for all integers i and j, and so that

$$\{a_1, a_2, \dots\} \subset \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\},$$

and

$$\{a_1, a_2, \dots\} \subset \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}.$$

Hence each of the two sets $\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ and $\{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}$ is dense in U. We now define $\varphi\colon U \to U$ by

$$\varphi(x_i) = y_i, \quad \text{for any integer } i.$$

On the other points of U we define φ by

$$\varphi(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} \varphi(x_{n_i})$$

where $\{x_{n_i}\}_{i\geqslant 1}$ is any Cauchy sequence taken from $\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$. It is easy to see that φ is one to one, maps U onto U, and preserves distances. □

Urysohn [1] gave an example of two bounded isometric subsets A and B of U with the property that no isometry from A onto B can be extended to an isometry from U onto itself. By considering countable dense subsets of A and B we can show that neither Theorem 4 nor Theorem 5 can be extended to arbitrary bounded sequences.

References

[1] P. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. 51 (1927), pp. 43–64 74–90.

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C-scattered and paracompact spaces *

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0. Introduction. The main problem considered in this paper is the problem of the topological product of paracompact spaces (Section 2). C-scattered spaces, which play an important rôle in this problem, are studied in Section 1 and some strong covering properties of C-scattered paracompact spaces are proved in Section 3. The results from this paper were partially announced in [25].

Each topological space considered in this paper is assumed to be completely regular.

The problem in a general setting reads as follows: what kind of separation and covering properties are preserved by the Cartesian product of finitely many spaces?

The Cartesian product of two normal (even paracompact) spaces need not be normal ([11], [19]). But, as J. Dieudonné [1] proved, the product $S\times T$ of a paracompact space S and a compact space T is always paracompact and hence normal. An excellent result of H. Tamano [23] reads: a completely regular space S is paracompact iff $S\times\beta S$ is normal. K. Morita [13] proved that if S is paracompact and such that each point has a nbd basis of the cardinality $\leqslant m$ and T is an m-compact normal space, then $S\times T$ is normal. This phenomenon appeared earlier also in the product of two $\aleph_0$-compact spaces, as is explicitly stated in the following Theorem of C. Ryll-Nardzewski [17]: if S is $\aleph_0$-compact and such that each point has a nbd basis of the cardinality $\leqslant m$ and T is m-compact, then $S\times T$ is $\aleph_0$-compact. An assumption concerning the cardinality of a basis plays an essential role also in the Product Theorem of K. Morita [13]: S is a normal m-paracompact space iff $S\times[0,1]^m$ is

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normal. H. Tamano [23] stated the following problem: to give an intrinsic topological description of the spaces S such that the product $S\times T$ is paracompact for any paracompact space T. Let us denote by Π the class of such S's. K. Morita [14] proved that if S is metrizable and $S\in\Pi$, then S is σ-locally compact; he also proved that if S is paracompact and σ-locally compact, then $S\in\Pi$. However, T. Ishii [5] has shown that there is a space in Π with one limit point only which is not σ-locally compact. The following Theorem of Z. Frolík [3] is very strange: the product of countably many paracompact spaces which are complete in the sense of Čech is paracompact and complete in the sense of Čech. On the other hand, from a theorem of K. Morita [14] it follows that if S is complete in the sense of Čech and $S\in\Pi$, then S is σ-locally compact. But every space that is complete in the sense of Čech and σ-locally compact must be C-scattered; this easily follows from the Baire Category Theorem. C-scattered spaces coincide with resolvable sets in compact Hausdorff spaces (Theorem 1.2). The class of all C-scattered spaces is perfect (Theorem 1.3) and it contains all locally compact and all scattered spaces. K-scattered spaces, where K is a class of spaces, were considered by A. H. Stone [21]. Product Theorem 2.1 implies also a result of J. Suzuki [22], Theorem 3 (here: Corollary 2.1) and Product Theorem 2.4 generalizes all known results concerning Π (Corollary 2.3, 2.4, 2.5 and Theorem of Katuta), but it is not yet a final solution of the problem of H. Tamano [23]. Every paracompact C-scattered space is absolutely paracompact (Theorem 3.1); however, the space of all rationals is not absolutely paracompact [7].

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1. C-scattered spaces. The are several important hierarchies of sets in a given basic topological space (e.g. Borelian, analytic and projective). Here a hierarchy of C-scattered spaces is investigated; the concept of a C-scattered space is a natural generalization of a locally compact and scattered space. The idea of a classification of scattered-type spaces belongs to G. Cantor and it is described in F. Hausdorff [4] and in A. H. Stone [21].

