On convex metric spaces VI

by

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§ 1. Introduction. The present paper is a continuation of some earlier studies of the following two problems:

I. Is every n-dimensional G-space a manifold?

II. Is every n-dimensional compact SC-WR-space a cell?

The first problem has been raised by H. Buseman ([4], p. 403). The second, communicated to us [7] by K. Borsuk, is a modification of an earlier one, raised by R. H. Bing [1].

Both problems, I and II, were solved affirmatively for $n \leqslant 3$: the first in [4] for $n \leqslant 2$ and in [5] for $n = 3$, the second in [7] for $n \leqslant 2$ and in [10] for $n = 3$.

In general, however, only some partial solutions are known (see § 3, [4] and [8]).

In [8] problem II has been solved positively by assuming that the space has a so-called CT-property. The aim of the present paper, announced earlier in [9], is to investigate that property in relation to the two problems I and II.

We show that the CT-property assumed locally at a point yields the positive solution of problems I and II (§ 4, Corollaries I and II).

Under the CT-property assumed locally at each point we obtain some strong local properties for G-spaces and SC-WR-spaces (§ 5, Corollaries I and II).

§ 6 is a kind of introduction to § 7, although Theorem 3 can perhaps be of some interest for itself.

In § 7 it is shown that G-spaces and SC-WR-spaces possess stronger local properties than those of § 5 and finally in § 8 we state an equivalent form of the local CT-property in terms of elementary geometry (Corollaries I and II).

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§ 2. Definitions and notation. The notions and notation not defined in the paper are derived from [7] and [8]. Let us recall some of them.

Let $\langle X, \varrho\rangle$ be a metric space with a metric ϱ. The set $\bar{B}(p, \varepsilon) = \{x: \varrho(p, x) \leqslant \varepsilon\}$ is called a *closed metric ball*, $B(p, \varepsilon) = \{x: \varrho(p, x) < \varepsilon\}$ is a *metric ball*; xzy means that z lies between x and y, i.e. that $\varrho(x, z) + \varrho(z, y) = \varrho(x, y)$. A triple $\{a, b, c\}$ is linear if one of points a, b, c lies between the other two. A segment with end points x, y is an arc joining x to y, isometric to a Euclidean segment; if unique, it is denoted by $\overline{xy}$. A metric space $\langle X, \varrho\rangle$ is *convex*, and then ϱ is *convex*, if every pair x, y can be joined by a segment. Segment $\overline{xz}$ is a prolongation of $\overline{xy}$ in $A \subset X$ (through y) if $y, z \in A$ and xyz. A prolongation is proper if $y \neq z$. A segment with no proper prolongation in A is called *maximal* in A. A ϱ-cone $C_\varrho(A, v)$ over $A \subset X$ with vertex $v \in X$ is the union of all segments $\overline{va}$, where $a \in A$. A subset $B_\varrho(A, v)$ of all points $a \in A$ such that $\overline{va}$ is maximal in A is a base of $C_\varrho(A, v)$.

We adopt the weakest form of local convexity [3]; namely, a metric ϱ is said to be *locally SC* (*strongly convex*) at a point $p \in X$ if there exists a neighbourhood U of p such that for each pair of points $x, y \in U$ there exists a unique segment $\overline{xy}$ (not necessarily in U).

A metric ϱ has the *CT-property* (*convex triangle property*) if a ϱ-cone $C_\varrho(\overline{xy}, v)$ is convex for every $x, y, v \in X$; a metric ϱ has the CT-property locally at a point p if there exists a neighbourhood U of p such that CT holds for every $x, y, v \in U$.

A metric ϱ is *WR* (*without ramifications*) if pqr, pqs and $p \neq q$ implies prs or psr. This property will be localized in a standard way.

A metric ϱ has the *local prolongation property* at p ([4], p. 33) if there exists a neighbourhood U of p such that for every $x, y \in U$ the segment $\overline{xy}$ has a prolongation in X.

If a property holds locally at every point $p \in X$, then we say that X has that *property locally*.

A metric space is a *G-space* in the sense of H. Buseman [4] if it is finitely compact, convex, WR and has the local prolongation property.

§ 3. Some characterizations. Following R. H. Bing and K. Borsuk [2], we say that a space X is *locally homogeneous* if for every two points $x_1, x_2 \in X$ there exists a homeomorphism h mapping a neighbourhood U of x_1 into X and satisfying the requirement $h(x_1) = x_2$.

It is known ([2], p. 106) that an n-dimensional complete, connected, locally contractible space is a *manifold* if and only if it is locally homogeneous and contains topologically a Euclidean n-ball.

It is also proved ([4], p. 49) that any G-space is locally homogeneous and locally SC (ibid. p. 39), whence it is locally contractible.

