and each $j$, let $V_j^n = \{d_j^n \cap h |\ h \text{ in } H_n\}$. It follows that $V_j^n$ is a discrete collection of open sets in $S$. Let $V = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} V_j^n$ and let $X_i$ be a set containing one point from each element of $V$. $K_i$ is contained in $\mathrm{CL}(X_i)$ and by Lemma 1.3 and Lemma 1.4, $X_i$ is $C$-developable in $S$. Thus $X = \bigcup_{i=1}^{\infty} X_i$ is dense in $S$ and by Lemma 1.4 is $C$-developable in $S$. Thus (iii) is true.

Suppose that (iii) is true. Let $K$ be a dense subset of $S$ which is $C$-developable in $S$. It follows that $K$, regarded as spaces, is metrizable. $S$ also has property $J$. For suppose that $D$ is an open set in $S$. Then $K \cap D$ is a dense subset of $D$. Denote by $G_1, G_2, \ldots$ the sequence of open covers of $S$ required for $K$ to be $C$-developable in $S$. For each $i$, let $K_i$ denote the set of all $p$ in $K$ such that each element of $G_i$ intersecting an element of $G_i$ containing $p$ is contained in $D$. And for each $i$, let $d_i = \{g \text{ in } G_i |\ g \cap K_i \neq \emptyset\}^*$. It follows that the sequence $d_1, d_2, \ldots$ has the desired properties. This completes the proof.

THEOREM 1.9 (Heath [9]). *The space $S$ is a semi-metric space if and only if for each point $x$ of $S$ there exists a non-increasing sequence $g_1(x)$, $g_2(x), \ldots$ of open sets in $S$ such that (1) $g_1(x), g_2(x), \ldots$ is a local base at $x$ and (2) if $y$ is a point of $S$ and $x_1, x_2, \ldots$ is a point sequence in $S$ such that for each $n$, $y$ is in $g_n(x_n)$, then $x_1, x_2, \ldots$ converges to $y$.*

THEOREM 1.10. *If $S$ is a semi-metric space, then $S$ has a dense developable subspace.*

Proof. By Theorem 1.5, it is sufficient to show that $S$ has a dense $\sigma$-discrete subspace.

For each point $p$ of $S$, let $g_1(p), g_2(p), \ldots$ be a sequence of open sets in $S$ as in Theorem 1.9. For each $j$, let $G_j = \{g_j(p) |\ p \text{ in } S\}$. Denote by $\Omega$ a well-ordering of the elements of $S$. For each $j$, let $K_j$ be the subset of $S$ such that: (1) The first element of $K_j$ is the first element of $S$ with respect to $\Omega$. (2) If $I$ is an initial segment of $K_j$, then the first element $p$ of $K_j - I$ is the first element of $S$ with respect to $\Omega$ such that $g_j(p) \cap I = \emptyset$ and $p$ is not in $g_j(q)$ for $q$ in $I$. (3) If $K'_j$ is a subset of $S$ having properties (1) and (2) then either $K'_j$ is $K_j$ or $K'_j$ is an initial segment of $K_j$.

It follows that $K = \bigcup_{i=1}^{\infty} K_i$ is dense in $S$. For suppose that $p$ is a point of $S$ and $D$ is an open set containing $p$. There exists an $n$ such that if $i \geqslant n$, $g_i(p)$ is contained in $D$. Thus suppose that for $i \geqslant n$, $g_i(p) \cap K = \emptyset$. Then for each $i \geqslant n$, $p$ is in $g_i(q_i)$ for some $q_i$ in $K_i$. If this were not true, $p$ would be in $K_j$ for some $j$. But by Theorem 1.9, the sequence $q_n, q_{n+1}, \ldots$ converges to $p$ and $p$ is a limit point of $K$. This is a contradiction. Thus $D$ contains a point of $K$.