A space S is said to be *scattered* (see [8], p. 78) if every closed subspace $R\neq 0$ has an isolated point $r\in R$. A space S is said to be *C-scattered* if every closed subspace $R\neq 0$ has a point $r\in R$ with a compact nbd in R. Clearly, every scattered space is C-scattered and every locally compact space also is C-scattered. The *C-derivative* R^* of a closed subset R of S is the set of all $r\in R$ without a compact nbd in R. Clearly, R^* is closed in R, and also in S. Hence the derivative operation $*$ has well-defined transfinite iterations:

$$R^{(0)}=R,\quad R^{(\alpha+1)}=(R^{(\alpha)})^*\quad\text{and}\quad R^{(\lambda)}=\bigcap_{\alpha<\lambda}R^{(\alpha)}$$

for the ordinal limit number λ.

The set $\hat{R}=\bigcap_{\alpha}R^{(\alpha)}$ is said to be the *C-kernel* of R. The rank $d(p)$ of a point p in a C-scattered space S is defined by the following condition: $d(p)=\alpha$ iff $p\in S^{(\alpha)}-S^{(\alpha+1)}$.

It is easy to verify that the following statements are equivalent:

(1) S is C-scattered,

(2) $S^{(\alpha)}=0$ for some ordinal α,

(3) $\hat{S}=0$,

(4) $\hat{R}=0$ for every closed set $R\subset S$,

(5) R^* is nowhere-dense in R for every closed set R in S,

(6) there is a transfinite decreasing sequence $\{F_\xi\colon \xi\leqslant\alpha\}$ of closed sets in S such that $F_0=S$, $F_\alpha=0$ and $F_\lambda=\bigcap_{\xi<\lambda}F_\xi$ for the ordinal limit number $\lambda\leqslant\alpha$, and $F_\xi-F_{\xi+1}$ is locally compact for each $\xi<\alpha$.

In the proof of the former equivalences it is useful to have the following property: if $R_1\subset R_2\subset S$ and R_1 and R_2 are closed in S, then $R_1^{(\alpha)}\subset R_2^{(\alpha)}$ for each α and, in particular, if R_2 is C-scattered, then R_1 is also C-scattered.

Let us remark that every open and obviously every closed subspace of a C-scattered space is C-scattered too. This fact is a consequence of a more general

THEOREM 1.1. *The family of all C-scattered subspaces of a space S is a ring (with the usual set-theoretical addition and subtraction).*

The proof follows from the following two lemmas:

LEMMA 1.1. *If $T=R\cup S$, where R and S are C-scattered, then T is C-scattered.*

Proof. Suppose F is a closed subspace of T and $F^*=F$. Without loss of generality we can assume that $F=T$, because $F\cap R$ and $F\cap S$ are C-scattered as closed subspaces in R and S, respectively, and $F=(F\cap R)\cup(F\cap S)$. If $R\neq 0$, then there is an open set U in T such that $\overline{U}\cap R$ is compact and non-void. So $\overline{U}\cap R$ is nowhere-dense in T, because $T^*=T$. Hence $V=U-(\overline{U}\cap R)$ is open in T and $V\neq 0$. But $V\subset S$, so V is open also in S and therefore V contains points of local compactness, because S is C-scattered. This contradicts $T^*=T$.

LEMMA 1.2. *If $T=S-R$, where R and S are C-scattered, then T is C-scattered.*

Proof. Let F be a closed subspace of T and H the closure of F in S. Then $F = H\cap T = H\cap (S-R) = (H\cap S)-R = (H\cap S)-(H\cap R)$, where $H\cap S$ and $H\cap R$ are C-scattered as closed subspaces of S and R, respectively. So, without loss of generality we can assume that $F = T$ and $H = S$. If $S-S^*\subset T$, then T has points of local compactness, because then $S-S^*$ is locally compact and open in T. Let $(S-S^*)\cap (S-T)\neq 0$. Since $S-T = S-(S-R) = R$, then $(S-S^*)\cap R$ is open in R. Since R is C-scattered, then $(S-S^*)\cap R$ contains a point p such that there is an open set U in S such that $p\in U$ and $\overline{U}\cap R$ is compact. Since $p\in S-S^*$, then there is an open nbd V of p in S such that $\overline{V}$ is compact. Let us put $W = U\cap V$. Then $\overline{W}$ and $\overline{W}\cap R$ are both compact. Clearly, $W\cap T = W-R = W-(\overline{W}\cap R)\subset T$. If $W-R = 0$, then $W\subset T$ and hence T has points of local compactness. If $W-R\neq 0$, then $W-(\overline{W}\cap R)$ is a locally compact non-void open subset of T and S too. Thus T has points of local compactness.

