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Reçu par la Rédaction le 28. 4. 1970

The space of rationals is not absolutely paracompact

by

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A Hausdorff space X is said to be *paracompact* (*metacompact*) if every open covering of X has a locally finite (point-finite) open refinement.

A Hausdorff space X is said to be *totally paracompact* (*totally metacompact*) if every open base of X contains a locally finite (point-finite) covering of X.

A family $\mathcal{B}$ of open sets in X is said to be an *outer base* of $Y \subset X$ if, for each $y \in Y$ and each open set G in X, such that $y \in G$ there exists a $B \in \mathcal{B}$ such that $y \in B \subset G$.

We call a subset Y of X *totally paracompact with respect to* X if every outer base of Y in X contains a locally finite (with respect to X) covering of Y.

It is easy to prove that if Y is totally paracompact with respect to $X \supset Y$, then Y is a totally paracompact subspace of X. A paracompact space X is said to be *absolutely paracompact* if, for every paracompact space Y such that X is embedded into Y as a closed subspace, X is totally paracompact with respect to Y.

For results on totally paracompact spaces we refer to [2] and [5].

In this paper we will prove that a space of E. Michael [3] is not totally metacompact and that the space of the rationals is not absolutely paracompact. It is known that the space of the rationals is totally paracompact, and that the space of the irrationals is paracompact but not totally paracompact (cf. [2]).

Let ω^ω denote the Baire space of sequences of non-negative integers. It is well known that ω^ω is homeomorphic to the space of all irrational numbers (cf. [4], p. 143).

Let

$$D = \{f \in \omega^\omega \mid \exists n\colon \forall k \geqslant n\colon f(k) = 0\}.$$

Then D is dense in ω^ω and D is homeomorphic to the space of rationals.

Let Φ be the set of all finite sequences of non-negative integers. For $\varphi = \{\varphi(i)\}_{i=0}^{n} \in \Phi$ and $k \in \omega$ we define

$$l(\varphi) = n+1\,,$$

$$B_\varphi = \{f \in \omega^\omega |\ \forall m < l(\varphi)\colon f(m) = \varphi(m)\}\,,$$

$$D_\varphi = D \cap B_\varphi \quad \text{and} \quad \overline{\varphi k} = \{\psi(i)\}_{i=0}^{n+1} \in \Phi\,,$$

where $\psi(i) = \varphi(i)$, for $i = 0, 1, \ldots, n$ and $\psi(n+1) = k$.

For φ and ψ in Φ we define

$$\varphi < \psi \quad \text{iff} \quad l(\varphi) < l(\psi) \text{ and } \forall k < l(\varphi)\colon \varphi(k) = \psi(k)\,.$$

Now let S be the space consisting of the same elements as ω^ω and having the following (finer) topology: *a subset of S is open iff it is of the form $U \cup V$, where U is open in ω^ω and $V \subset \omega^\omega \setminus D$.*

It is known that the space S is paracompact (cf. [1], p. 216).

It is easy to see that the relative topology of D is the same both in ω^ω and in S.

The family

$$\mathcal{B}_0 = \{B_\varphi |\ \varphi \in \Phi\}\,,$$

is an outer base for D in S, because it is an open base for the topology in ω^ω.

Now, for every $k > 0$ and $\varphi \in \Phi$, we put

$$B_\varphi^k = B_{\overline{\varphi 0}} \cup (B_{\overline{\varphi k}} \setminus D) \quad \text{and} \quad \mathcal{B} = \{B_\varphi^k | k > 0 \text{ and } \varphi \in \Phi\}\,.$$

Clearly, every B_φ^k is an open set in S. Moreover, $\mathcal{B}$ is an outer base of D in S. For, if $f \in D \cup B_\varphi$, then there is a $\psi \in \Phi$ such that $\psi > \varphi$, $\psi(k) = f(k)$ for every $k < l(\psi)$ and $f\big(l(\psi)\big) = 0$.

Hence $f \in B_{\overline{\psi 0}} \subset B_\psi^k$ for each $k > 0$ and $B_\psi^k \subset B_\varphi$.

Now we prove the following

LEMMA. *If $\mathcal{A} \subset \mathcal{B}$ is a covering of D in S, then $\mathcal{A}$ is not point-finite, i.e. there exists a point $f \in S$ such that f belongs to infinitely many sets from $\mathcal{A}$.*

Proof. We will define by induction an infinite sequence of sets from $\mathcal{A}$ having a non-void intersection.