Now, let $X_1 = K_1$ and for each $i > 1$, let $X_i = K_i - \big(\mathrm{CL}(\bigcup_{j=1}^{i-1} X_j) \cap K_i\big)$. Since $K$ is dense in $S$ it follows that $X = \bigcup_{i=1}^{\infty} X_i$ is dense in $S$. Consider $X_i$ for each $i$. By the definition of $K_i$, no point of $X_i$ is a limit point of $X_i$. And by the definition of $X_i$, no point of $\bigcup_{j=i+1}^{\infty} X_j$ is a limit point of $X_i$. Thus if $X_i$ has a limit point $q$ in $X$, $q$ must be in $\bigcup_{j=1}^{i-1} X_j$. But for each point $p$ in $X_i$, there exists an $n$ such that $p$ is not in $g_n(q)$ for $q$ in $\bigcup_{j=1}^{i-1} X_j$. If this were not true, the sequence $q_1, q_2, \ldots$, where for each $n$, $q_n$ is in $\bigcup_{j=1}^{i-1} X_j$ and $p$ is in $g_n(q_n)$, would converge to $p$ and hence $p$ would be a limit point of $\bigcup_{j=1}^{i-1} X_j$. Thus for each $n$, let $X_i^n = \{p \text{ in } X_i \mid p \text{ is not in } g_n(q) \text{ for } q \text{ in } \bigcup_{j=1}^{i-1} X_j\}$. Note for each $n$, $X_i^n$ has no limit point in $X$.

Thus $X = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} X_i^n$ is a $\sigma$-discrete subspace of $S$ which is dense in $S$. This completes the proof.

THEOREM 1.11. *If $S$ is a first countable $\sigma$-space then $S$ has a dense subset which is developable in $S$.*

Proof. Let $N = \bigcup_{i=1}^{\infty} N_i$ be a network for $S$ such that for each $i$, $N_i$ is locally finite. For each $i$, let $K_i$ be a point set containing one point from each element of $N_i$. It follows that for each $i$, $K_i$ is discrete in $S$ and $K = \bigcup_{i=1}^{\infty} K_i$ is dense in $S$. Thus by Lemma 1.2 and Lemma 1.4, $K$ is a dense subset of $S$ which is developable in $S$.

THEOREM 1.12. *If $S$ is a Nagata space then $S$ has a dense subset which is $C$-developable in $S$ and which, regarded as space, is metrizable.*

Proof. Heath has shown in [10] that each Nagata space is a $\sigma$-space. Thus as in the proof of Theorem 1.11, let $K = \bigcup_{i=1}^{\infty} K_i$ be a dense subset of $S$ such that for each $i$, $K_i$ is discrete in $S$. Since each Nagata space is paracompact [4] (hence collectionwise normal), for each $i$, there exists a discrete collection $U_i$ of open sets in $S$ covering $K_i$ such that each element of $U_i$ contains only one point of $K_i$. Thus by Lemma 1.3 and Lemma 1.4, $K$ is a dense subset of $S$ which is $C$-developable in $S$. And since $K$ (regarded as space) is a Moore space with a development which satisfies Axiom C at each point of $K$, then $K$ is metrizable [14].



**II.** THEOREM 2.1 (Heath [9]). *The space $S$ is a Nagata space if and only if for each point $p$ of $S$ there exists a sequence $g_1(p), g_2(p), \ldots$ of open sets in $S$ such that (1) $g_1(p), g_2(p), \ldots$ is a non-increasing local base at $p$ and (2) if $x$ is a point of $S$ and $R$ is an open set containing $x$ then there exists an $n$ such that if $g_n(x) \cap g_n(q) \neq \emptyset$ for some point $q$ of $S$ then $q$ is in $R$.*

DEFINITION 2.2 (Borges [2]). The space $S$ is a *$w\Delta$-space* if and only if there is a sequence $G_1, G_2, \ldots$ of open covers of $S$ such that if $x$ is a point of $S$ and for each $i$, $p_i$ is in $\mathrm{st}(x, G_i)$, then the sequence $p_1, p_2, \ldots$ has a cluster point.