COROLLARY 1.1. *If S is a C-scattered space, then the family of all C-scattered subspaces of S is a Boolean Algebra (with the usual set-theoretical addition and complementation).*

A set R in a space T is said to be *resolvable* (see [8], p. 96), if there is a decreasing transfinite sequence $\{F_\alpha\colon\ \alpha<\beta\}$ of closed sets in T such that $R = (F_0-F_1)\cup (F_2-F_3)\cup \ldots$

If R is a resolvable set in some compact space T, then R is C-scattered. Indeed, if F is a closed subset in R, then $F = 0$ or there is a minimal α such that $F\cap (F_\alpha - F_{\alpha+1})\neq 0$. Clearly, every $p\in F\cap (F_\alpha - F_{\alpha+1})$ is a point of local compactness of F. The converse statement is in the following

THEOREM 1.2. *If R is a C-scattered set in a space S and T is a compactification of S, then R is resolvable in T, as well as being resolvable in S.*

Proof. It suffices to prove that R is resolvable in S. Let $R^{(\alpha)}$ be the αth iteration of the C-derivative of R in R. Since R is C-scattered, we have $R = \bigcup_{\alpha<\beta}(R^{(\alpha)}-R^{(\alpha+1)})$, for some ordinal β. It is easy to verify the following equality

$$R^{(\alpha)}-R^{(\alpha+1)} = \overline{R^{(\alpha)}-R^{(\alpha+1)}} - \left(\overline{R^{(\alpha)}-R^{(\alpha+1)}} - (R^{(\alpha)}-R^{(\alpha+1)})\right).$$

Let us put $F_\alpha = \overline{R^{(\alpha)}-R^{(\alpha+1)}}$ and $H_\alpha = \overline{R^{(\alpha)}-R^{(\alpha+1)}} - (R^{(\alpha)}-R^{(\alpha+1)})$. Since $R^{(\alpha)}-R^{(\alpha+1)}$ is locally compact, then it is locally closed in S (for locally closed sets see [8], p. 65). Hence H_α is closed in S (see [8], p. 65, Corollary). Now we have $R = \bigcup_{\alpha<\beta}(F_\alpha - H_\alpha)$. From the definition of F_α and H_α we see that $H_\alpha\subset F_\alpha$. To see that $F_{\alpha+1}\subset H_\alpha$, we will prove at first that $\overline{R^{(\alpha+1)}}\cap (R^{(\alpha)}-R^{(\alpha+1)}) = 0$. If there were a point p such that $p\in \overline{R^{(\alpha+1)}}\cap R^{(\alpha)}$ and $p\notin R^{(\alpha+1)}$, then for each open nbd U of p in S we would have

$U\cap R^{(\alpha+1)}\neq 0$. But $R^{(\alpha+1)}$ is a closed set in R, and so $U\cap R\cap R^{(\alpha+1)}\neq 0$ and $p\in R^{(\alpha)}\subset R$ imply that $p\in R^{(\alpha+1)}$, which gives a contradiction. Now we have $H_\alpha = \overline{R^{(\alpha)}-R^{(\alpha+1)}} - (R^{(\alpha)}-R^{(\alpha+1)}) = \overline{R^{(\alpha)}} - (R^{(\alpha)}-R^{(\alpha+1)}) \supset \overline{R^{(\alpha+1)}} - (R^{(\alpha)}-R^{(\alpha+1)}) = \overline{R^{(\alpha+1)}} = \overline{R^{(\alpha+1)}-R^{(\alpha+2)}} = F_{\alpha+1}$, because of the proved equality. If $\lambda\leqslant\beta$ is a limit ordinal, then $F_\lambda = \overline{R^{(\lambda)}-R^{(\lambda+1)}} = \overline{R^{(\lambda)}} = \bigcap_{\alpha<\lambda} R^{(\alpha)} \subset \bigcap_{\alpha<\lambda}\overline{R^{(\alpha)}} = \bigcap_{\alpha<\lambda}\overline{R^{(\alpha)}-R^{(\alpha+1)}} = \bigcap_{\alpha<\lambda} F_\alpha = \bigcap_{\alpha<\lambda} H_\alpha$. Hence the transfinite sequence

$$F_0, H_0, F_1, H_1, \ldots, F_\alpha, H_\alpha, \ldots \quad (\alpha<\beta)$$

is decreasing; so R is resolvable in S. The proof is finished.

Remark. Theorem 1.1 now follows also from Theorem 1.2 and a known fact that the family of all resolvable subsets in a given space is a ring (see [8], p. 100, Theorem 1).

