θ-continuous extensions of maps on τX

by

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In [5] was given a characterization of continuous maps $f\colon X\to Y$ of a Hausdorff space X into an H-closed Hausdorff space Y possessing a continuous extension $\tau f\colon \tau X\to Y$ on the Katětov H-closed extension τX of the space X, and it was shown there that not every continuous map has such an extension. However, under certain assumptions on the space Y, even each continuous map $f\colon X\to Y$ has a unique θ-continuous extension $\tau f\colon \tau X\to Y$. Since θ-continuity (a notion due to Fomin [3]) seems to be a reasonable generalization of continuity in the theory of H-closed spaces, the question was raised in [5] which continuous maps $f\colon X\to Y$ of a Hausdorff space X into an H-closed space Y possess a θ-continuous extension $f^*\colon \tau X\to Y$ and when f^* is unique. In this paper some results in this direction are given. Also the case of non-unique extensions is discussed.

1. Preliminaries. The Katětov H-closed extension of a Hausdorff space X was defined originally in [4] as the set τX consisting of the points of X and all open ultrafilters without adherence points in X with topology generated by open subsets of X and sets of the form $U_\xi = \{\xi\}\cup U$, where ξ is a point of $\tau X - X$ and $U\in\xi$. Observe that

$$\mathrm{Cl}_{\tau X} U_\xi = \mathrm{Cl}_X U \cup \{\xi\in\tau X - X\colon\ U\in\xi\}.$$

A map $f\colon X\to Y$ is called *θ-continuous* [3] if for each $x\in X$ and for each open neighbourhood U_y of $y=f(x)$ there exists an open neighbourhood U_x of x such that $f(\mathrm{Cl}\,U_x)\subset \mathrm{Cl}\,U_y$. Clearly, a continuous map is θ-continuous. These notions coincide if Y is regular.

2. The form of the continuous extension τf. The proper maps, i.e. maps $f\colon X\to Y$ of a Hausdorff space X into an $\boldsymbol{H}$-closed space Y having a continuous (consequently, unique) extension $\tau f\colon \tau\boldsymbol{X}\to Y$, were characterized in [5], theorem 4.4, by the condition

(1) for each $y\in Y$ and each open neighbourhood $\boldsymbol{U}_y$ of y, there exists an open neighbourhood V_y of y such that

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\,V_y)\subset \mathrm{Cl} f^{-1}(\boldsymbol{U}_y).$$

The existence of a continuous extension τf of a proper map f was proved in [5] by a categorial argument, but no formula describing the form of τf was given. To write this formula, suppose that for $\xi \in \tau X - X$, the symbol $\mathfrak{U}(\xi)$ denotes the family of all open subsets U of Y containing $f(U')$ for some $U' \in \xi$.

2.1. *The map $\tau f\colon \tau X \to Y$ defined by the formula $\tau f(x) = f(x)$ for $x \in X$ and $\tau f(\xi) = \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(\xi)\}$ for $\xi \in \tau X - X$ is the (unique) continuous extension of a proper map $f\colon X \to Y$.*

Proof. First let us prove that the above definition of τf is correct, i.e. $\bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(\xi)\}$ is a one-point set for each $\xi \in \tau X - X$. Since $\mathfrak{U}(\xi)$ is a centred family of open sets ($U \supset f(U')$ and $V \supset f(V')$ imply $U \cap V \supset f(U' \cap V') \neq \emptyset$) in the H-closed space Y, there exists a point $y \in \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(\xi)\}$. This point is uniquely determined by ξ and, moreover, we have

(2) $$f^{-1}(U_y) \in \xi \quad \text{for each } U_y.$$

To prove (2) observe that

(3) $$\mathrm{Cl}\, U_y \cap f(U) \neq \emptyset \quad \text{for each } U_y \text{ and } U \in \xi,$$

for if $\mathrm{Cl}\, U_y \cap f(U) = \emptyset$, then $f(U) \subset Y - \mathrm{Cl}\, U_y$; thus $Y - \mathrm{Cl}\, U_y \in \mathfrak{U}(\xi)$ and, by the definition of y, we get $y \in \mathrm{Cl}(Y - \mathrm{Cl}\, U_y) \subset Y - U_y$, a contradiction. Now, if $f^{-1}(U_y) \notin \xi$ for some U_y (contrary to (2)), then there exists, by a known property of ultrafilters, a $U \in \xi$ such that $f^{-1}(U_y) \cap U = \emptyset$. But f being a proper map, there exists a V_y with $\mathrm{Int} f^{-1}(\mathrm{Cl}\, V_y) \subset \mathrm{Cl} f^{-1}(U_y)$. Then the open set $U' = U - f^{-1}(\mathrm{Cl}\, V_y)$ is dense in U and therefore $U' \in \xi$, since open ultrafilters contain all dense and open subsets of each element. By the definition of U', we have $U' \cap f^{-1}(\mathrm{Cl}\, V_y) = \emptyset$, whence $\mathrm{Cl}\, V_y \cap f(U') = \emptyset$, contrary to (3). Thus (2) is proved, which means that $f^{-1}(U_y) \in \xi$ for each U_y whenever $y \in \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(\xi)\}$. Since Y is a Hausdorff space and ξ does not contain disjoint open sets, there is one such y only. Therefore we are allowed to define $\tau f(\xi)$ to be the only point in the set $\bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(\xi)\}$.

It is easy to see that τf is continuous: taking for $x \in X$ and for an arbitrary $U_{f(x)}$ the set $U_x = f^{-1}(U_{f(x)})$ as an open (also in τX) neighbourhood of x, it follows that $\tau f(U_x) = f(U_x) = U_{f(x)}$; and taking as U_y for $\xi \in \tau X - X$ an arbitrary neighbourhood of $y = \tau f(\xi)$, we have $f^{-1}(U_y) \in \xi$ by (2), so that $U_\xi = \{\xi\} \cup f^{-1}(U_y)$ is an open neighbourhood of ξ in τX for which

$$\tau f(U_\xi) = f(U_\xi \cap X) \cup \tau f\big(U_\xi \cap (\tau X - X)\big) = f\big(f^{-1}(U_y)\big) \cup \tau f(\xi) = U_y.$$

Remark. It is proved, in fact, that $\{y\} = \{\tau f(\xi)\} = \bigcap \{\mathrm{Cl} f(U)\colon U \in \xi\}$, since from (2) it follows that $f^{-1}(U_y) \cap U \neq \emptyset$ for each $U \in \xi$, whence $U_y \cap f(U) \neq \emptyset$ for each U_y and $U \in \xi$. This means in particular that f has the property $\bigcap \{\mathrm{Cl} f(U)\colon U \in \xi\} \neq \emptyset$ for each $\xi \in \tau X - X$. This property of a map f is equivalent to (1), as shown by Błaszczyk and Mioduszewski [2], theorem 2'.

3. Necessary conditions for the existence of θ-continuous extensions. The equivalence of the following conditions, although almost obvious, is proved for use in various situations later on.

3.1. *For a continuous map $f\colon X \to Y$ and arbitrary points $\xi \in \tau X - X$ and $y \in Y$ the following conditions are equivalent:*

(a) *for each U_y there exists a $U \in \xi$ such that $f(U) \subset \mathrm{Cl}\, U_y$,*

(b) $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \in \xi$ *for each* U_y,

(c) $y \in \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(\xi)\}$.

Proof. (a)$\Rightarrow$(b). Observe that if (a) holds, then for each U_y and each $U' \in \xi$ formula (3) holds, i.e. $\mathrm{Cl}\, U_y \cap f(U') \neq \emptyset$.

To prove this, take for U_y, according to (a), a set $U \in \xi$ such that $f(U) \subset \mathrm{Cl}\, U_y$. Then

$$\mathrm{Cl}\, U_y \cap f(U') \supset \mathrm{Cl}\, U_y \cap f(U \cap U') = f(U \cap U'),$$

the equality being a consequence of the choice of U. Since $U \cap U' \in \xi$, the set $f(U \cap U')$ is non-empty and (3) follows.

To prove (b), suppose on the contrary that $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \notin \xi$ for some U_y. Then $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \cap U = \emptyset$ for some $U \in \xi$, and $U' = U - f^{-1}(\mathrm{Cl}\, U_y)$ is an element of ξ such that $U' \cap f^{-1}(\mathrm{Cl}\, U_y) = \emptyset$. This implies $\mathrm{Cl}\, U_y \cap f(U') = \emptyset$ for some U_y and $U' \in \xi$, which contradicts (3).

(b)$\Rightarrow$(c). Suppose, contrary to (c), that $y \notin \mathrm{Cl}\, U$ for some $U \in \mathfrak{U}(\xi)$. Then $U \supset f(U')$ for some $U' \in \xi$ and, since $y \in U_y = Y - \mathrm{Cl}\, U$, we have $\mathrm{Cl}\, U_y \cap f(U') \subset (Y - U) \cap f(U') = \emptyset$. Thus $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \cap U' = \emptyset$, which implies $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \notin \xi$, a contradiction of (b).