Putting together those results we conclude that

3.1. *An n-dimensional G-space is a manifold if and only if it contains an n-cell.*

Similar results were obtained in [8] for SC-WR-compact spaces. In particular,

3.2. *An n-dimensional SC-WR compact space is a cell if and only if it contains a convex n-cell.*

§ 4. Local CT-property at a point. Each 2-dimensional compact SC-WR-space has the CT-property and so it is a 2-cell [8]. In a 2-dimensional G-space the CT-property holds locally [4], p. 81 and [8], 11.8, and so such a space is a 2-manifold. A common generalization of these two results will follow from

THEOREM 1. *Let $\langle X, \varrho\rangle$ be a metric space of finite dimension. If $\bar{B}(p, r)$ is a compact metric ball in which a metric ϱ is SC-WR-CT, then for every $0 < \varepsilon \leqslant r$ metric ball $B(p, \varepsilon)$ contains a convex cell Q such that $\dim Q = \dim \bar{B}(p, r)$.*

The proof will follow from the six lemmas below:

4.1. *If A is closed and $C_\varrho(A, v) \subset B(p, r)$, then $C_\varrho(A, v)$ is closed. If, moreover, A is convex, then $C_\varrho(A, v)$ is convex.*

For the proof see [8], 4.3 and 10.3.

Recall that the Bcone M denotes the space obtained from the Cartesian product $M \times [0, 1]$ by the identification of the set $M \times 1$ to one point.

4.2. *If $B_\varrho(A, v)$ is closed, $C_\varrho(A, v) \subset B(p, r)$ and $v \notin B_\varrho(A, v)$ then $C_\varrho(A, v)$ is homeomorphic to the Bcone $B_\varrho(A, v)$.*

The proof runs as in [8], 6.1.

The following two lemmas will be used in the inductive construction of a convex cell Q.

4.3. *If $C_\varrho(\overline{ab}, v) \subset B(p, r)$ and a triple $\{a, b, v\}$ is not linear, then $C_\varrho(\overline{ab}, v)$ is a convex disk.*

The proof follows from [8], 6.4 and from 4.1 and 4.2 above.

4.4. *If Q_k is a convex k-cell, $k \geqslant 2$, $v \notin Q_k$, $C_\varrho(Q_k, v) \subset B(p, r)$ and $\varrho(v, Q_k) < \varrho(v, \operatorname{Bd} Q_k)$, then $B_\varrho(Q_k, v) = Q_k$ and $C_\varrho(Q_k, v)$ is a convex $(k+1)$-cell.*

Proof. It follows from 4.1 that the ϱ-cone $C = C_\varrho(Q_k, v)$ is closed and convex. Since the set C is contained in $B(p, r)$, we infer that $\langle C, \varrho\rangle$ is a compact SC-WR-CT-space. Thus there exists a point $a \in \operatorname{Int} Q_k$ such that $\overline{va} \cap \operatorname{Bd} Q_k = 0$. Applying 14.2 from [8] to the space C, we get the equality $B_\varrho(Q_k, v) = Q_k$. Now, according to 4.2, the ϱ-cone C is a $(k+1)$-cell.

The following lemma results directly from the triangle inequality:

4.5. *If $0 < 4\eta < r$, $A \subset B(p, \eta)$, and $\varrho(v, A) \leqslant \eta$, then $C_\varrho(A, v) \subset B(p, 4\eta)$.*

The last lemma asserts a kind of homogeneity of $B(p, r)$

4.6. *If* $0 < 4\eta < r$, $q \in B(p, \eta)$, $0 < \eta' \leqslant \eta$, *then* $\dim B(q, \eta') = \dim \bar{B}(p, r)$.

Although $\bar{B}(p, r)$ is not an SC-WR-space, the ϱ-homotopy $H_{p,k}$ defined in [8], § 5 is a homeomorphism for any k and t such that $kt-1 \neq 0$; hence $B(p, \eta)$ contains a homeomorphic copy of $\bar{B}(p, r)$. For the same reason the ϱ-homotopy $H_{q,k}$ defined in the ϱ-cone $C_\varrho(\bar{B}(p, \eta), q)$ is a homeomorphism and by a suitable choice of k and t we obtain a homeomorphic copy of $\bar{B}(p, r)$ in $B(q, \eta')$.

Proof of Theorem 1. We may suppose that $0 < 4\varepsilon \leqslant r$. If $\dim \bar{B}(p, r) = n > 0$, then, in view of 4.6, it suffices to show that $B(p, \varepsilon)$ contains a convex cell Q satisfying the following condition:

(C) if $a \in \operatorname{Int} Q$, then there exists a μ, $0 < \mu < \varrho(a, \operatorname{Bd} Q)$, such that $B(a, \mu) \subset Q$.