A space R is said to be *absolutely resolvable* if R is resolvable in each space S such that R is embedded in S. So we can say that a space is absolutely resolvable iff it is C-scattered.

COROLLARY 1.2. *If S is a compactification of a scattered (or C-scattered) space R, then the remainder $S-R$ is C-scattered.*

PROBLEM 1.1. *If R is a C-scattered space with $\operatorname{ind} R = 0$, then is there a compactification S of R such that the remainder $S-R$ is scattered?*

From a positive solution of the former problem follows a positive solution of the main problem of Z. Semadeni [18]: does there exist for every scattered space a compactification which is also scattered?

A mapping f from S into T is said to be perfect if it is continuous and closed and if $f^{-1}(t)$ is compact for each $t\in f(S)$.

THEOREM 1.3. *If f is a perfect mapping from S onto T, then S is C-scattered iff T is C-scattered. In other words: the class of all C-scattered spaces is perfect.*

Proof. ($\Rightarrow$) Let S be C-scattered and A a non-void closed subset of T. Then there is an irreducible closed subset B of S such that $f(B) = A$. Without loss of generality we can assume that $B = S$ and $A = T$. Let U be an open set in S such that $\overline{U}$ is compact and non-void. Then $T-f(S-U)$ is open in T and $T-f(S-U)\subset f(U)\subset f(\overline{U})$, whence $T-f(S-U)$ has a compact closure in T. If $T-f(S-U)$ were void, then we would have $T = f(S-U)$, which is impossible, because $S-U$ is closed and different from S and S is irreducible. Hence the points of $T-f(S-U)$ are some points of local compactness of T.

($\Leftarrow$) If T is C-scattered and A is a non-void closed subset of S, then $f(A)$ is a closed subset of T. Without loss of generality we can assume that $A = S$ and $f(A) = T$. There is an open non-void set U in T such that $\overline{U}$ is compact. But $f^{-1}(U)\subset f^{-1}(\overline{U})$ and $f^{-1}(\overline{U})$ is compact. So the points of $f^{-1}(U)$ are some points of local compactness of S.

The proof is complete.

Corollary 1.3. *If $T=f(S)$, where T is a scattered space and f is a perfect mapping, then S is a C-scattered space.*

Remark. It can be proved by the same arguments as in the proof of Theorem 1.3 that (a) if f is a perfect mapping from a scattered space S onto a space T, then T is also scattered, and (b) if f is a perfect mapping from a space S onto a scattered space T such that $f^{-1}(t)$ is scattered for each $t\in T$, then S is also a scattered space.

Corollary 1.4. *If S is a C-scattered space and T is a compact space, then $S\times T$ is C-scattered.*

Proof. The projection $f\colon S\times T\to S$ is a perfect mapping, because T is compact. Hence by Theorem 1.3 the space $S\times T$ is C-scattered.

Theorem 1.4. *If S and T are C-scattered spaces, then $S\times T$ is also C-scattered.*

Proof. Let R be any non-void closed subspace of $S\times T$. Let P be the projection of R to S. Then $Q=\overline{P}$ is C-scattered as a closed subspace of S. Without loss of generality we can assume that $Q=S$. Since S^* is closed and nowhere-dense in S, then $S-S^*$ is open and dense in S. Hence $P\cap(S-S^*)\neq 0$, because P is dense in S. Let us take $p\in P\cap(S-S^*)$ and an open nbd U of p in S such that $\overline{U}$ is compact and $\overline{U}\cap S^*=0$. Now $(U\times T)\cap R\neq 0$ and $\overline{U}\times T$ is C-scattered by Corollary 1.4. Since $0\neq(U\times T)\cap R\subset(\overline{U}\times T)\cap R$, $(U\times T)\cap R$ is open in R and $(\overline{U}\times T)\cap R$ is a C-scattered closed subset in R, then $(U\times T)\cap R$ contains some point of local compactness of R, because $\big((\overline{U}\times T)\cap R\big)^*$ is nowhere-dense in $(\overline{U}\times T)\cap R$. The proof is finished.