Let

$$n_0 = \inf\{l(\varphi)|\ \exists k\colon B_\varphi^k \in \mathcal{A}\}\,.$$

We choose a $B_{\varphi_0}^{k_0} \in \mathcal{A}$ such that $l(\varphi_0) = n_0$. Then obviously

$$D_{\overline{\varphi_0 k_0}} \cap \bigcup \{B_\varphi^k \in \mathcal{A}|\ l(\varphi) < l(\varphi_0)\} = \emptyset.$$

If $D_{\overline{\varphi_0 k_0}} \cap B_\varphi^k \neq \emptyset$ for some $B_\varphi^k \in \mathcal{A}$ with $l(\varphi) = l(\varphi_0)$, then $D_{\overline{\varphi_0 k_0}} \cap B_{\overline{\varphi 0}} \neq \emptyset$, because $D_{\overline{\varphi_0 k_0}} \cap (B_{\overline{\varphi k}} \setminus D) = \emptyset$. But then $\overline{\varphi 0} = \overline{\varphi_0 k_0}$ and hence $k_0 = 0$. This is a contradiction. We have thus shown that

$$D_{\overline{\varphi_0 k_0}} \cap \bigcup \{B_\varphi^k \in \mathcal{A}|\ l(\varphi) \leqslant l(\varphi_0)\} = \emptyset\,.$$

Observe that

$$l(\varphi) > l(\varphi_0) \text{ and } B_\varphi^k \subset B_{\overline{\varphi_0 k_0}} \quad \text{if} \quad B_\varphi^k \in \mathcal{A} \text{ and } B_\varphi^k \cap D_{\overline{\varphi_0 k_0}} \neq \emptyset.$$

Now assume that n_t and $B_{\varphi_t}^{k_t} \in \mathcal{A}$ have been defined in such a way that

$$l(\varphi_t) = n_t \quad \text{and} \quad D_{\overline{\varphi_t k_t}} \cap \bigcup \{B_\varphi^k \in \mathcal{A}|\ l(\varphi) \leqslant l(\varphi_t)\} = \emptyset\,.$$

Now, observe that

$$l(\varphi) > l(\varphi_t) \text{ and } B_\varphi^k \subset B_{\overline{\varphi_t k_t}} \quad \text{if} \quad D_{\overline{\varphi_t k_t}} \cap B_\varphi^k \neq \emptyset \text{ and } B_\varphi^k \in \mathcal{A}\,.$$

Since $D_{\overline{\varphi_t k_t}}$ is covered by $\mathcal{A}$, there is a $B_\varphi^k \in \mathcal{A}$ such that $D_{\overline{\varphi_t k_t}} \cap B_\varphi^k \neq \emptyset$.

Now, let

$$n_{t+1} = \inf\{l(\varphi)|\ \exists k\colon B_\varphi^k \in \mathcal{A} \text{ and } D_{\overline{\varphi_t k_t}} \cap B_\varphi^k \neq \emptyset\}\,,$$

and choose a $B_{\varphi_{t+1}}^{k_{t+1}} \in \mathcal{A}$ such that $l(\varphi_{t+1}) = n_{t+1}$ and $B_{\varphi_{t+1}}^{k_{t+1}} \subset B_{\overline{\varphi_t k_t}}$. It follows from the last inclusion that $\overline{\varphi_{t+1} k_{t+1}} > \varphi_t k_t$, and so we have $D_{\overline{\varphi_{t+1} k_{t+1}}} \subset D_{\overline{\varphi_t k_t}}$.

According to the minimality property of n_{t+1} and the last inclusion, we have

$$D_{\overline{\varphi_{t+1} k_{t+1}}} \cap \bigcup \{B_\varphi^k \in \mathcal{A}|\ l(\varphi) < l(\varphi_{t+1})\} = \emptyset\,.$$

Since we must prove

$$D_{\overline{\varphi_{t+1} k_{t+1}}} \cap \bigcup \{B_\varphi^k \in \mathcal{A}|\ l(\varphi) \leqslant l(\varphi_{t+1})\} = \emptyset\,,$$

it is sufficient to show that

$$D_{\overline{\varphi_{t+1} k_{t+1}}} \cap \bigcup \{B_\varphi^k \in \mathcal{A}|\ l(\varphi) = l(\varphi_{t+1})\} = \emptyset\,.$$

Suppose, to the contrary, that there exists a $B_\varphi^k \in \mathcal{A}$ such that

$$l(\varphi) = l(\varphi_{t+1}) \quad \text{and} \quad D_{\overline{\varphi_{t+1} k_{t+1}}} \cap B_\varphi^k \neq \emptyset\,.$$

Then $D_{\overline{\varphi_{t+1} k_{t+1}}} \cap B_{\overline{\varphi 0}} \neq \emptyset$, because $D_{\overline{\varphi_{t+1} k_{t+1}}} \cap (B_{\overline{\varphi k}} \setminus D) = \emptyset$. But then $\overline{\varphi_{+1}^t k_{t+1}} = \overline{\varphi 0}$, because $l(\varphi) = l(\varphi_{t+1})$. Hence $k_{t+1} = 0$. This is a contradiction.