(c)$\Rightarrow$(a). Suppose, contrary to (a) that there exists a U_y such that $f(U) \not\subset \mathrm{Cl}\, U_y$ for each $U \in \xi$. Then $\big(X - f^{-1}(\mathrm{Cl}\, U_y)\big) \cap U \neq \emptyset$ for each $U \in \xi$, whence $X - f^{-1}(\mathrm{Cl}\, U_y) = f^{-1}(Y - \mathrm{Cl}\, U_y) \in \xi$. But then $Y - \mathrm{Cl}\, U_y \in \mathfrak{U}(\xi)$ and, by (c), $y \in \mathrm{Cl}(Y - \mathrm{Cl}\, U_y) \subset Y - U_y$, a contradiction.

The conditions just examined determine the form of θ-continuous extensions of f as follows:

3.2. *If f^* is a (not necessarily unique) θ-continuous extension of $f\colon X \to Y$ on τX, and $y = f^*(\xi)$, then each of the conditions* (a), (b) *and* (c) *holds.*

Proof. It suffices to prove that, in fact, (a) holds. To do this take an arbitrary neighbourhood U_y of the point $y = f^*(\xi)$. By the θ-continuity of f^*, there exists an open neighbourhood $U_\xi = \{\xi\} \cup U$, $U \in \xi$, such that $f^*(\mathrm{Cl}\, U_\xi) \subset \mathrm{Cl}\, U_y$. For this U thus $f(U) \subset f(\mathrm{Cl}\, U) = f^*(\mathrm{Cl}\, U_\xi \cap X) \subset \mathrm{Cl}\, U_y$.

Remark. The condition (c) states, in fact, that if f is extended θ-continuously to f^*, the point $f^*(\xi)$ must be chosen among points of the set $\bigcap\{\mathrm{Cl}\,U\colon U\in\mathfrak{U}(\xi)\}$ as in the case of continuous extension (however, in this case there is one such point only). Evidently, it is natural to ask for conditions on the map f which ensure that there is only one point in $\bigcap\{\mathrm{Cl}\,U\colon U\in\mathfrak{U}(\xi)\}$. By condition (b) this becomes an easy exercise: we must guarantee that for each $\xi\in\tau X-X$ there exists exactly one point y for which the family $\mathcal{F}_y=\{\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y)\colon U_y$ running over all open neighbourhoods of $y\}$ is contained in ξ. The existence of such a point y is a consequence, in virtue of (b)$\Leftrightarrow$(c), of the H-closedness of Y.

3.3. *An ultrafilter $\xi\in\tau X-X$ contains exactly one $\mathcal{F}_y$ iff for arbitrary different points y' and y'' there exist $U_{y'}$ and $U_{y''}$ such that*

$$\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_{y'}\cap\mathrm{Cl}\,U_{y''})=\emptyset\,. \tag{4}$$

Proof. ($\Rightarrow$). Suppose, on the contrary, that there exist y and y' such that $\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'})\neq\emptyset$ for each U_y and $U_{y'}$. Then $\mathcal{F}=\{\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'})\colon y\in U_y$ and $y'\in U_{y'}\}$ is a centred family of open subsets of X, since

$$\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'})\cap\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y\cap\mathrm{Cl}\,V_{y'})$$
$$\supset\mathrm{Int}f^{-1}\big(\mathrm{Cl}(U_y\cap V_y)\cap\mathrm{Cl}(U_{y'}\cap V_{y'})\big)\neq\emptyset\,.$$

The family $\mathcal{F}$ has no adherence points in X:

$$\bigcap\{\mathrm{Cl}\,U\colon U\in\mathcal{F}\}\subset\bigcap\{f^{-1}(\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'})\colon y\in U_y \text{ and } y'\in U_{y'}\}$$
$$=f^{-1}\big(\bigcap\{\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'}\colon y\in U_y \text{ and } y'\in U_{y'}\}\big)=\emptyset$$

since

$$\bigcap\{\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'}\colon y\in U_y \text{ and } y'\in U_{y'}\}=\{y\}\cap\{y'\}=\emptyset\,.$$

Thus there exists an ultrafilter ξ without adherence points in X, i.e. a point of $\tau X-X$, containing $\mathcal{F}$ and, in consequence, both $\mathcal{F}_y$, and $\mathcal{F}_{y'}$, contrary to the assumption.

($\Leftarrow$). Since

$$\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y)\cap\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_{y'})=\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'})=\emptyset$$

for some U_y and $U_{y'}$ whenever $y\neq y'$, no filter contains both $\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y)$ and $\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_{y'})$. In consequence, each $\xi\in\tau X-X$ contains at most one family $\mathcal{F}_y$ and such a family indeed exists.

Remarks. Call a map $f\colon X\to Y$ a *Urysohn map* if it satisfies the condition (4). Recall that a Urysohn space (called sometimes completely Hausdorff) is a space in which each two different points y and y' can be separated by closed neighbourhoods, i.e. $\mathrm{Cl}\,U_y\cap\mathrm{Cl}\,U_{y'}=\emptyset$ for some U_y and $U_{y'}$. Thus each map into a Urysohn space is a Urysohn map. The Urysohn maps are of interest they possess a unique θ-continuous extension on τX, as theorem 3.5 brings out.

Call a map $f\colon X\to Y$ *r.o.-proper* (r.o. standing for "regularly open") if it has property (1) with respect to r.o. subsets of Y.

3.4. *A map $f\colon X\to Y$ into an H-closed space is a Urysohn map iff it is r.o.-proper.*

Proof ($\Rightarrow$). Let U_y be an r.o. neighbourhood of y. Since f is a Urysohn map, then for each $y'\in Y-U_y$ there exist an open neighbourhood $U_{y'}$ of y' and an open neighbourhood $U_y(y')$ of y such that

$$\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y(y')\cap\mathrm{Cl}\,U_{y'})=\emptyset\,. \tag{5}$$

The family $\{U_{y'}\colon y'\in Y-U_y\}$ is an open covering of $Y-U_y$, being H-closed as a regularly closed subset of the H-closed space Y. Hence there exists a finite family $\{U_{y_1},\ldots,U_{y_n}\}$ such that

$$Y-U_y\subset\mathrm{Cl}\,U_{y_1}\cup\ldots\cup\mathrm{Cl}\,U_{y_n}. \tag{6}$$

Take $V_y=U_y(y_1)\cap\ldots\cap U_y(y_n)$ — an open neighbourhood of y. Then

$$f^{-1}(\mathrm{Cl}\,V_y-U_y)\subset f^{-1}\big(\mathrm{Cl}\,V_y\cap(\mathrm{Cl}\,U_{y_1}\cup\ldots\cup\mathrm{Cl}\,U_{y_n})\big)$$
$$=f^{-1}(\mathrm{Cl}\,V_y\cap\mathrm{Cl}\,U_{y_1})\cup\ldots\cup f^{-1}(\mathrm{Cl}\,V_y\cap\mathrm{Cl}\,U_{y_n})$$
$$\subset f^{-1}(\mathrm{Cl}\,U_y(y_1)\cap\mathrm{Cl}\,U_{y_1})\cup\ldots\cup f^{-1}\big(\mathrm{Cl}\,U_y(y_n)\cap\mathrm{Cl}\,U_{y_n}\big)$$

the first inclusion being a consequence of (6), and the second one a consequence of the definition of V_y. But $f^{-1}(\mathrm{Cl}\,U_y(y_k)\cap\mathrm{Cl}\,U_{y_k})$ is, by (5), a nowhere-dense closed subset of X for each $k=1,\ldots,n$, whence $\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y-U_y)=\emptyset$, or equivalently $\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y)\subset\mathrm{Cl}f^{-1}(U_y)$, which means that f is r.o.-proper.

($\Leftarrow$). Let y and y' be different points of Y. Then there exists an r.o. neighbourhood U_y of y such that $y'\notin\mathrm{Cl}\,U_y$ since Y is a Hausdorff space. The map f being r.o.-proper, take for this U_y a V_y such that

$$\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y)\subset\mathrm{Cl}f^{-1}(U_y)\,. \tag{7}$$

The set $U_{y'}=Y-\mathrm{Cl}\,U_y$ is an open neighbourhood of y' and $\mathrm{Cl}\,U_{y'}=Y-U_y$ since U_y is an r.o. subset of Y, so that

$$\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y\cap\mathrm{Cl}\,U_{y'})=\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y)\cap\mathrm{Int}f^{-1}(Y-U_y)$$
$$=\mathrm{Int}f^{-1}(\mathrm{Cl}\,V_y-U_y)=\emptyset$$

the last equality being a consequence of (7). Thus f is a Urysohn map.