We proceed by induction. Put $\eta = \varepsilon/4^{n+1}$ and take a point $q \neq p$ such that $Q_1 = \overline{pq} \subset B(p, \eta)$. If Q_1 does not satisfy (C), we find a point $a \in \operatorname{Int} Q_1$, $0 < \mu < \min(\eta, \varrho(a, \operatorname{Bd} Q_1))$, and a point $v \in B(a, \mu) \backslash Q_1$. Then the triple $\{p, q, v\}$ is not linear, and by 4.3 the ϱ-cone $Q_2 = C_\varrho(\overline{pq}, v)$ is a convex disk. According to 4.5 we have $Q_2 \subset B(p, 4\eta)$. Now suppose that Q_m is a convex m-cell contained in $B(p, 4^{m-1}\cdot\eta)$, $2 \leqslant m \leqslant n$. If Q_m satisfies (C), we take $Q = Q_m$ and the proof is finished; otherwise we find a point $a \in \operatorname{Int} Q_m$, a number μ with $0 < \mu < \min(\eta, \varrho(a, \operatorname{Bd} Q_m))$, and a point $v \in B(a, \mu) \backslash Q_m$. Then, by 4.5, $C_\varrho(Q_m, v) \subset B(p, 4^m\cdot\eta) \subset B(p, \varepsilon)$. From 4.4 we infer that $Q_{m+1} = C_\varrho(Q_m, v)$ is a convex $(m+1)$-cell contained in $B(p, 4^m\cdot\eta)$.

In this way we must come at last to a convex cell $Q \subset B(p, \varepsilon)$ satisfying (C).

From Theorem 1, 3.1 and 3.2 we infer two important corollaries:

COROLLARY I. *Every n-dimensional G-space which has the CT-property locally at a point is a manifold.*

This has been suggested by K. Borsuk and A. Lelek.

COROLLARY II. *Every n-dimensional SC-WR-compact space which has the CT-property locally at a point is a cell.*

This is a generalization of the Main Theorem in [8].

§ 5. Interior points. Every point of a 2-dimensional G-space is an interior point of a sufficiently small disk ([4], p. 51). The same holds for every interior point of a 2-dimensional SC-WR-cell. In fact, in both cases a sufficiently small disk may even have the form of a 2-simplex, i.e. of a ϱ-cone over a segment. In both cases the interior of a disk is also convex.

A generalization of these results will follow from

THEOREM 2. *Let $\langle X, \varrho\rangle$ be a metric space of finite dimension and let $\bar{B}(p, r)$ be a closed metric ball in which a metric ϱ is compact, SC-WR-CT. If for any $x \in \bar{B}(p, r)$ a segment $\overline{xp}$ has a prolongation (through p), then, for every $0 < \varepsilon \leqslant r$, the metric ball $B(p, \varepsilon)$ contains a convex cell Q such that $\dim Q = \dim \bar{B}(p, r)$ and $p \in \operatorname{Int} Q$.*

The proof will be modelled on those of § 4.

If $C_\varrho(A, u)$ and $C_\varrho(A, v)$ are contained in $B(p, r)$ and if $\overline{uv} \cap A \neq 0$, we put

$$C_\varrho(A, u, v) = C_\varrho(A, u) \cup C_\varrho(A, v) .$$

We have

5.1. *If A is closed, then $C_\varrho(A, u, v)$ is closed. If, moreover, A is convex, then $C_\varrho(A, u, v)$ is convex.*

Proof. By 4.1 both ϱ-cones of the union $C_\varrho(A, u, v)$ are closed; hence $C_\varrho(A, u, v)$ is closed. Moreover, if A is convex, both ϱ-cones are convex. So in order to prove the convexity of $C_\varrho(A, u, v)$ it suffices to suppose that $a \in C_\varrho(A, u)$, $b \in C_\varrho(A, v)$, and to prove that $\overline{ab} \subset C_\varrho(A, u, v)$. Let $a_1, b_1 \in A$ and let $a \in \overline{ua_1}$, $b \in \overline{vb_1}$. Consider a ϱ-cone $C_1 = C_\varrho(\overline{uv}, b_1)$. We have $\overline{ub} \subset C_1$ and, denoting by c a point from $\overline{uv} \cap A$, we have $\overline{cb_1} \subset C_1$. By the convexity of A we have $\overline{cb_1} \subset A$. Denote by b_2 a point common to $\overline{cb_1}$ and $\overline{ub}$, the existence of b_2 being obvious (comp. [8], § 11), and consider the ϱ-cone $C_2 = C_\varrho(\overline{ub}, a_1)$. We have

$$C_2 = C_\varrho(\overline{ub_2}, a_1) \cup C_\varrho(\overline{b_2 b}, a_1) = C_\varrho(a_1\overline{b_2}, u) \cup C_\varrho(a_1\overline{b_2}, b) \subset C_\varrho(A, u, v) .$$

Evidently, $a, b \in C_2$, whence $\overline{ab} \subset C_2$, and this ends the proof.