DEFINITION 2.3 (Hodel [11]). The space $S$ has a *$G_\delta^*$-diagonal* if and only if there is a sequence $G_1, G_2, \ldots$ of open covers of $S$ such that for any two distinct points $x$ and $y$ of $S$, there is an $n$ such that $y$ is not in $\mathrm{CL}(\mathrm{st}(x, G_n))$.

THEOREM 2.4 (Hodel [11]). *The following are equivalent:*

(i) *$S$ is a Moore space.*

(ii) *$S$ is a regular $w\Delta$-space with a $G_\delta^*$-diagonal.*

DEFINITION 2.5. The space $S$ is a *$w\delta$-space* if and only if for each point $p$ of $S$ there is a sequence $g_1(p), g_2(p), \ldots$ of open sets in $S$ each term of which contains $p$ such that if $x$ is a point of $S$ and for each $i$, $p_i$ is in $\mathrm{st}(x, G_i)$ where $G_i = \{g_i(p) \mid p \text{ in } S\}$, and $g_i(p_i) \cap g_i(x) \neq \emptyset$, then the sequence $p_1, p_2, \ldots$ has a cluster point.

THEOREM 2.6 (Compare with Theorem 2.1). *If $S$ is a $w\delta$-space with a $G_\delta^*$-diagonal then for each point $p$ in $S$ there exists a non-increasing sequence $g_1(p), g_2(p), \ldots$ of open sets in $S$ such that (1) $g_1(p), g_2(p), \ldots$ is a local base at $p$ and (2) if $x$ is a point of $S$ and $R$ is an open set containing $x$ then there exists an $n$ such that if $q$ is in $\mathrm{st}(x, G_n)$, where $G_n = \{g_n(p) \mid p \text{ in } S\}$, and $g_n(q) \cap g_n(x) \neq \emptyset$, then $q$ is in $R$.*

Proof. For each point $p$ in $S$, let $r_1(p), r_2(p), \ldots$ be a sequence of open sets as in the definition of a $w\delta$-space. Let $H_1, H_2, \ldots$ be a sequence of open covers of $S$ as in the definition of a $G_\delta^*$-diagonal. For each point $p$ in $S$, let $g_1(p), g_2(p), \ldots$ be a sequence of open sets in $S$ each term of which contains $p$ such that for each $i$, (1) $g_i(p) \subset r_i(p)$, (2) $g_i(p)$ is contained in an element of $H_i$, and (3) $g_i(p) \supset g_{i+1}(p)$.

Now suppose that there exists a point $x$ of $S$ contained in the open set $R$ such that for each $i$, $q_i$ is in $\mathrm{st}(x, G_i)$, where $G_i = \{g_i(p) \mid p \text{ in } S\}$, $g_i(q_i) \cap g_i(x) \neq \emptyset$ and $q_i$ is not in $R$. The sequence $q_1, q_2, \ldots$ has a cluster point $y$ since $S$ is a $w\delta$-space. $y$ cannot be $x$, thus since $S$ has *a* $G_\delta^*$-diagonal there exists an $n$ such that $y$ is not in $\mathrm{CL}(\mathrm{st}(x, G_n))$. Thus there exists an open set $u$ in $S$ such that $u$ contains $y$ and does not intersect $\mathrm{st}(x, G_n)$. But $\mathrm{st}(x, G_i) \supset \mathrm{st}(x, G_{i+1})$, for each $i$. Which means that $u$ does not contain $q_i$ for $i \geqslant n$. Therefore $y$ is not a cluster point of $q_1, q_2, \ldots$ and this is a contradiction. It follows that part (2) in the statement of the theorem is satisfied.

For each point $p$ in $S$, the sequence $g_1(p), g_2(p), \ldots$ is a non-increasing sequence of open sets. It also forms a local base at $p$. For suppose $p$ is contained in the open set $R$. From the above there exists an $n$ such that if $x$ is in $\mathrm{st}(p, G_n)$ and $g_n(x) \cap g_n(p) \neq \emptyset$, then $x$ is in $R$. If $x$ is in $g_n(p)$ then $x$ is in $\mathrm{st}(p, G_n)$ and $g_n(x) \cap g_n(p) \neq \emptyset$. Thus $g_n(p) \subset R$. This completes the proof.