Corollaries. A Urysohn map $f\colon X\to Y$ into an H-closed space Y can be extended to $\tau f\colon\tau X\to Y$ by the formula

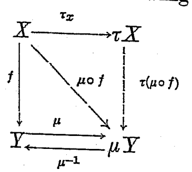
$$\{\tau f(\xi)\}=\bigcap\{\mathrm{Cl}\,U\colon U\in\mathfrak{U}(\xi)\}$$

(recall that by (c) of 3.1 and 3.3 this is the only possible θ-continuous extension — this enables us to use the symbol τf for it). This extension happens, in fact, to be θ-continuous, which may be calculated by using the technique of the proof of 3.4, but may be obtained also by the following categorial considerations.

For a Hausdorff space Y let μY denote the set Y endowed with the Hausdorff topology generated by r.o. subsets of Y. Let $\mu\colon Y\to\mu Y$ be the corresponding contraction, i.e. the identity on the underlying set Y. The map $\mu^{-1}\colon \mu Y\to Y$ will be called an *expansion*. If Y is H-closed, then μY is also H-closed, even a minimal Hausdorff space [4]. The map μ^{-1} is θ-continuous. By 3.4, each Urysohn map $f\colon X\to Y$ into an H-closed space Y is a composition $X\xrightarrow{f}Y\xrightarrow{\mu}\mu Y\xrightarrow{\mu^{-1}}Y$ of a continuous proper map $X\xrightarrow{f}Y\xrightarrow{\mu}\mu Y$ into a minimal space μY and a θ-continuous expansion $\mu Y\xrightarrow{\mu^{-1}}Y$. Note that the converse is also true: a composition $X\to Y\to Y'$ of a proper map $X\to Y$ into a minimal space Y and a θ-continuous expansion $Y\to Y'$ (Y' is then H-closed as a θ-continuous image of the H-closed space Y) is a Urysohn map, since, by 3.4, $X\to Y$ is a Urysohn map and expansions between H-closed topologies preserve regularly closed sets [5].

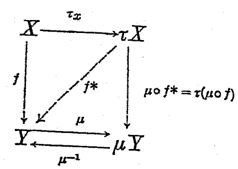
3.5. *Each Urysohn map $f\colon X\to Y$ into an H-closed space Y possesses a unique θ-continuous extension $\tau f\colon \tau X\to Y$.*

Proof. Take the decomposition $X\xrightarrow{f}Y\xrightarrow{\mu}\mu Y\xrightarrow{\mu^{-1}}Y$ of f. By theorem 4.4 of [5] there exists a continuous (consequently unique) extension $\tau(\mu\circ f)\colon \tau X\to\mu Y$ of the proper map $X\xrightarrow{f}Y\xrightarrow{\mu}\mu Y$. The composition $\tau X\xrightarrow{\tau(\mu\circ f)}\mu Y\xrightarrow{\mu^{-1}}Y$ of the continuous $\tau(\mu\circ f)$ and θ-continuous μ^{-1} is a θ-continuous extension of f, as may be seen in the following diagram:



(observe that $\mu^{-1}\circ\tau(\mu\circ f)\circ\tau_X=\mu^{-1}\circ\mu\circ f=f$).

The extension of f is unique, for if f^* is any θ-continuous extension of f, then $\mu\circ f^*\circ\tau_X=\mu\circ f$, since f^* is an extension of f. This means that $\mu\circ f^*$ is an extension of $\mu\circ f$.



Since the extension of $\mu\circ f$ is unique, thus $\mu\circ f^*=\tau(\mu\circ f)$. Finally, $f^*=\mu^{-1}\circ\tau(\mu\circ f)$, and so the uniqueness of f^* is proved.

Remark. By compact-like spaces we mean spaces for which μY are compact (then Y are necessarily H-closed). These are exactly the Urysohn H-closed spaces. Thus each map into a compact-like space possesses a unique θ-continuous extension to τX (cf [5], p. 24).

4. Extending non-Urysohn maps. A map $f\colon X\to Y$ into an H-closed space Y has a θ-continuous extension $f\colon \tau X\to Y$ iff the map $\mu f\colon X\to Y\to\mu Y$ into the minimal space μY has such an extension. The case of compact-like Y (i.e. compact μY) being trivial, in the sequel Y is always assumed to be a minimal and non-compact space.

Example. Let $\overline{Y}$ be a subset of the plane consisting of the points $(-1,0)$, $(1,0)$, $(\pm 1/n,1/m)$, $(0,1/m)$, where $n,m=1,2,\ldots$ Let the topology on Y be generated by plane sets of the form

$$U_k^-=\{(-1,0)\}\cup\{(-1/n,1/m)\colon n=1,2,\ldots \text{ and } m>k\},$$
$$U_k^+=\{(1,0)\}\cup\{(1/n,1/m)\colon n=1,2,\ldots \text{ and } m>k\},$$

where $k=1,2,\ldots$ and the usual open neighbourhoods of points distinct from $(-1,0)$ and $(1,0)$. The space Y is a minimal Hausdorff space and is non-compact, which is easy to check (it may be found in this or other forms in the literature).

Since Urysohn maps into minimal spaces coincide with proper ones (a consequence of 3.4), the problem reduces to the following: which non-proper maps $f\colon X\to Y$ of a Hausdorff space X into a minimal and non-compact space Y possess a θ-continuous extension $f^*\colon \tau X\to Y$ and when is f^* unique?

Example. Let N be the subspace of $\overline{Y}$ consisting of points $(0,1/m)$ where $m=1,2,\ldots$ The embedding $i\colon N\subset\overline{Y}$ is not proper at the points $(-1,0)$ and $(1,0)$ since $\operatorname{Int} i^{-1}(\operatorname{Cl}U_k^-)=\operatorname{Int} i^{-1}(\operatorname{Cl}U_k^+)=N-\{(0,1),\ldots$ $\ldots,(0,1/k)\}\neq\emptyset$ for each $k=1,2,\ldots$; meanwhile $\operatorname{Cl} i^{-1}(U_k^-)=\operatorname{Cl} i^{-1}(U_k^+)$ $=\emptyset$ for each $k=1,2,\ldots$

Set $D_f(V_y,U_y)=\operatorname{Int} f^{-1}(\operatorname{Cl}V_y-U_y)$. Observe that

(8) $D_f(V_y^1,U_y)\cap\ldots\cap D_f(V_y^k,U_y)\supset D_f(V_y^1\cap\ldots\cap V_y^k,U_y)$ for each open neighbourhood $V_y^1,\ldots,V_y^k$ and U_y of y

and

(9) if $U_y'\subset U_y''$, then $D_f(V_y,U_y'')\subset D_f(V_y,U_y')$ for each open neighbourhood V_y,U_y' and U_y'' of y.

A point $y\in Y$ is said to be *f-non-proper* if there exists an open neighbourhood U_y such that

(10) $D_f(V_y,U_y)\neq\emptyset$ for each open neighbourhood V_y.

In other words, f-non-proper points are the points at which f is not proper (such a point is a non-regular point of Y, i.e. there exists a U_y such that $\mathrm{Cl}V_y \not\subset U_y$ for each V_y).

Neighbourhoods U_y for which (10) holds are called f-non-proper.

4.1. *The f-non-proper neighbourhoods of an f-non-proper point form a neighbourhood basis.*

Proof. Observe that the intersection $U_y \cap U'_y$ of an f-non-proper neighbourhood U_y and an arbitrary open neighbourhood U'_y is f-non-proper since $D_f(V_y, U_y \cap U'_y)$ contains by (9) the set $D_f(V_y, U_y)$ which is by (10) non-empty for each V_y. Thus the sets $U_y \cap U'_y$ form a neighbourhood basis of y.

4.2. *A point $y \in Y$ is f-non-proper iff there exists another point $y' \in Y$ such that*

(11) $$\mathrm{Int} f^{-1}(\mathrm{Cl}U_y \cap \mathrm{Cl}U_{y'}) \neq \emptyset \quad \text{for each } U_y \text{ and } U_{y'}.$$

Proof. Since Y has a basis consisting of r.o. sets, this is a "localization" of theorem 3.4, and it was in fact proved there.

This means that each f-non-proper point y has "neighbours" y' which are also f-non-proper. Points y and y' for which (11) holds are said to be f-*tangent*.

EXAMPLE. The points $(-1, 0)$ and $(1, 0)$ are i-tangent.

The family $\mathcal{F}(U_y, f) = \{D_f(V_y, U_y)\colon V_y$ running over all open neighbourhoods of $y\}$ will be called the f-*defect* of U_y (more precisely, of the pair (y, U_y)). The notion of the f-defect is interesting for the f-non-proper points and their f-non-proper neighbourhoods only, since in this case all members of $\mathcal{F}(U_y, f)$ are non-empty. In the sequel only this non-trivial case is discussed.