By a dBcone M we denote the space obtained from the Cartesian product $M \times [-1, 1]$ by the identification of the set $M \times 1$ to one point and $M \times -1$ to another one.

5.2. *If $B_\varrho(A, u) = B_\varrho(A, v) = A$, A is compact and convex, $\overline{uv} \cap A \neq 0$ and $u, v \notin A$, then $C_\varrho(A, u, v)$ is homeomorphic to the dBcone A.*

Proof. By 4.2 $C_\varrho(A, u)$ and $C_\varrho(A, v)$ are homeomorphic to the Bcone A; hence it suffices to show that the common part of $C_\varrho(A, u)$ and $C_\varrho(A, v)$ is equal to A. Let $a \in \overline{uv} \cap A$. Evidently, A being the base, $\overline{ua} \cap \overline{va} = a$ and $\overline{uv} \cap A = a$. We shall show that for any $b_1, b_2 \in A$ we have $\overline{ub_1} \cap \overline{vb_2} \subset A$. Consequently, $\overline{ub_1} \cap \overline{vb_2}$ consist of at most one point. For supposing the contrary, we would have a point $c \in (\overline{ub_1} \cap \overline{vb_2}) \backslash A$, whence $b_1 \neq c \neq b_2$. By the CT-property, the segments $\overline{b_1 a}$ and $\overline{cv}$ would have a common point z, and by the convexity of A the point z would belong to A; this is a contradiction, because the segment $\overline{vb_2}$ would meet the base A in two different points b_2 and z.

5.3. *If $p \in \operatorname{Int}\overline{ab}$, $\overline{ab} \cap \overline{uv} = p$, $u \neq p \neq v$ and $C_\varrho(\overline{ab}, u, v) \subset B(p, r)$, then $C_\varrho(\overline{ab}, u, v)$ is a convex disk and p is its interior point.*

Proof. In view of 5.2 it suffices to show that $B_\varrho(\overline{ab}, u) = B_\varrho(\overline{ab}, v) = \overline{ab}$. Observe that neither the triple $\{a, b, u\}$ nor the triple $\{a, b, v\}$ is linear, for otherwise p would be a ramification point. For the same reason the triples $\{u, v, a\}$ and $\{u, v, b\}$ are not linear either. It is known (comp. [7], 5.6) that $B_\varrho(\overline{ab}, v) = \overline{a_1 b_1} \subset \overline{ab}$. If we had $a_1 \neq a$, then vaa_1, aa_1p and $a_1 \neq p$. We would then get a contradiction, because in the convex disk $C_\varrho(\overline{uv}, a)$ a_1 is an interior point (comp. [8], 6.2) and $C_\varrho(\overline{uv}, a)$ is star-like ([8], 9.1), i.e. every segment passing through an interior point meets a boundary of $C_\varrho(\overline{uv}, a)$ in one point at most. In an analogous way we show that $b_1 = b$ and that $B_\varrho(\overline{ab}, u) = \overline{ab}$.

5.4. *If Q_k is a convex k-cell, where $k \geqslant 2$, $p \in \operatorname{Int} Q_k$, $p \in \overline{uv}$, $u, v \notin Q_k$, $\varrho(u, p) < \varrho(u, \operatorname{Bd} Q_k)$, $\varrho(v, p) < \varrho(v, \operatorname{Bd} Q_k)$, then $Q_{k+1} = C_\varrho(Q_k, u, v)$ is a convex $(k+1)$-cell and $p \in \operatorname{Int} Q_{k+1}$.*

Proof. By 4.1, $C_\varrho(Q_k, u)$ is a compact SC-WR-CT-space. Applying 14.2 from [8], we have $B_\varrho(Q_k, u) = Q_k$. Analogously, $B_\varrho(Q_k, v) = Q_k$. From 5.1 and 5.2 we infer that $C_\varrho(Q_k, u, v)$ is a convex $(k+1)$-cell. Evidently, p is an interior point of this cell.

Proof of Theorem 2. Now the proof is similar to that of Theorem 1. Suppose that $0 < 4\varepsilon \leqslant r$ and let $\dim \bar{B}(p, r) = n$. Applying 4.6, we have to show that $B(p, \varepsilon)$ contains a convex cell Q satisfying the following condition:

(C′) $p \in \operatorname{Int} Q$ and there exists a μ, $0 < \mu < \varrho(p, \operatorname{Bd} Q)$, such that $B(p, \mu) \subset Q$.