Problem 1.2. *What are the conditions under which a C-scattered space S with $\operatorname{ind}S=0$ has a perfect mapping onto a scattered space?*

The condition $\operatorname{ind}S=0$ is necessary, because the real line has no perfect mapping onto a scattered space. Some additional condition is necessary even for locally compact spaces; it is shown by the following

Theorem 1.5. *Let $m>2^{\aleph_0}$, $p\in\{0,1\}^m$ and $S=\{0,1\}^m-\{p\}$, where $\{0,1\}^m$ is the generalized Cantor space of weight m. If f is a perfect mapping from S onto T, then T is not scattered.*

Proof. Suppose there is a perfect mapping f from S onto a scattered space T. Let T_0 be the set of all isolated points of T. Then $\{f^{-1}(t)\colon t\in T_0\}$ is a family of open sets of T which are pairwise disjoint. Since S is open in $\{0,1\}^m$, $f^{-1}(t)$ is open also in $\{0,1\}^m$ for every $t\in T_0$. So T_0 is countable, because $\{0,1\}^m$ has the Souslin property: every family of pairwise disjoint open sets is countable. Now we claim that $w(T)\leqslant 2^{\aleph_0}$. Since T is locally compact and scattered, then by [24], Corollary 3, p. 569 $\operatorname{ind}T=0$. Hence the family $\mathcal{B}$ of all clopen subsets of T is a basis of topology of T. It is easy to see that the mapping g from $\mathcal{B}$ into the power set $\mathcal{P}T_0$ of T_0 defined by putting $g(B)=B\cap T_0$, where $B\in\mathcal{B}$ is one-to-one, because T_0 is dense in T. So $|\mathcal{B}|\leqslant 2^{\aleph_0}$, and hence the weight $w(T)\leqslant 2^{\aleph_0}$. To obtain a contradiction let us remark that $\{0,1\}^m$ contains a one-point-compactification of some isolated set S_0 of cardinality m, with $\{p\}$ as a remainder. So S_0 is a closed and isolated set in S. Hence $f(S_0)$ is also a closed and isolated set in T. The cardinality of $f(S_0)$ is m, because f is perfect. Hence $w(T)\geqslant w\big(f(S_0)\big)=w(S_0)=m>2^{\aleph_0}$, which yields the desired contradiction. The proof is complete.

K. Morita [12] proved that if f is a closed continuous mapping from a paracompact locally compact space S onto a space T, then the set of all points of local incompactness of T is isolated and closed in T; hence the C-derivative T^* of T is an isolated set, so T is C-scattered.

Problem 1.3. *Is a closed continuous image of a C-scattered (or scattered) space C-scattered (resp. scattered)?*

Theorem 1.6. *Let S be a paracompact C-scattered space. If $S^{(\alpha+1)}=0$, then there is a locally finite closed covering $\{S_i\colon i\in I\}$ of S such that $S_i^{(\alpha)}$ is compact for each $i\in I$. If $S^{(\alpha)}=0$ and α is a limit ordinal, then there is a locally finite closed covering $\{S_i\colon i\in I\}$ of S such that for each $i\in I$ there is a $\beta<\alpha$ such that $S_i^{(\beta)}=0$.*

Proof. If $S^{(\alpha+1)}=0$, then $S^{(\alpha)}$ is locally compact and hence for each $p\in S^{(\alpha)}$ there is an open nbd U_p in S such that $\overline{U}_p\cap S^{(\alpha)}$ is compact. Now $\{U_p\colon p\in S^{(\alpha)}\}\cup\{S-S^{(\alpha)}\}$ is an open covering of S. The desired covering is its refinement. If $S^{(\alpha)}=0$ and α is the limit, then the family $\{S-S^{(\beta)}\colon \beta<\alpha\}$ is an open covering of S. The desired covering of S is its refinement. The proof is complete by Theorem 4 in [2], p. 210.

Theorem 1.7. *If R is a metrizable space, then R is C-scattered iff R is an F_σ- and G_δ-absolute space.*

Proof. ($\Rightarrow$) From Theorem 4′ of A. H. Stone [21] it follows that if R is a metrizable C-scattered space, then R is σ-locally compact. Hence, by Theorem 2 of [20], R is F_σ-absolute. It remains to prove that R is G_δ-absolute. We prove this by transfinite induction using Theorem 1.6. If $R^{(1)}=0$, then R is locally compact and hence it is G_δ-absolute. If $R^{(\alpha+1)}=0$, then $R^{(\alpha)}$ is locally compact. Since $R-R^{(\alpha)}$ is locally G_δ-absolute by the inductive assumption, then $R-R^{(\alpha)}$ is also G_δ-absolute, because it is paracompact. Hence R, as the union of two G_δ-absolute sets $R^{(\alpha)}$ and $R-R^{(\alpha)}$, is G_δ-absolute. If $R^{(\alpha)}=0$ for limit α, then R has by Theorem 1.6 a locally finite closed covering $\{R_i\colon i\in I\}$ such that for each $i\in I$ there is a $\beta<\alpha$ such that $R_i^{(\beta)}=0$. So every R_i is G_δ-absolute by the inductive assumption. Hence R is also G_δ-absolute.