4.3. *$\mathcal{F}(U_y, f)$ is a centred family of open sets of X which has no adherence points in X.*

Proof. Since, by (8), $\mathcal{F}(U_y, f)$ is submultiplicative, it is, all the more, a centred family.

To prove that $\mathcal{F}(U_y, f)$ has void adherence, observe that

(12) $$\bigcap\{\mathrm{Cl}\,\mathrm{Int} f^{-1}(\mathrm{Cl}V_y - U_y)\colon y \in V_y\} \subset \bigcap\{f^{-1}(\mathrm{Cl}V_y)\colon y \in V_y\} = f^{-1}(y)$$

(the inclusion follows from $\mathrm{Cl}\,\mathrm{Int} f^{-1}(\mathrm{Cl}V_y - U_y) \subset f^{-1}(\mathrm{Cl}V_y)$ and the equality from $\{y\} = \bigcap\{\mathrm{Cl}V_y\colon y \in V_y\}$.

Simultaneously

(13) $$\bigcap\{\mathrm{Cl}\,\mathrm{Int} f^{-1}(\mathrm{Cl}V_y - U_y)\colon y \in V_y\} \subset f^{-1}(\mathrm{Cl}U_y - U_y) \subset f^{-1}(Y - U_y).$$

Suppose that $x \in X$ is an adherence point of $\mathcal{F}(U_y, f)$. Then (12) implies $x \in f^{-1}(y)$ and (13) implies $x \in f^{-1}(Y - U_y)$. Thus $f^{-1}(y) \cap f^{-1}(Y - U_y) \neq \emptyset$ — a contradiction.

The family $\mathcal{F}(y, f) = \bigcup\{\mathcal{F}(U_y, f)\colon U_y$ running over all f-non-proper open neighbourhoods of $y\}$ will be called the f-*defect* of y.

Remark. $\mathcal{F}(y, f)$ is a centred family whenever the family of f-non-proper neighbourhoods is subadditive, i.e. for each U'_y and U''_y there exists a U_y such that $U'_y \cup U''_y \subset U_y$.

The proof of this, in view of the submultiplicativity of $\mathcal{F}(U_y, f)$, reduces to the proof of the inequality

(14) $$D_f(V^1_y, U^1_y) \cap \ldots \cap D_f(V^k_y, U^k_y) \neq \emptyset \quad \text{for arbitrary}$$
$$D_f(V^1_y, U^1_y) \in \mathcal{F}(U^1_y, f), \ldots, D_f(V^k_y, U^k_y) \in \mathcal{F}(U^k_y, f).$$

To prove this, take an f-non-proper $U_y \supset U^1_y \cup \ldots \cup U^k_y$, which exists by hypothesis. Then, by (9),

(15) $$D_f(V^i_y, U_y) \subset D_f(V^i_y, U^i_y) \quad \text{for each } i = 1, \ldots, k.$$

Let $V_y = V^1_y \cap \ldots \cap V^k_y$. Then, by (8)

(16) $$D_f(V_y, U_y) \subset D_f(V^i_y, U_y) \quad \text{for each } i = 1, \ldots, k.$$

Now, (14) follows immediately from (15) and (16), so the centration of $\mathcal{F}(y, f)$ is proved.

In what follows it will be assumed that the f-defects of f-non-proper points are centred, to avoid technical difficulties.

Finally, note that $\mathcal{F}(y, f)$ has no adherence points in X, since $\mathcal{F}(y, f)$ contains the families $\mathcal{F}(U_y, f)$, having, in view of 4.3, void adherence.

Let $\hat{\mathcal{F}}(y, f)$ denote the open filter generated by $\mathcal{F}(y, f)$, i.e. $\hat{\mathcal{F}}(y, f) = \{U\colon U$ open in X and $U \supset D_f(V_y, U_y)$ for some $D_f(V_y, U_y) \in \mathcal{F}(y, f)\}$.

f-tangent points y and y' are said to be f-*symmetric* if $\hat{\mathcal{F}}(y, f) = \hat{\mathcal{F}}(y', f)$.

EXAMPLE. From 4.2 it follows that a non-proper map $f\colon X \to \overline{Y}$ has at least two f-non-proper points which are f-tangent. Each point $y \in \overline{Y}$ except $(-1, 0)$ and $(1,0)$ is regular in the space $\overline{Y}$; thus y is regular for each map $f\colon X \to \overline{Y}$. It follows, that for each non-proper map $f\colon X \to \overline{Y}$ of an arbitrary Hausdorff space X into $\overline{Y}$, the points $(-1, 0)$ and $(1, 0)$ are just the only f-non-proper points. Since these points are topologically equivalent, there is $\hat{\mathcal{F}}\big((-1, 0), f\big) = \hat{\mathcal{F}}\big((1, 0), f\big)$. Minimal spaces, which can be constructed from copies of $\overline{Y}$ in a simple way without destroying the equivalence of the points $(-1, 0)$ and $(1, 0)$ possess similar properties; the one-point minimal and non-compact extension of the infinite disjoint sum $\bigoplus\{\overline{Y}_t\colon t \in T\}$ of copies of $\overline{Y}$ (this extension is constructed by adding to $\bigoplus \overline{Y}_t$ a point p with topology generated by open sets of $\bigoplus \overline{Y}_t$ and sets of the form $U_A = \{p\} \cup \{\overline{Y}_t\colon t \in T - A\}$, A running over the family of finite subsets of T), has the following property: $\hat{\mathcal{F}}\big((-1, 0)_t, f\big) = \hat{\mathcal{F}}\big((1, 0)_t, f\big)$ for each $f\colon X \to \bigoplus \overline{Y}_t \cup \{p\}$ whenever f is not proper at points $(-1, 0)_t$

and $(1,0)_t$ of $\overline{Y}_t$ (the points $(-1,0)_t$ and $(1,0)_t$ are, as in $\overline{Y}$, the only non-regular points of $\oplus\overline{Y}_t \cup \{p\}$) and, moreover, $(-1,0)_t$ is f-tangent to $(1,0)_t$ only.

However, f-tangent points are not necessarily f-symmetric. Let $\overline{Y}'$ be the "right half" of $\overline{Y}$, i.e. the subspace of $\overline{Y}$ consisting of points with non-negative first coordinates. Let $\overline{Y}_1$ be the "left odd quarter", i.e. the subspace of $\overline{Y}$ consisting of points with non-positive first coordinates and odd second ones and let $\overline{Y}_2$ be the "left even quarter". Stick $\overline{Y}_1$ to $\overline{Y}'$ along $\overline{Y}' \cap \overline{Y}_1$, i.e. along the set of points of $\overline{Y}$ of the form $(0, 1/2n-1)$; in other words, consider the pushout diagram

$$\begin{array}{ccc} \overline{Y}' \cap \overline{Y}_1 & \longrightarrow & \overline{Y}' \\ \downarrow & & \downarrow \\ \overline{Y}_1 & \dashrightarrow & Z \end{array}$$

then stick $\overline{Y}_2$ to $Z \subset \overline{Y}$ along $\overline{Y}' \cap \overline{Y}_2 \subset Z$, i.e. along the set of points of $\overline{Y}$ of the form $(0, 1/2n)$, that is, take the pushout diagram

$$\begin{array}{ccc} \overline{Y}' \cap \overline{Y}_2 & \longrightarrow & \overline{Y}_2 \\ \downarrow & & \downarrow \\ Z & \dashrightarrow & W \end{array}$$

It is easy to check that W is a Hausdorff, minimal and non-compact space. There are three non-regular points in W — the point $(1,0)$ of $\overline{Y}' \subset W$, the point $(-1,0)_1$ of $\overline{Y}_1 \subset W$ and $(-1,0)_2$ of $\overline{Y}_2 \subset W$. The point $(1,0)$ cannot be separated by disjoint closed neighbourhoods from $(-1,0)_1$ and $(-1,0)_2$ (W is therefore not compact), since the sets $\mathrm{Cl}\, U_{(1,0)}$ meet all except a finite number of the points of the set $N = \{(0, 1/n):\ n = 1, 2, \ldots\} \subset W$; meanwhile, the sets $\mathrm{Cl}\, U_{(-1,0)_1}$ and $\mathrm{Cl}\, U_{(-1,0)_2}$ meet N at an infinite number of points (all except a finite number of the points $(0, 1/2n-1)$ and $(0, 1/2n)$, respectively).