COROLLARY 2.7. *If $S$ is a $w\delta$-space with a $G_\delta^*$-diagonal then $S$ is a semi-metric space.*

Proof. Use Theorem 2.6 to satisfy Theorem 1.9.

THEOREM 2.8. *The following are equivalent:*

(i) *$S$ is a Nagata space.*

(ii) *$S$ is a paracompact $w\delta$-space with a $G_\delta^*$-diagonal.*

Proof. Suppose that (ii) is true. For each point $p$ of $S$, let $g_1(p), g_2(p), \ldots$ be a non-increasing sequence of open sets as in Theorem 2.6. For each $i$, let $G_i = \{g_i(p) \mid p \text{ in } S\}$. Since $S$ is paracompact, for each $i$, there exists an open cover $H_i$ of $S$ such that if $p$ is a point of $S$ then $\mathrm{st}(p, H_i)$ is contained in an element of $G_i$. For each point $p$ of $S$, let $g_1'(p), g_2'(p), \ldots$ be a sequence of open sets in $S$ each term of which contains $p$ such that for each $i$, (1) $g_i'(p) \subset g_i(p)$, (2) $g_i'(p)$ is contained in some element of $H_i$, and (3) $g_{i+1}'(p) \subset g_i'(p)$. Now suppose that $x$ is a point of $S$ and $R$ is an open set containing $x$. There exists an $n$ such that if $p$ is in $\mathrm{st}(x, G_n)$ and $g_n(p) \cap g_n(x) \neq \emptyset$, then $p$ is in $R$. But if $g_n'(p) \cap g_n'(x) \neq \emptyset$, then $p$ is in $\mathrm{st}(x, G_n)$ and $g_n(p) \cap g_n(x) \neq \emptyset$. Thus if $g_n'(p) \cap g_n'(x) \neq \emptyset$ for some point $p$ in $S$, then $p$ is in $R$. Thus Theorem 2.1 is satisfied and $S$ is a Nagata space.

Suppose that (i) is true. Then $S$ is stratifiable and therefore paracompact [4]. $S$ also has a $G_\delta^*$-diagonal since Hodel proved in [11] that each regular semi-stratifiable space has a $G_\delta^*$-diagonal. To see that $S$ is a $w\delta$-space, for each point $p$ of $S$ let $g_1(p), g_2(p), \ldots$ be a sequence of open sets as in Theorem 2.1. Without loss of generality, require that $g_{i+1}(p) \subset g_i(p)$ for each $i$. It then follows that if $x$ is a point of $S$ and $R$ is an open set containing $x$, then there exists an $m$ such that if $n \geqslant m$ and $g_n(x) \cap g_n(q) \neq \emptyset$ for some point $q$ in $S$ then $q$ is in $R$. Thus suppose $x$ is a point of $S$ and there exists a point sequence $p_1, p_2, \ldots$ in $S$ such that for each $i$, $p_i$ is in $\mathrm{st}(x, G_i)$, where $G_i = \{g_i(p) \mid p \text{ in } S\}$, and $g_i(p_i) \cap g_i(x) \neq \emptyset$. Then $x$ is a cluster point of $p_1, p_2, \ldots$ For if $R$ is an open set containing $x$, there exists an $m$ such that if $n \geqslant m$ then $p_n$ is in $R$. Thus $S$ is a $w\delta$-space.



COROLLARY 2.9. *The semi-metric space $S$ is a Nagata space if and only if it is a paracompact $w\delta$-space.*

Remark 2.10. A $w\Delta$-space is a $w\delta$-space. Thus it follows from [2] and Theorem 2.8 that the class of $w\delta$-spaces contains the class of Moore spaces, $M$-spaces, and first countable stratifiable spaces.

QUESTION 2.11. Is a regular $w\delta$-space with a $G_\delta^*$-diagonal a $\sigma$-space? Or even must such a space $S$ have a dense subset which is developable in $S$?

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*Reçu par la Rédaction le 12. 8. 1970*