(⇐) If R is F_σ-absolute, then it is σ-locally compact, i.e. $R = \bigcup_{i<\omega} R_i$ where R_i's are locally compact, by Theorem 2 of [20]. Since R is G_δ-absolute, then R fulfils the Baire Category Theorem. Let F be a closed subspace of R. It is clear that F has also the former properties. So without loos of generality we can assume that $F = R$. Hence there is an $i < \omega$ such that $\mathrm{Int}(R_i) \neq 0$. So R has points of local compactness.

The proof is complete.

2. The topological product of paracompact spaces. It is well known that the topological product of two paracompact spaces need not be normal, and hence it need not be paracompact. Thus the following question is reasonable: what kind of intrinsic topological properties have the spaces from Π, i.e. the class of all paracompact spaces S such that $S \times T$ is paracompact for each paracompact space T? H. Tamano [23] raised the following problem: find a topological property related to the class Π. Here this problem is not yet solved, but we shall define a very wide class contained in Π which we suspect to be equal to Π (see Theorem 2.5 below).

We now claim that connectedness plays no rôle in the problem of the product of paracompact spaces. We apply to products the following theorem of K. Nagami [16]: S is paracompact iff there is a perfectly zero-dimensional space $\dot{S}$ and a perfect mapping from $\dot{S}$ onto S (perfectly zero-dimensional means such that every open covering has a refinement by pairwise disjoint open sets). Hence we have a

Reduction principle. *$S \times T$ is paracompact iff $\dot{S} \times T$ is paracompact iff $S \times \dot{T}$ is paracompact iff $\dot{S} \times \dot{T}$ is paracompact.*

The proof follows from the fact that $f \times g$ is a perfect mapping if f and g are perfect (see [3]), and from the invariance of paracompactness by perfect mappings.

A space S is said to be *locally Π* if every point $p \in S$ has an open nbd U such that $\overline{U}$ belongs to Π.

Theorem 2.1. *The class Π has the following properties:*

(2.1.1) *If $S \in \Pi$ and R is a F_σ-set in S, then $R \in \Pi$.*

(2.1.2) *If $S \in \Pi$ and $T \in \Pi$, then $S \times T \in \Pi$.*

(2.1.3) *If S is paracompact and locally Π, then $S \in \Pi$.*

(2.1.4) *Π is a perfect class; in particular: $S \in \Pi$ iff $\dot{S} \in \Pi$.*

(2.1.5) *If $R \subset S$, R is compact, S is paracompact and $S - R$ is locally Π, then $S \in \Pi$.*

(2.1.6) *If R is a closed σ-locally compact G_δ-set in a paracompact space S and $S - R$ is locally Π, then $S \in \Pi$.*

Proof. (2.1.1) Since R is a F_σ-set in S, then $R \times T$ is a F_σ-set in $S \times T$. Hence, if $S \times T$ is paracompact, then so is $R \times T$, because paracompactness is F_σ-hereditary.

(2.1.2) If $S \in \Pi$, $T \in \Pi$ and R is a paracompact space, then $R \times S$ is paracompact, because $S \in \Pi$. Further, $R \times S \times T$ is also paracompact, because $T \in \Pi$. Hence $S \times T \in \Pi$.

(2.1.3) Let $\{S_i: i \in I\}$ be a locally finite closed covering of S such that $S_i \in \Pi$ and let T be a paracompact space. Then $\{S_i \times T: i \in I\}$ is a locally finite closed covering of $S \times T$ by paracompact sets. Then $S \times T$ is paracompact by a theorem of K. Morita [13].

(2.1.4) If f is a perfect mapping from a paracompact space R onto a paracompact space S and T is a paracompact space, then $f \times \mathrm{id}_T$ is a perfect mapping from $R \times T$ onto $S \times T$, by a theorem of Z. Frolík [3]. Hence $R \times T$ is paracompact iff $S \times T$ is paracompact, because the class of all paracompact spaces is perfect.