With respect to the embedding $j\colon N \subset W$ the point $(1,0)$ is j-tangent to both $(-1,0)_1$ and $(-1,0)_2$ but it is not j-symmetrical. However, $(1,0)$ becomes even symmetrical to $(-1,0)_1$ with respect to the map $j_1\colon N \to W$ defined by $j_1(n) = (0, 1/2n-1)$ and similarly to $(-1,0)_2$ with respect to $j_2\colon N \to W$ given by $j_2(n) = (0, 1/2n)$.

The f-symmetry relation induces a decomposition of the set of all f-non-proper points of Y into disjoint classes the members of which are in a sense equivalent with respect to the possibility of extending the map f onto τX, as the following considerations show. Any f-non-proper point $y \in Y$ has an associated subset $T_{[y]}$ of $\tau X - X$, consisting of ultra-

filters and containing $\hat{\mathcal{F}}(y,f)$ (the set $T_{[y]}$ depends only on the f-symmetry class of y, abbreviated to $[y]$, since $\hat{\mathcal{F}}(y,f) = \hat{\mathcal{F}}(y',f)$ for f-symmetrical points y and y'). Moreover, looking for a θ-continuous extension f^* of f, each point of the class $[y]$ is a possible value of f^* at points from $T_{[y]}$, by 3.1. To get a θ-continuous extension in such a way let us assume that $\hat{\mathcal{F}}(y,f)$ and $\hat{\mathcal{F}}(y',f)$ are *uniformly separated* whenever y is not f-symmetric with y', which means that

(17) Each f-non-proper $y \in Y$ has a neighbourhood U_y such that for each $y' \notin [y]$ there exists a $U_{y'}$ such that

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y \cap \mathrm{Cl}\, U_{y'}) = \emptyset.$$

A map $f\colon X \to Y$ which has property (17) is called *pseudo-proper*. Each proper map is pseudo-proper since in this case there are no f-non-proper points. Each pseudo-proper map with a finite number of f-non-proper points may be understood as proper with respect to the symmetry classes, which means that for each distinct class $[y]$ and $[y']$ there exist open sets U and U' such that $[y] \subset U$ and $[y'] \subset U'$ and $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U \cap \mathrm{Cl}\, U') = \emptyset$ (in other words, the composition of f and the quotient map $q\colon X \to X/_{[\ ]}$ induced by the decomposition of X consisting of symmetry classes and f-proper points is an r.o.-proper map.

Example. Each map $f\colon X \to \overline{Y}$ has at most one symmetry class, and thus it is pseudo-proper. The space $P = \oplus \overline{Y}_t \cup \{p\}$ has a similar property — the symmetry classes of an arbitrary map $f\colon X \to P$ consist of pairs $S_t = \{(-1,0)_t, (1,0)_t\}$ where $(-1,0)_t$ and also $(1,0)_t$ are f-non-proper points. Since distinct classes S_t and $S_{t'}$ can be separated in P by disjoint closed neighbourhoods, namely the closed and open sets $\overline{Y}_t$ and $\overline{Y}_{t'}$, a map f has property (17). Thus each map $f\colon X \to P$ is pseudo-proper, although there may exist an infinite number of f-symmetry classes.

Observe that pseudo-proper maps have the following property:

4.4. *$T_{[y]} \cap T_{[y']} = \emptyset$ for each pair of distinct points y and y' which are not f-tangent.*

Proof. The points y and y' being not f-tangent, neighbourhoods U_y and $U_{y'}$ can be chosen, according to (17), such that

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \cap \mathrm{Int} f^{-1}(\mathrm{Cl}\, U_{y'}) = \emptyset\,. \tag{18}$$

Moreover, U_y and $U_{y'}$ may be assumed to be f-non-proper, since by 4.1 the f-non-proper neighbourhoods form a basis of the neighbourhoods of an f-non-proper point.

Since $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y)$ contains the set $D_f(U_y, U_y) = \mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y - U_y)$, which is an element of $\mathcal{F}(y,f)$, we conclude that $\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_y) \in \hat{\mathcal{F}}(y,f)$.

Each ultrafilter ξ of $T_{[y]}$ contains, by definition, the filter $\hat{\mathcal{F}}(y,f)$. Thus $\mathrm{Int}f^{-1}(\mathrm{Cl}U_y)\in\xi$ for each $\xi\in T_{[y]}$ and $\mathrm{Int}f^{-1}(\mathrm{Cl}U_{y'})\in\xi'$ for each $\xi'\in T_{[y']}$. Hence, in view of (18), no ultrafilter can be an element of both $T_{[y]}$ and $T_{[y']}$, which implies $T_{[y]}\cap T_{[y']}=\emptyset$.

Each pseudo-proper map has a θ-continuous extension on τX. To simplify the proof, consider some properties of extensions of a map $f\colon X\to Y$ to $f^*\colon \tau X\to Y$ defined by the formula:

$$(19)\qquad \begin{aligned} &f^*(x)=f(x) &&\text{for } x\in X\,,\\ &f^*(\xi)\in\bigcap\{\mathrm{Cl}U\colon U\in\mathcal{U}(\xi)\} &&\text{for } \xi\in\tau X-X\,. \end{aligned}$$

Recall that by 3.2 each θ-continuous extension of f must be defined as in (19). By 1.1 the intersection $\bigcap\{\mathrm{Cl}U\colon U\in\mathcal{U}(\xi)\}$ is a one-point set equal to $\bigcap\{\mathrm{Cl}f(U)\colon U\in\xi\}$ whenever the last intersection is non-empty. Call an ultrafilter ξ of $\tau X-X$ f-proper if $\bigcap\{\mathrm{Cl}f(U)\colon U\in\xi\}\neq\emptyset$.

4.5. *Each extension defined by* (19) *is continuous and θ-continuous* [1] *at each point of X and each f-proper point of $\tau X-X$.*

Proof. Let $x\in X$. Take an arbitrary open neighbourhood $U_{f(x)}$ of the point $f(x)$. Then $U_x=f^{-1}(U_{f(x)})$ is an open (in τX) neighbourhood of x with $f^*(U_x)=f(U_x)=U_{f(x)}$ from which the continuity of f^* at x follows. To prove the θ-continuity of f^* at this point, observe that $\mathrm{Cl}_{\tau X}U_x=\mathrm{Cl}_XU_x\cup\{\xi\in\tau X-X\colon\ U_x\in\xi\}$. Thus $f^*(\mathrm{Cl}_{\tau X}U_x)=f(\mathrm{Cl}_XU_x)\cup\{f^*(\xi)\colon U_x\in\xi\}$. But $f(\mathrm{Cl}_XU_x)=f\big(\mathrm{Cl}_Xf^{-1}(U_{f(x)})\big)\subset\mathrm{Cl}U_{f(x)}$ by the continuity of f. Consider a filter ξ, for which $U_x\in\xi$. The set $U_x=f^{-1}(U_{f(x)})$ being an element of ξ, $U_{f(x)}$ is an element of $\mathcal{U}(\xi)$. Thus $f^*(\xi)\in\mathrm{Cl}U_{f(x)}$ by (19). We conclude that $f^*(\mathrm{Cl}U_x)\subset\mathrm{Cl}U_{f(x)}$, which means the θ-continuity at $x\in X$.

For a proper point $\xi\in\tau X-X$ let U_y be an open neighbourhood of $y=f^*(\xi)$. Then, by 1.1, there is $f^{-1}(U_y)\in\xi$. Thus $U_\xi=\{\xi\}\cup f^{-1}(U_y)$ is an open (in τX) neighbourhood of ξ for which $f^*(U_\xi)=\{y\}\cup U_y=U_y$, whence the continuity of f^* at ξ follows. To prove θ-continuity at ξ observe that $f^*(\mathrm{Cl}U_\xi)\subset\mathrm{Cl}U_y$ for the same reasons as in the analogous case for points of X.

To make the situation clear, characterize the remaining points of τX, i.e. the f-non-proper ultrafilters of $\tau X-X$.

4.6. *An ultrafilter $\xi\in\tau X-X$ is f-non-proper iff there exist a point $y\in Y$ and an open neighbourhood U_y of y such that $\mathcal{F}(U_y,f)\subset\xi$. These are the only points of τX at which f^* is not continuous.*

[1] A map which is continuous at a point need not be θ-continuous at that point. Let $(X,\mathcal{T})$ be a space which has a non-regular point x. Let $\mathcal{T}'$ be the topology on X generated by $\mathcal{T}$ and all one-point sets, except $\{x\}$, of X. The set-theoretical identity $e_X\colon(X,\mathcal{T})\to(X,\mathcal{T}')$ is continuous but not θ-continuous at x, although $(X,\mathcal{T}')$ is a regular space.