Put $\eta = \varepsilon/4^{n+1}$ and choose $Q_1 = \overline{ab}$ such that $p \in \operatorname{Int} \overline{ab}$ and $\overline{ab} \subset B(p, \eta)$. The existence of $\overline{ab}$ is ensured by 4.6 and by the assumed prolongation of segments through p. If Q_1 does not satisfy (C′), we can find $0 < \mu < \min(\eta, \varrho(p, \operatorname{Bd} Q_1))$ and a point $v \in B(p, \mu) \setminus Q_1$. Then prolong the segment $\overline{vp}$ through p and find a point u in $B(p, \mu)$ such that $p \in \overline{uv}$ and $u \neq p \neq v$. Evidently, $\overline{ab} \cap \overline{uv} = p$, for otherwise p would be a ramification point. By 4.5 and 5.3 the set $Q_2 = C_\varrho(\overline{ab}, u, v)$ is a convex disk contained in $B(p, 4\eta)$ and p is an interior point of Q_2. Suppose that Q_m is a convex m-cell contained in $B(p, 4^{m-1} \cdot \eta)$, $2 \leqslant m \leqslant n$, and that $p \in \operatorname{Int} Q_m$. If Q_m satisfies (C′), put $Q = Q_m$ and the proof is finished. If Q_m does not satisfy (C′), we can find a number μ, $0 < \mu < \min(\eta, \varrho(a, \operatorname{Bd} Q_m))$, and a point v such that $v \in B(p, \mu) \setminus Q_m$. Then we prolong the segment $\overline{vp}$ through p and find a point u in $B(p, \mu)$ such that $p \in \overline{uv}$ and $u \neq p \neq v$. By 4.5 $C_\varrho(Q_m, u, v) \subset B(p, 4^m \cdot \eta) \subset B(p, \varepsilon)$, whence by 4.4, $B_\varrho(Q_m, v) = Q_m$. Consequently, $u \notin Q_m$. Applying 5.4, we see that $Q_{m+1} = C_\varrho(Q_m, u, v)$ is a convex $(m+1)$-cell and $p \in \operatorname{Int} Q_{m+1}$. In this way we must come at last to a convex cell $Q \subset B(p, \varepsilon)$ satisfying (C′).

From this theorem and from 3.1 and 3.2 we now infer two corollaries.

COROLLARY I. *Every point of an n-dimensional G-space with the local CT-property is an interior point of a sufficiently small convex n-cell.*

COROLLARY II. *Every n-dimensional compact SC-WR-space with the local CT-property is a cell and each interior point of it is an interior point of a sufficiently small convex n-cell.*

Remarks.

1. In both cases sufficiently small cells are SC-WR-cells and so, by [8], 9.1, their interiors are convex open cells.

2. In both cases suficiently small cells satisfy condition (C) from § 4 and so their interiors are open subsets of X.

By a slight modification of the proof of Theorem 2 one can show that

3. Every interior point of an n-dimensional SC-WR-CT-cell Q is an interior point of a convex hull of an $(n+1)$-tuple contained in Q.

The same result follows also directly from Theorem 4 below.

§ 6. Straight CT-spaces. A G-space is called a *straight space* ([4], § 8) if every pair of its points determines a unique straight line, i.e. a set isometric to a Euclidean line.

A straight space X is *Desarguesian* ([4], § 13 and § 14) if X can be mapped topologically on an open convex subset C of the n-dimensional affine space A^n in such a way that each straight line in X goes into the intersection of C with a line in A^n.

Taking any Euclidean metrization of A^n in which the affine lines are Euclidean straight lines and calling a homeomorphism which preserves the metric betweenness a linear homeomorphism, one can say that a straight space is Desarguesian if it can be imbedded in E^n by a linear homeomorphism.

As is known ([4], p. 68), a 2-dimensional straight space is Desarguesian if and only if it satisfies the Desargues property. A higher-dimensional straight space is Desarguesian if and only if any three points of it lie in a plane, i.e. in a 2-dimensional subset of X which, with the metric of X, is a G-space ([4], p. 76).

The following theorem explains the role of the CT-property taken globally:

THEOREM 3. *For $n \geqslant 3$, any n-dimensional straight space $\langle X, \varrho \rangle$ is Desarguesian if and only if it has the CT-property.*

Take three non-linear points $a, b, c \in X$. By 4.3 a ϱ-cone $D = C_\varrho(\overline{ab}, c)$ is a convex disk. Let $v \in \operatorname{Int} D$, and let, for any $x \neq v$, R_x denote a ray through x with the origin v, i.e. $v, x \in R_x$ and there exists an isometric mapping i of R_x onto non-negative reals R^+ with $i(v) = 0$.