(2.1.5) Let T be any paracompact space. According to (2.1.4) and to the Reduction Principle we can assume that S and T are perfectly zero-dimensional. Let $\mathcal{A}$ be any open covering of $S \times T$. For each $t \in T$ the set $R \times \{t\}$ is compact, whence there is a clopen nbd U_t of R in S and a clopen nbd V_t of t in T such that $R \times \{t\} \subset U_t \times V_t \subset \bigcup \mathcal{A}_t$, where $\mathcal{A}_t$ is some finite subfamily of $\mathcal{A}$. Now $\{V_t: t \in T\}$ is an open covering of T, whence there is a discrete clopen covering $\mathcal{B}$ of T which refines $\{V_t: t \in T\}$. For each $B \in \mathcal{B}$ we pick a $t_B \in T$ such that $B \subset V_{t_B}$. Clearly, $\{U_{t_B} \times B: B \in \mathcal{B}\}$ is a clopen discrete refinement of $\{U_t \times V_t: t \in T\}$, and it covers $R \times T$. But also $\{(S - U_{t_B}) \times B: B \in \mathcal{B}\}$ is a clopen discrete family in $S \times T$. Since $(S - U_{t_B}) \cap R = 0$ and $S - U_{t_B}$ is closed, $S - U_{t_B}$ is paracompact and locally Π. Hence, by (2.1.3), $S - U_{t_B}$ belongs to Π and so $\mathcal{A}$ has a locally finite open refinement $\mathcal{A}_B$ in $S \times T$ such that $\bigcup \mathcal{A}_B = (S - U_{t_B}) \times B$, for every $B \in \mathcal{B}$. Finally,

$$\bigcup \{\mathcal{A}_B: B \in \mathcal{B}\} \cup \{(U_{t_B} \times B) \cap A: A \in \mathcal{A}_{t_B} \text{ and } B \in \mathcal{B}\}$$

is an open locally finite refinement of $\mathcal{A}$ and it covers $S \times T$. So $S \times T$ is paracompact and hence $S \in \Pi$.

(2.1.6) Let $\mathcal{A}$ be any open covering of $S \times T$, where T is a paracompact space. At first we will construct an open refinement of $\mathcal{A}$, σ-locally finite in $S \times T$, which covers $R \times T$, and further an open refinement of $\mathcal{A}$, σ-locally finite in $S \times T$, which covers $(S - R) \times T$.

Since R is closed in S and σ-locally compact, there is a family $\{R_i: i \in I_n \text{ and } n < \omega\}$ of compact sets such that for each $n < \omega$ the family $\{R_i: i \in I_n\}$ is locally finite in S and $R = \bigcup_{n<\omega} \bigcup_{i \in I_n} R_i$. For each $n < \omega$ we take a locally finite family $\{Q_i: i \in I_n\}$ of open sets in S such that

$Q_i \supset R_i$ for each $i \in I_n$. For each $t \in T$ and $i \in \bigcup_{n<\omega} I_n$ the set $R_i \times \{t\}$ is compact and so there is an open nbd $U_{i,t}$ of R_i in S and an open nbd $V_{i,t}$ of t in T such that $U_{i,t} \subset Q_i$ and $R_i \times \{t\} \subset U_{i,t} \times V_{i,t} \subset \bigcup \mathcal{A}_{i,t}$, where $\mathcal{A}_{i,t}$ is some finite subfamily of $\mathcal{A}$. For each $i \in \bigcup_{n<\omega} I_n$ the family $\{V_{i,t}: t \in T\}$ is an open covering of T. Hence it has an open locally finite refinement $\mathcal{B}_i$. For each $B \in \mathcal{B}_i$ we pick a $t_B \in T$ such that $B \subset V_{i,t_B}$. Clearly, for each $i \in \bigcup_{n<\omega} I_n$ the family $\{(U_{i,t_B} \times B) \cap A: A \in \mathcal{A}_{i,t_B}$ and $B \in \mathcal{B}_i\}$ is an open refinement of $\mathcal{A}$, locally finite in $S \times T$, which covers $R_i \times T$. But the family $\{(U_{i,t_B} \times B) \cap A: A \in \mathcal{A}_{i,t_B}, B \in \mathcal{B}_i$ and $i \in I_n\}$ is also locally finite for each $n < \omega$, because we have chosen Q_i's for this reason. Hence we have a σ-locally finite refinement of $\mathcal{A}$, open in $S \times T$, which covers $R \times T$.

Since R is a G_δ-set in S, $S - R$ is an F_σ-set and so $S - R = \bigcup_{n<\omega} F_n$, F_n are closed in S and $F_n \cap R = 0$ for each $n < \omega$. Since S is normal, for each F_n there is an open set W_n in S such that $F_n \subset W_n \subset \overline{W_n} \subset S - R$. But then $\overline{W_n} \in \Pi$ and so $\mathcal{A}$ has a locally finite refinement $\mathcal{C}_n$, open in $S \times T$ which covers $F_n \times T$. So $\mathcal{A}$ has a σ-locally finite open refinement $\bigcup_{n<\omega} \mathcal{C}_n$ which covers $(S - R) \times T$.

The proof of Theorem 2.1 is complete.