Proof. ($\Rightarrow$). Let ξ be an f-non-proper ultrafilter. Take an arbitrary point $y\in\bigcap\{\mathrm{Cl}U\colon U\in\mathcal{U}(\xi)\}$ (such a point exists, as was shown in the proof of 2.1). The set $\bigcap\{\mathrm{Cl}f(U)\colon U\in\xi\}$ being empty for the f-non-proper ξ, there exists an element $U\in\xi$ for which $y\notin\mathrm{Cl}f(U)$. Thus there exists an open neighbourhood U_y for which $U_y\cap f(U)=\emptyset$. This implies $f^{-1}(U_y)\notin\xi$ and, in consequence, $X-\mathrm{Cl}f^{-1}(U_y)=\mathrm{Int}f^{-1}(Y-U_y)\in\xi$. On the other hand $\mathrm{Int}f^{-1}(\mathrm{Cl}V_y)\in\xi$ for each neighbourhood V_y by 2.1 (b). Thus

$$\mathrm{Int}f^{-1}(Y-U_y)\cap\mathrm{Int}f^{-1}(\mathrm{Cl}V_y)=\mathrm{Int}f^{-1}(\mathrm{Cl}V_y-U_y)$$

is an element of ξ for each V_y, which means $\mathcal{F}(U_y,f)\subset\xi$.

($\Leftarrow$). To prove the converse implication, observe that from $\mathcal{F}(U_y,f)\subset\xi$ it follows, in particular, that $\mathrm{Int}f^{-1}(\mathrm{Cl}V_y)\in\xi$ for each V_y, whence, by 2.1 (a), each $\mathrm{Cl}V_y$ contains some $f(U)$, U being an element of ξ. Thus there is no point y' in $\bigcap\{\mathrm{Cl}f(U)\colon U\in\xi\}$ — in the opposite case $y'\in\mathrm{Cl}V_y$ for each V_y and, in consequence, $y=y'$, which is a contradiction to $y\notin\bigcap\{\mathrm{Cl}f(U)\colon U\in\xi\}$.

Finally, observe that f^* is not continuous at f-non-proper ultrafilters of $\tau X-X$. Since an f-non-proper ultrafilter ξ contains the filter $\mathcal{F}(U_y,f)$ for some neighbourhood U_y of $y=f^*(\xi)$, in particular $U=\mathrm{Int}f^{-1}(\mathrm{Cl}U_y-U_y)\in\xi$. Thus $U_\xi=\{\xi\}\cup U$ is an open neighbourhood of ξ and for each basic open neighbourhood $U'_\xi=\{\xi\}\cup U'$ we have $f^*(U'_\xi)\supset f^*(U'_\xi\cap U_\xi)=\{y\}\cup f(U'\cap U)\not\subset U_y$, the last inequality being a consequence of $f(U)\subset Y-U_y$ and $U'\cap U\neq\emptyset$. Thus $f^*(U'_\xi)\not\subset U_y$ for each basic U'_ξ, which contradicts the continuity of f^* at ξ.

Now pass to the proof of the announced theorem:

4.7. *Each pseudo-proper map $f\colon X\to Y$ possesses a θ-continuous extension $f^*\colon\tau X\to Y$.*

Proof. To define the map f^* on the points of $\tau X-X$, take the decomposition of the remainder into disjoint subsets $T_{[y]}$, $[y]$ running over the f-symmetry classes (these sets are disjoint by 4.4) and $T=(\tau X-X)-\bigcup T_{[y]}$. By 4.6, the elements of T are exactly the f-proper ultrafilters. Thus we define

$$(20)\qquad f^*(\xi)=\bigcap\{\mathrm{Cl}f(U)\colon U\in\xi\}\qquad\text{for } \xi\in T.$$

Now, extend f^* to the whole τX by the formula

$$(21)\qquad f^*(T_{[y]})=y,\ y \text{ being an arbitrary but fixed representative of } [y].$$

Each ultrafilter $\xi\in T_{[y]}$ contains $\hat{\mathcal{F}}(y,f)$ and thus all sets $\mathrm{Int}f^{-1}(\mathrm{Cl}U_y)$, and this is by 2.1. equivalent to $y\in\bigcap\{\mathrm{Cl}U\colon U\in\mathcal{U}(\xi)\}$. But $y=f^*(\xi)$ by (21), and thus from (20) and (21) we conclude that f^* is an extension of the form (19). It follows from 4.5 that f^* is θ-continuous (and even continuous) at $X\cup T$. To end the proof, check θ-continuity at $\xi\in T_{[y]}$.

The point y being f-non-proper, we may restrict the considerations to the basis of open neighbourhoods of y, consisting of f-non-proper neighbourhoods. Let U_y be an arbitrary neighbourhood from this basis. It remains to find a neighbourhood U_ξ of ξ such that

$$f^*(\mathrm{Cl}\,U_\xi) \subset \mathrm{Cl}\,U_y \,. \tag{22}$$

Take a neighbourhood $\widetilde{U}_y$ according to (17); consequently, each $y' \notin [y]$ has a neighbourhood $U_{y'}$ with

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\,\widetilde{U}_y \cap \mathrm{Cl}\,U_{y'}) = \emptyset \,. \tag{23}$$

It may be assumed that $U_y \subset \widetilde{U}_y$.

Now, as $\xi \in T_{[y]}$, there exists a V_y for which $\mathcal{F}(V_y, f) \subset \xi$; thus $U = \mathrm{Int} f^{-1}(\mathrm{Cl}\,U_y - V_y)$ is an element of ξ.

Take $U_\xi = \{\xi\} \cup U$. Then $\mathrm{Cl}\,U_\xi = \mathrm{Cl}_X U \cup \mathrm{Cl}\,U_\xi \cap (\tau X - X)$. We show that an ultrafilter of $\mathrm{Cl}\,U_\xi \cap (\tau X - X)$ cannot be carried to $Y - \mathrm{Cl}\,U_y$ under f^*, i.e.

$$f^*\big(\mathrm{Cl}\,U_\xi \cap (\tau X - X)\big) \subset \mathrm{Cl}\,U_y \,. \tag{24}$$

To prove this, take an arbitrary point $y' \in (Y - \mathrm{Cl}\,U_y) - [y]$. Then there exists a $U_{y'}$ with

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\,U_y \cap \mathrm{Cl}\,U_{y'}) = \emptyset \tag{25}$$

for an f-proper y' it is the neighbourhood $U_{y'}$ chosen for the neighbourhood $Y - \mathrm{Cl}\,U_y$ of y', which satisfies

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\,U_{y'}) \cap \mathrm{Cl} f^{-1}(Y - \mathrm{Cl}\,U_y) = X - \mathrm{Int} f^{-1}(\mathrm{Cl}\,U_y);$$

for an f-non-proper point y' of $Y - \mathrm{Cl}\,U_y$, not f-tangent to y, it is the neighbourhood $U_{y'}$ from (23) (recall that $U_y \subset \widetilde{U}_y$).

Now, suppose, on the contrary, that there exists an ultrafilter $\bar{\xi}$ of $\mathrm{Cl}\,U_\xi \cap (\tau X - X)$, i.e. an ultrafilter containing U and such that

$$f^*(\bar{\xi}) = \bar{y} \in Y - \mathrm{Cl}\,U_y \,. \tag{26}$$

Then, since f^* satisfies (19), we have

$$\mathrm{Int} f^{-1}(\mathrm{Cl}\,U_{\bar{y}}) \in \xi \quad \text{for each } U_{\bar{y}} \,. \tag{27}$$

Suppose that $\bar{y} \in (Y - \mathrm{Cl}\,U_y) - [y]$. Then (25) together with (27) imply that ξ contains disjoint sets — a contradiction.

Thus $\bar{y} \notin (Y - \mathrm{Cl}\,U_y) - [y]$, or equivalently

$$\bar{y} \in \mathrm{Cl}\,U_y \cup [y] \,. \tag{28}$$

From (26) and (27) it follows that $\bar{y} \in [y]$. Since $\bar{y} = f^*(\bar{\xi}) \in [y]$, it follows from (21) of the definition of f^* that $\bar{y} = y$. But on account of (26), $\bar{y} \in Y - \mathrm{Cl}\,U_y$ — a contradiction.

Thus (24) is proved.

Now, to prove (22), observe that for the chosen $U_\xi = \{\xi\} \cup \mathrm{Int} f^{-1}(\mathrm{Cl}\,U_y - V_y)$ we have

$$f^*(\mathrm{Cl}\,U_\xi) = f^*(X \cap \mathrm{Cl}\,U_\xi) \cup f^*\big((\tau X - X) \cap \mathrm{Cl}\,U_\xi\big) \,.$$

But

$$f^*(X \cap \mathrm{Cl}\,U_\xi) = f\big(\mathrm{Cl}(U_\xi \cap X)\big) = f(\mathrm{Cl}\,U) = f\big(\mathrm{Cl}\,\mathrm{Int} f^{-1}(\mathrm{Cl}\,U_y - V_y)\big) \subset \mathrm{Cl}\,U_y,$$

which implies together with (24) that $f^*(\mathrm{Cl}\,U_\xi) \subset \mathrm{Cl}\,U_y$ and (22) is proved. This ends the proof of the θ-continuity of f^*.