Put

$$P = \bigcup R_x, \quad \text{where} \quad x \in \operatorname{Bd} D.$$

In a sequence of lemmas we shall show that P is a 2-dimensional G-space.

6.1. *If $x, y \in X$, $z \in R_x \cap R_y$ and $z \neq v$, then $R_x = R_y$.*

The proof is obvious.

6.2. $D \subset P$.

Indeed, $P \supset \mathrm{Bd}D$ and $v \in P$. If $z \in \mathrm{Int}D$ and $z \neq v$, then the segment $\overline{vz}$ has a prolongation in the SC-WR-disk D to a point $x \in \mathrm{Bd}D$ (see [8], 7.4). By 6.1 $R_z = R_x$. Consequently, $z \in R_x \subset P$.

6.3. *P is closed in X, whence P is a finitely compact space.*

The proof follows from the lemma on the convergence of geodesics in a G-space ([4], p. 40) and from the compactness of $\mathrm{Bd}D$.

6.4. $\dim P = 2$.

As a matter of fact, P is homeomorphic to the cone over $\mathrm{Bd}D$, i.e. to the set $\mathrm{Bd}D \times R^+$ with $\mathrm{Bd}D \times 0$ identified with a point.

6.5. *P is convex.*

Let $p, q \in P$. We have to show that $\overline{pq} \subset P$. If either one of the points p, q is equal to v, or $p, q \in D$, or the triple $\{p, q, v\}$ is linear — the proof is trivial. So it remains to consider the case where $p \in R_x$, $q \in R_y$, $x, y \in \mathrm{Bd}D$ and neither the triple $\{x, y, v\}$ nor the triple $\{p, q, v\}$ is linear. Now we have three possibilities to consider: 1° vxp and vyq, 2° vpx and vyq, 3° vxp and vqy. 1° A ϱ-cone $C_\varrho(\overline{pq}, v)$ is an SC-WR-disk, so for any $t \in \overline{pq}$ the segments $\overline{vt}$ and $\overline{xy}$ have a common point z (comp. [8], § 11). The point z belongs to $\overline{xy}$, so $z \in D$ and $z \neq v$. By 6.2 there exists a point $z' \in \mathrm{Bd}D$ such that $z \in R_{z'}$; evidently, $R_{z'} \subset P$. But $z \in R_{z'} \cap R_t$, whence, by 6.1, we have $t \in P$. 2° We have vxx and vyq. Since neither $\{v, x, y\}$ nor $\{v, x, q\}$ is linear, we infer from 1° that $\overline{xq} \subset P$ and $v \notin \overline{xq}$. Consequently, a convex disk $C_\varrho(\overline{xq}, v)$ is contained in P, and so by $\overline{pq} \subset C_\varrho(\overline{xq}, v)$ we see that $\overline{pq}$ is in P. 3° In view of the symmetry of the assumption, the proof is analogous to that of 2°.

6.6. *P has the local prolongation property.*

We have to show that to every point $p \in P$ there corresponds a positive number ϱ_p such that for any two distinct points $x, y \in P$ with $\varrho(p, x) < \varrho_p$ and $\varrho(p, y) < \varrho_p$ there exists a point $z \in P$ such that $z \neq y$ and xyz. If $p \in \mathrm{Int}D$, the proof follows from [8], 7.4 and $\varrho_p \geqslant \varrho(p, \mathrm{Bd}D)$, so we may suppose that $p \notin \mathrm{Int}D$; in particular, $p \neq v$. Take a point $x \in \mathrm{Int}D$ not linear with p and v and find a point $y \in D$ such that $v \in \overline{xy}$, $y \neq v$. Now, taking a point $z \in R_p$ such that $p \in \overline{vz}$ and $z \neq p$, we see that a triple $\{x, y, z\}$ is not linear, whence $C_\varrho(\overline{xy}, z)$ is a convex disk. By 6.5 such a disk is contained in P and it is obvious that p is its interior point (comp. [8], 6.4). By an application of [8], 7.4 the proof is completed.

6.7. *P is a straight space.*

The proof follows from 6.5, 6.6 and from the assumption that X is a straight space.