From our construction we see that

$$\varphi_0 < \varphi_1 < \varphi_2 < \ldots$$

Now let $f \in S$ be defined by

$$f(n) = \varphi_n(n)\,;$$

then

$$f \in \bigcap_{t \in \omega} B_{\varphi_t}^{k_t}.$$

So $\mathcal{A}$ is not point-finite, and the proof of the Lemma is complete.

As consequences of this Lemma we have the following two theorems:

THEOREM 1. *The space S is not totally metacompact.*

In fact, $\mathcal{B}' = \mathcal{B} \cup \{\{x\}|\ x \in S\setminus D\}$ is an open base of S and $\mathcal{B}'$ contains no point-finite covering of S, because, by the Lemma, $\mathcal{B}$ contains no point-finite covering of D.

THEOREM 2. *The space D (which is homeomorphic to the space of the rationals) is not absolutely paracompact.*

This is clear, because D is a closed subspace of the paracompact space S and $\mathcal{B}$ is a "wrong" outer base of D in S.

Remarks. (1) S is a paracompact space with the property that S^d (the set of all limit points of S) is totally paracompact, but S itself is not totally paracompact.

(2) Every C-scattered paracompact space is absolutely paracompact [6]. Every paracompact locally compact space and every $F_\sigma \cap G_\delta$-absolute metrizable space is C-scattered paracompact [6]. Hence, there are many absolutely paracompact spaces.

(3) Since there is a homeomorphism h from ω^ω onto the space of all irrationals in the unit interval I (cf. [4], p. 143), $h(S)$ is a closed subspace of the space $I_{h(D)}$ (cf. [1], p. 216). Since $I_{h(D)}\setminus h(S)$ is a discrete open set, $I_{h(D)}$ is not totally metacompact.

E. Michael (cf. [3] or [1], p. 218) proved that $\omega^\omega \times I_{h(D)}$ is not normal. Here, as we have seen above, neither ω^ω (cf. [2]) nor $I_{h(D)}$ is totally metacompact.

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Reçu par la Rédaction le 12. 5. 1970

A metrization theorem for developable spaces*

by

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1. Preliminaries. The usual approach to the problem of metrizability of developable spaces has been through the use of the rich global properties enjoyed by the developable spaces, and the additional assumption of normality. In [2] the application of a significant local property, the first countability of a developable space, was made in giving another (non-normal) dimension to this problem. The additional global property used in this application was sequential mesocompactness. A family of subsets $\{F_\alpha:\ \alpha \in A\}$ of a space X is said to be cs-*finite* if for each convergent sequence, $\{P_i\}$ in X, $F_\alpha \cap \{P_i:\ i \in N\} \neq \emptyset$ for at most finitely many $\alpha \in A$. Accordingly, a Hausdorff space X is called *sequentially mesocompact* provided: every open covering of X has a cs-finite open refinement.

In this paper, the use of both the local and global properties of developable spaces is made to yield a metrization theorem, Theorem 2.1, for developable spaces which improves both of the following theorems.

THEOREM 1.1 ([1], Theorem 10). *A developable space is metrizable if and only if it is collectionwise normal.*

THEOREM 1.2 ([2], Theorem 4.2). *A developable space is metrizable if and only if it is sequentially mesocompact.*

A non-normal simultaneous generalization of sequentially mesocompact spaces and collectionwise normal spaces is introduced.

DEFINITION. A Hausdorff space X is said to have *property* (ω) if for each discrete collection of closed sets $\{F_\alpha:\ \alpha \in A\}$ in X, there exists a cs-finite collection of open sets $\{G_\alpha:\ \alpha \in A\}$ such that $F_\alpha \subset G_\alpha$, for each $\alpha \in A$ and $G_\alpha \cap F_\beta = \emptyset$, if $\alpha \neq \beta$.

Let $\{F_\alpha:\ \alpha \in A\}$ be any discrete collection of closed sets in a space X. Suppose X is sequentially mesocompact, and consider the open covering $\mathcal{U} = \{X - \bigcup_{\gamma \neq \alpha} F_\gamma:\ \alpha \in A\}$ of X. Let $\mathcal{G} = \{G_\beta:\ \beta \in B\}$ be a cs-finite open

* This work was supported by the Texas Christian University Research Foundation and by Texas A&M University.