Remark. It is proved, in fact, that each map f^* of the form (19), sending $T_{[y]}$ into a fixed point of $[y]$ is a θ-continuous extension of the pseudo-proper map f. Thus a pseudo-proper map $f\colon X \to Y$ has a unique extension $\tau f\colon \tau X \to Y$ iff each f-symmetry class is a one-point set. However, a pseudo-proper map having one-point f-symmetry classes only is proper, which may easily be deduced from the definition (condition (17) implies that f is a Urysohn map, and thus, in this case, a proper map).

It has been proved, as an example, that each map $f\colon X \to \overline{Y}$ of a Hausdorff space X into the minimal space $\overline{Y}$ is pseudo-proper. A non-proper map $f\colon X \to \overline{Y}$ has one f-symmetry class S, consisting of points $(-1, 0)$ and $(1, 0)$ and thus there are two extensions of f, one of them sending the set T_S into $(-1, 0)$, the second one into $(1, 0)$ (these are the only extensions of f). As a corollary it follows that a map $f\colon X \to \overline{Y}$ has a unique θ-continuous extension on τX iff it is proper (and then τf is continuous). The space P has the same property.

A minimal space Y is called *θ-extensor* if each map $f\colon X \to Y$ of a Hausdorff space X into Y has a θ-continuous extension $f^*\colon \tau X \to Y$ on τX. A *proper θ-extensor* is a θ-extensor for which f^* is unique only for proper f.

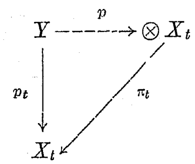
The Tychonoff product of θ-extensors (proper θ-extensors) is a θ-extensor (proper θ-extensor). For the proof some categorial considerations are necessary.

Let $\theta\mathcal{M}$ denotes the category of minimal Hausdorff spaces and all their θ-continuous maps. The product, in the categorial sense, (see [6]) of a family $\{X_t\colon t \in T\}$ of spaces from $\theta\mathcal{M}$ is a space $\otimes X_t$ from $\theta\mathcal{M}$ together with a family of θ-continuous maps $\pi_t\colon \otimes X_t \to X_t$ (called projections) which has the following property:

($\otimes$) for each Y from $\theta\mathcal{M}$ and each family of θ-continuous maps $p_t\colon Y \to X_t$ there exists a unique θ-continuous map $p\colon Y \to \otimes X_t$ such that

$$\pi_t \circ p = p_t \quad \text{for each } t \in T \,, \tag{29}$$

i.e. the following diagram



is commutative.

$\theta\mathcal{M}$ is a category with products, i.e.

4.8. *For each family $\{X_t: t \in T\}$ of spaces from $\theta\mathcal{M}$ there exists a product $\otimes X_t$ — it is simply the Tychonoff product of the spaces X_t.*

Proof. It is well known that the Tychonoff product of minimal spaces is a minimal space. The projections $\pi_t: \otimes X_t \to X_t$, being continuous, are morphisms from $\theta\mathcal{M}$.

To prove $(\otimes)$, let $p_t: Y \to X_t$ be θ-continuous maps. There exists a unique map $Sp: SY \to S\otimes X_t$ of the underlying sets SY and $S\otimes X_t$ of the spaces Y and $\otimes X_t$ for which diagram (29) commutes. To prove that $p: Y \to \otimes X_t$ is θ-continuous, take an arbitrary $y \in Y$ and an arbitrary open basic neighbourhood $U = \pi_{t_1}^{-1}(U_1) \cap \ldots \cap \pi_{t_n}^{-1}(U_n)$ of $p(y)$, U_k being an arbitrary open subset of X_{t_k} for $k = 1, \ldots, n$.

From $\pi_t \circ p = p_t$ it follows that $p_t(y) = \pi_t(p(y)) \in \pi_t(U)$; thus $p_{t_k}(y) \in \pi_{t_k}(U) = U_k$ for $k = 1, \ldots, n$. The maps p_t being θ-continuous there exist neighbourhoods U_y^k of y such that

(30) $$p_{t_k}(\mathrm{Cl}\, U_y^k) \subset \mathrm{Cl}\, U_k \quad \text{for} \quad k = 1, \ldots, n.$$

Take $U_y = U_y^1 \cap \ldots \cap U_y^n$ and calculate that $p(\mathrm{Cl}\, U_y) \subset \mathrm{Cl}\, U$, which means that p is θ-continuous.

Calculate:

(31) $$(\pi_{t_k} \circ p)(\mathrm{Cl}\, U_y) = \pi_{t_k}\big(p(\mathrm{Cl}\, U_y)\big) = p_{t_k}(\mathrm{Cl}\, U_y) \subset p_{t_k}(\mathrm{Cl}\, U_y^k) \subset \mathrm{Cl}\, U_k \quad \text{for} \quad k = 1, \ldots, n$$

(the second equality is a consequence of $\pi_t \circ p = p_t$, the first inclusion follows from the definition of U_y and the second from (30)).

Thus

(32) $$p(\mathrm{Cl}\, U_y) \subset \pi_{t_k}^{-1}(\mathrm{Cl}\, U_k) \quad \text{for each } k = 1, \ldots, n,$$

and since

$$\pi_{t_1}^{-1}(\mathrm{Cl}\, V_1) \cap \ldots \cap \pi_{t_n}^{-1}(\mathrm{Cl}\, V_n) = \mathrm{Cl}\big(\pi_{t_1}^{-1}(V_1) \cap \ldots \cap \pi_{t_n}^{-1}(V_n)\big)$$

for each open subsets $V_k \subset X_{t_k}$, it follows from (32) that

$$p(\mathrm{Cl}\, U_y) \subset \pi_{t_1}^{-1}(\mathrm{Cl}\, U_1) \cap \ldots \cap \pi_{t_n}^{-1}(\mathrm{Cl}\, U_n) = \mathrm{Cl}\big(\pi_{t_1}^{-1}(U_1) \cap \ldots \cap \pi_{t_n}^{-1}(U_n)\big) = \mathrm{Cl}\, U;$$

thus p is θ-continuous and the theorem is proved.

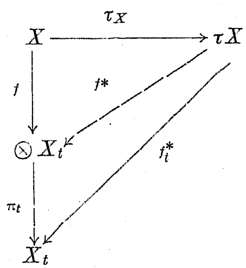
Remark. The theorem remains valid when $\mathcal{M}$ is replaced by an arbitrary subcategory $\mathcal{C}'$ of the category $\mathcal{C}$ of all topological spaces and all their continuous maps whenever $\mathcal{C}'$ is closed under the formation of the Tychonoff product (in this case $\theta\mathcal{C}'$ denotes the category consisting of spaces from $\mathcal{C}'$ and all their θ-continuous maps) — this assumption is true for $\mathcal{C}' = \mathcal{M}$ and this is the only step of the proof which makes use of the form of $\mathcal{M}$.

Now, by an easy categorial argument, we prove that

4.9. *The product of θ-extensors is a θ-extensor.*

Proof. Take an arbitrary map $f: X \to \otimes X_t$ into the product of θ-extensors X_t. The map $f_t = \pi_t \circ f: X \to X_t$ can be extended to a θ-continuous $f_t^*: \tau X \to X_t$ for each t.

(*)



The maps f_t^* induce, in view of 4.8, a θ-continuous $f^*: \tau X \to \otimes X_t$, such that $\pi_t \circ f^* = f_t^*$. Observe that f^* is an extension of f: $\pi_t \circ f^* \circ \tau_X = f_t^* \circ \tau_X = \pi_t \circ f$ for each t, and thus $f^* \circ \tau_X = f$.

4.10. *The product of proper θ-extensors is a proper θ-extensor.*

Proof. Let $f: X \to \otimes X_t$ be an arbitrary non-proper map, X_t being proper θ-extensors. Then, in view of 3.4, there exist different points $x = \{x_t\}$ and $x' = \{x_t'\}$ of $\otimes X_t$ such that

(33) $$\mathrm{Int} f^{-1}(\mathrm{Cl}\, U_x) \cap \mathrm{Int} f^{-1}(\mathrm{Cl}\, U_{x'}) \neq \emptyset \quad \text{for each open neighbourhood } U_x \text{ and } U_{x'}.$$

In particular, $\mathrm{Cl}\, U_x \cap \mathrm{Cl}\, U_{x'} \neq \emptyset$ for each U_x and $U_{x'}$ and thus for some $\bar{t}$ the points $x_{\bar{t}} = \pi_{\bar{t}}(x)$ and $x_{\bar{t}}' = \pi_{\bar{t}}(x')$ cannot be separated by closed disjoint neighbourhoods in $X_{\bar{t}}$ (for otherwise x and x' could be separated by closed disjoint neighbourhoods — in consequence of the formula $\pi_{\bar t}^{-1}(\mathrm{Cl}\,U) = \mathrm{Cl}\,\pi_{\bar t}^{-1}(U)$ applied to disjoint closed neighbourhoods $\mathrm{Cl}\,U_t$ and $\mathrm{Cl}\,U_{t'}$ of x_t and x'_t, which in this case exist since $x_t \neq x'_t$ for some t).