§ 7. Local linear homeomorphism. Although the interior of an SC-WR-cell is not finitely compact, it has some properties of a straight space. In particular, it is convex (cf. [8], § 9) and has a local prolongation property ([8], 7.2). Moreover, any segment joining two interior points has a unique prolongation on both sides to the segment whose end-points lie on the boundary of a cell. So, in view of a result of H. Buseman ([4], (13.1) and (14.1)), it is natural that the following theorem holds:

THEOREM 4. *If $\langle Q, \varrho\rangle$ is an SC-WR-CT n-cell and $n \geqslant 3$, then* $\mathrm{Int}Q$ *is a Desarguesian space, i.e. there exists a linear homeomorphism of* $\mathrm{Int}Q$ *into E^n.*

It is not surprising that the construction of such a specialized homeomorphism will be quite long (comp. [4], p. 65). Neither can we hope that there exists a remetrization preserving linearity and such that the interior of an SC-WR-CT-cell becomes a straight space (if it were so, Theorem 3 could be applied). Fortunately, H. Buseman's proof of theorems (13.1) and (14.1) (cf. [4], p. 68–80) works almost literally in our case, and so we confine ourselves to pointing out some slight modifications in it.

Following H. Buseman, decompose the proof into two parts:

Part one. 2-dimensional case. For any two points a, b in the interior D of an SC-WR-disk Q let $g(a, b)$ denote the segment with end-points on the boundary of Q which contains a and b. The existence of $g(a, b)$ follows from [8], 7.4. Considering $g(a, b)$ as a geodesic and thus modifying the meaning of a geodesic in relation to Buseman's book [4], we assume the Desargues Property in the form written in [4], p. 67–68. Now we have

7.1. *If D is interior of an SC-WR-disk in which the Desargues Property holds, then there exists a linear homeomorphism of D into an affine plane A^2.*

By [8], 9.1, D is convex and by [8], 12.1 and 11.8 D possesses Pasch's property. The convergence of geodesics in the modified sense is obvious (comp. [8], 2.5). Now the rest of the proof is the same as in [4], p. 68–75.

Part two. Higher-dimensional case. Let $\langle Q, \varrho\rangle$ be an SC-WR-CT n-cell, where $n \geqslant 3$. We call a subset L of Q flat if L is a convex cell and the boundary of L is a subset of the boundary of Q. If L has dimension r, we call it briefly r-flat. We admit also a trivial flat consisting of one or zero points.

We begin with

7.2. *Any three non-linear points of* Int$Q$ *lie in a* 2*-flat.*

Proof. Take three non-linear points $a, b, c \in \operatorname{Int} Q$. By 4.3 the ϱ-cone $D = C_\varrho(\overline{ab}, c)$ is a convex disk. Let $v \in \operatorname{Int} D$ and, for any $x \neq v$, let the set R_x denote a maximal prolongation in Q of a segment vx. Put

$$P = \bigcup R_x, \quad \text{where} \quad x \in \operatorname{Bd} D.$$

One can easily see that P is a disk and that the boundary of P lies in BdQ. Finally, the convexity of P follows from the argument analogous to that in the proof of 6.5.

7.3. *If p is an interior point of a flat L and $x \in L$, then the maximal prolongations of $\overline{px}$ in Q and in L are equal.*

The proof follows from the fact the maximal prolongation of a segment $\overline{px}$ in an SC-WR-cell with p in the interior of such a cell meets the boundary once only ([8], 9.1 and 7.4).

7.4. *The intersection of two flats is a flat.*

Proof. The intersection B of two flats L' and L'' is evidently compact and convex. So, except for a trivial case where B contains one point at most, it is a cell. It remains to show that $\operatorname{Bd} B \subset \operatorname{Bd} Q$. As follows from [8], 9.1 and 7.4, for any $x \in \operatorname{Bd} B$ and any $p \in \operatorname{Int} B$ the segment $\overline{px}$ is not prolongable in B. Now if $x \in \operatorname{Bd} B \backslash \operatorname{Bd} Q$, then the segment $\overline{px}$ would be prolongable in Q (up to the boundary of Q), and so by 7.3 it would be prolongable in L' and in L'', whence it would be prolongable also in B: a contradiction.

7.5. *Any $r+1$ points of* Int$Q$ *which do not lie on any r'-flat $L_{r'}$, where $r' < r$, lie on at most one r-flat L_r.*

The proof goes exactly as in [4], (14.2) with only a slight modification at the end. Namely, under Buseman's notation, we obtain the following formulation: Unless $L' = L''$, one of these flats, say L'', contains a point p not contained in the other, L'. Now, no segment $\overline{px}$ with $x \in \operatorname{Int} B$ can intersect Int$B$ twice, because otherwise, by 7.3, $\overline{xp} \subset B \subset L'$. So Int$B$ is a base of the ϱ-cone $V = C_\varrho(\operatorname{Int} Q, p)$. This base contains an r-dimensional cell and so, by 4.2, V contains an $(r+1)$-dimensional cell. On the other hand, $V \subset L''$: a contradiction.

Now the proof of Theorem 4 follows as in [4], p. 76–80.

Theorem 4 together with Corollaries I and II from § 5 implies the following two corollaries:

COROLLARY I. *Every point of an n-dimensional G-space, where $n \geqslant 3$, with the local CT-property is an interior point of a convex n-cell which is linearly homeomorphic to a Euclidean n-cell.*

COROLLARY II. *Every n-dimensional SC-WR compact space, where $n \geqslant 3$, with the local CT-property is a cell and each of its interior points is an interior point of a convex n-cell which is linearly homeomorphic to a Euclidean n-cell.*

§ 8. Locally Desarguesian spaces. An n-dimensional separable metric space $\langle X, \varrho\rangle$ is called *locally Desarguesian* if each point $p \in X$ has a convex neighbourhood U which can be transformed by a linear homeomorphism onto an open subset of E^n.

It is obvious that

8.1. *Every locally Desarguesian space is a topological manifold* (*without boundary*).

8.2. *Every locally Desarguesian space has the local CT-property.*

By § 5 Remark 2 and 8.2, Corollary I from § 7 can be expressed in the following form

COROLLARY I. *Every n-dimensional G-space, where $n \geqslant 3$, is locally Desarguesian if and only if it has the local CT-property.*

In a compact SC-WR-space the end-point of a maximal segment is called a *frontier point* and the set of all frontier points of X is denoted by $F(X, \varrho)$ (comp. [7], p. 185).

It is well known that $F(X, \varrho)$ is contained in the set $L(X, \varrho)$ of all homotopically labile points (comp. [8], 7.1). Recently B. Krakus [6] has shown that in an n-dimensional compact SC-WR-space $F(X, \varrho)$ is a closed $(n-1)$-dimensional set such that $F(X, \varrho) = L(X, \varrho)$ and $S(X, \varrho) = X \backslash L(X, \varrho)$ is convex. It follows that the set $S(X, \varrho)$ is an n-dimensional open and convex subset of X. Moreover, $S(X, \varrho)$ has the local prolongation property.

If X is an SC-WR-cell, then $S(X, \varrho) = \operatorname{Int} X$ ([8], 7.2).

Whence and from Corollaries II of § 4 and of § 7 we infer that

COROLLARY II. *A subset $S(X, \varrho)$ of a compact n-dimensional SC-WR-space is locally Desarguesian if and only if it has the local CT-property.*

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An algebraic equivalent of a multiple choice axiom

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Adopting the notation of Bleicher [1], let FS_1 be the following statement:

For every set $\mathcal{C}$ of non-empty sets there exists a function f defined on $\mathcal{C}$ such that, for each $T \in \mathcal{C}$, $f(T)$ is a non-empty finite subset of T.

It has been shown (op. cit.) that FS_1 can be derived in a suitable set theory without the axiom of choice (e.g., the system S of Mostowski [4]) from the assumption that there exists a field F such that, for every vector space V over F, each subspace of V is a direct summand of V.

Now a vector space over the rationals is the same thing as a torsion-free divisible abelian group. So, clearly, if we assume the apparently stronger condition that, for every abelian group A, each torsion-free divisible subgroup of A is a direct summand of A, then FS_1 can be effectively proved. It turns out, in fact, that this condition is equivalent to FS_1.

THEOREM. *FS_1 effectively implies that, for every abelian group A, each torsion-free divisible subgroup of A is a direct summand of A.*

Proof. Let D be a torsion-free divisible subgroup of A. We will construct a homomorphism $h: A \to D$ such that $h(d) = d$ for each $d \in D$. To this end, let f be a multiple choice function for the set of all non-empty subsets of A, and let g be a multiple choice function for the set of all non-empty sets of homomorphisms from subgroups of A into D. The following recursion defines a chain of homomorphisms h_α from subgroups B_α of A into D such that $D \subseteq B_\alpha$ and $h_\alpha(d) = d$ for each $d \in D$:

$$h_0 = \mathrm{id}_D .$$

If α is a limit ordinal,

$$h_\alpha = \bigcup_{\beta<\alpha} h_\beta .$$

If $\alpha = \beta+1$: Let H be the set of all homomorphic extensions of h_β to the subgroup generated by B_β and the elements in $f(A \setminus B_\beta)$. The proof that H is non-empty is completely constructive since $f(A\setminus B_\beta)$ is finite, even if D were not torsion-free. Let $g(H) = \{h'_1, \ldots, h'_n\}$. Then we define

$$h_\alpha = \frac{1}{n}(h'_1 + \ldots + h'_n) .$$