Let U and U' be arbitrary open neighbourhoods of $x_{\bar t}$ and $x'_{\bar t}$. Then

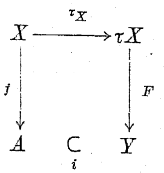
$$\mathrm{Int}(\pi_{\bar t}\circ f)^{-1}(\mathrm{Cl}\,U \cap \mathrm{Cl}\,U') = \mathrm{Int}f^{-1}(\pi_t^{-1}(\mathrm{Cl}\,U \cap \mathrm{Cl}\,U'))$$
$$= \mathrm{Int}f^{-1}(\mathrm{Cl}\,\pi_t^{-1}(U) \cap \mathrm{Cl}\,\pi_t^{-1}(U')) \neq \emptyset$$

the inequality being a consequence of (33) applied to neighbourhoods $\pi_{\bar t}^{-1}(U)$ and $\pi_{\bar t}^{-1}(U')$ of x and x'. Thus $\pi_{\bar t}\circ f$ is not a proper map, since it was shown to be non-Urysohn at points $x_{\bar t}$ and $x'_{\bar t}$.

Now, $\pi_{\bar t}\circ f$ being a non-proper map into the proper θ-extensor $X_{\bar t}$, there exist two different θ-continuous extensions $f^*_{\bar t}$ and $f^{**}_{\bar t}$ of f on τX. For $t \neq \bar t$ let f^*_t denote an arbitrary θ-continuous extension of f. Then the maps f^* and f^{**} of the diagram $(*)$ induced by $\{f^*_t, f^*_{\bar t}\}$ and $\{f^*_t, f^{**}_{\bar t}\}$ are different extensions of f. The map f being, by assumption, an arbitrary non-proper map $f\colon X \to \bigotimes X_t$, this proves that $\bigotimes X_t$ is a proper θ-extensor.

4.11. *An H-closed subspace of a θ-extensor (proper θ-extensor) is a θ-extensor (proper θ-extensor).*

Proof. Let A be an H-closed subspace of a θ-extensor Y and let $f\colon X \to A$ be an arbitrary map. Then there exists a θ-continuous extension $F\colon \tau X \to Y$ of the composite map $i \circ f$:



Prove that $F(\tau X) \subset A$. Suppose, on the contrary, that there exists a point $\xi \in \tau X$ for which $F(\xi) \notin A$ (ξ is a point of the remainder, since $F(X) = f(X) \subset A$). Since F is a θ-continuous extension of $i \circ f$ on τX, we conclude from 3.2 that $\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y) \in \xi$ for each neighbourhood of $y = F(\xi)$. But A being an H-closed subspace of Y, there exist a neighbourhood U_y such that $\mathrm{Cl}\,U_y \cap A$ is nowhere-dense in A (see [5]); thus ξ contains $\mathrm{Int}f^{-1}(\mathrm{Cl}\,U_y) = \emptyset$ — a contradiction.

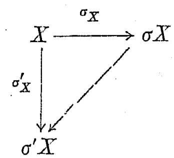
Since $F(\tau X) \subset A$, this implies the existence of a θ-continuous extension of f on τX for an arbitrary map f, and so A is proved to be a θ-extensor.

To prove that $A \subset Y$ is a proper θ-extensor whenever Y is such, observe that the map $i \circ f$ is not proper, whenever f is not; thus there exist two different extensions F and F' of $i \circ f$ on τX since Y is a proper θ-extensor. The non-proper f having at least two extensions, A is proved to be a proper θ-extensor.

Remark. Since it is not known whether there exists a universal minimal space playing the same role as the Tychonoff cube in the theory of completely regular spaces and their compact extensions, it cannot be deduced from 4.9 and 4.11 that each minimal space is a θ-extensor. On the other hand, it is not known whether there exist minimal spaces which are not θ-extensors. By 4.9 and 4.11 such a space cannot be found in the product $\bigotimes \overline{Y}_t$ of copies of $\overline{Y}$, which is a θ-extensor, even a proper θ-extensor, as shown before. Also W and P, the former considered minimal spaces, can be proved to be proper θ-extensors.

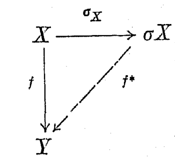
5. A categorial description of σX. In [1] an extension of a semi-regular space X to a minimal space σX was constructed (only semi-regular spaces possess minimal extensions) and it was proved there that $\sigma X = \mu\tau X$. The extension σX turned out not to be maximal in the usual sense, i.e. the diagram

(σ)



where $\sigma' X$ is an arbitrary minimal extension of X, cannot be completed by a continuous map $\sigma X \to \sigma' X$, in general. In [5], th. 6.1, the maximality of σX with respect to θ-continuous maps was proved; more precisely, the diagram

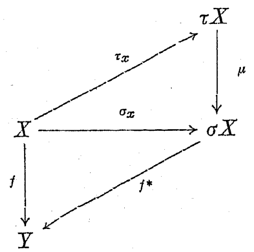
(θσ)



where f is a proper map into a minimal Hausdorff space Y can be completed by a θ-continuous map f^*. In particular, σX proves to be a θ-maximal extension of X to a minimal space in the sense of diagram (σ), where $\sigma X \to \sigma' X$ is assumed to be θ-continuous only, since dense embeddings are proper maps. However, the completing map f^* of $(\theta\sigma)$ has not been proved to be unique and the question was raised in [5] whether this is true. In the following theorem a positive answer is given.

5.1. *There exists a unique map f^* completing diagram* ($\theta\sigma$).

Proof. Since $\sigma X = \mu\tau X$, the map μf^* of f^* from diagram ($\theta\sigma$) is a θ-continuous extension of f on τX, as may be seen in the following diagram



By 3.2 and 3.1 (c) there is only one θ-continuous extension of f on τX, the set $\bigcap\{\mathrm{Cl}\,U\colon U \in \mathfrak{U}(\xi)\}$ being a one-point set for each $\xi \in \tau X - X$ for the proper map f. Thus f^* is unique, μ being the identity on the underlying sets $S\tau X = S\sigma X$.

Thus σX can be characterized by the following property: for each proper $f\colon X \to Y$ there exists a unique θ-continuous map σf completing diagram ($\theta\sigma$), since it has been proved in [5] that the maximality of σX with respect to θ-continuous maps, described by ($\theta\sigma$) even without uniqueness, characterizes σX.

6. Weakly θ-continuous extensions of maps on τX. The question arises whether each map $f\colon X \to Y$ of a Hausdorff space X into H-closed space Y has an extension on τX which is "continuous-like" in a sense less restrictive than θ-continuity. A natural generalization of both continuity and θ-continuity is weak θ-continuity (a notion due to Fomin [3]) also defined as follows:

A map $f\colon X \to Y$ is called *weakly θ-continuous* if for each $x \in X$ and for each open neighbourhood U_y of $y = f(x)$ there exists an open neighbourhood U_x of x such that $f(U_x) \subset \mathrm{Cl}\,U_y$.

6.1. *Each map $f\colon X \to Y$ of a Hausdorff space X into an H-closed space Y possesses a weakly θ-continuous extension $f^*\colon \tau X \to Y$ on τX. The extension is unique iff f is a Urysohn map.*

Proof. It is easy to verify that each extension of the form (19) is weakly θ-continuous. By 4.5 it suffices to prove that f^* is weakly θ-continuous at points of $\tau X - X$. To prove this, let U_y be an arbitrary neighbourhood of $y = f^*(\xi)$, ξ being an arbitrary point of $\tau X - X$. Then $\mathrm{Int}\,f^{-1}(\mathrm{Cl}\,U_y)$ is an element of ξ and for $U_\xi = \{\xi\} \cup \mathrm{Int}\,f^{-1}(\mathrm{Cl}\,U_y)$ we have $f^*(U_\xi) = f(\mathrm{Int}\,f^{-1}(\mathrm{Cl}\,U_y)) \cup \{f^*(\xi)\} \subset \mathrm{Cl}\,U_y$, which means that f^* is weakly θ-continuous at ξ.

A map f has a unique weakly θ-continuous extension iff all sets $\bigcap\{\mathrm{Cl}\,U\colon U \in \mathfrak{U}(\xi)\}$ are one-point sets. By 3.3 this is the case iff f is a Urysohn map.

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