From (2.1.3) and (2.1.5) follows the following:

COROLLARY 2.1 (J. Suzuki [22], Theorem 3). *If R is a locally compact closed subset of a paracompact space S such that $S - R$ is locally Π, then $S \in \Pi$.*

DEFINITION. A set R is said to be *well-situated in a space S*, if for every paracompact space T, every open covering of $R \times T$ in $S \times T$ has an open locally finite refinement in $S \times T$ which covers $R \times T$. By Π^* we will denote the class of all spaces R which are well-situated in every paracompact space S such that R is embedded in S as a closed subset.

Putting $R = S$, we see that $\Pi^* \subset \Pi$. Using the example of E. Michael [11] we can assert that the space of all rational numbers does not belong to Π^*. However, it belongs to Π; this follows from (2.1.6) for $R = S$. Hence $\Pi^* \neq \Pi$. Although Π^* is not F_σ-hereditary, it is F-hereditary; the proof is analogical to the proof that paracompactness is F-hereditary.

PROBLEM 2.1. *Is the class Π^* perfect?*

A space S is said to be *locally Π^** if every point $p \in S$ has an open nbd U in S such that $\overline{U}$ is in Π^*.

THEOREM 2.2. *If R is paracompact and locally Π^*, then $R \in \Pi^*$.*

Proof. Let S be a paracompact space such that R is a closed subset of S. There is a closed covering $\{R_i: i \in I\}$ of R, and locally finite in R such that $R_i \in \Pi^*$, because R is paracompact and Π^* is F-hereditary. Since S is paracompact and R is closed in S, there is a locally finite open family $\{U_i: i \in I\}$ in S such that $U_i \supset R_i$ for each $i \in I$. Since R_i is closed

also in S, then R_i is well-situated in S. Let T be any paracompact space and $\mathcal{A}$ be any open covering of $R \times T$ in $S \times T$. Then $\mathcal{A}$, as an open covering of $R_i \times T$, has an open refinement $\mathcal{A}_i$ locally finite in $S \times T$ and such that $R_i \times T \subset \bigcup \mathcal{A}_i \subset U_i \times T$. Clearly, $\mathcal{A}' = \bigcup \{\mathcal{A}_i: i \in I\}$ is an open refinement of $\mathcal{A}$, locally finite in $S \times T$, and it covers $R \times T$. Hence $R \in \Pi^*$. The proof is complete.

THEOREM 2.3. *If R is a closed C-scattered subset of a paracompact space S, then R is well-situated in S; i.e., every paracompact C-scattered space belongs to Π^*.*

The proof proceedes by transfinite induction.

If $R^{(0)} = 0$, then $R = 0$ and hence the theorem is true.

If $R^{(\alpha+1)} = 0$, then $R^{(\alpha)}$ is locally compact. By Theorem 1.6 there is a locally finite closed covering $\{R_i: i \in I\}$ of R in R such that every $R_i^{(\alpha)}$ is compact. According to Theorem 2.2 it suffices to prove that each R_i is well-situated in S. To simplify the notation let us put $F = R_i$. Let T be any paracompact space and $\mathcal{A}$ be any open covering of $F \times T$ in $S \times T$. For each $t \in T$ the set $F^{(\alpha)} \times \{t\}$ is compact; hence there is an open set U_t in S and an open set V_t in T such that $F^{(\alpha)} \times \{t\} \subset U_t \times V_t \subset \overline{U}_t \times \overline{V}_t \subset \bigcup \mathcal{A}_t$, where $\mathcal{A}_t$ is some finite subfamily of $\mathcal{A}$ and $t \in T$. Clearly, $\{V_t: t \in T\}$ is an open covering of T. Since T is paracompact, there is an open locally finite refinement $\mathcal{B}$ of $\{V_t: t \in T\}$. Without loss of generality we can assume that $\mathcal{B}$ is irreducible. For each $B \in \mathcal{B}$ we pick a $t_B \in T$ such that $B \subset V_{t_B}$. Clearly, $\{V_{t_B}: B \in \mathcal{B}\}$ is also a covering of T. Now for each $B \in \mathcal{B}$ we will define an open set B' in T such that $\overline{B'} \subset B$ and $\bigcup \{B': B \in \mathcal{B}\} = T$. To do that, let us take a well-ordering relation $<$ of $\mathcal{B}$. Assume that for some $B_0 \in \mathcal{B}$, for every $B < B_0$, the set B' is already defined in such a way that $\overline{B'} \subset B$ and $\bigcup \{B_1': B_1 < B\} \cup \bigcup \{B_1: B_1 \geqslant B\} = T$. The set $B_0'' = T - (\bigcup \{B': B < B_0\} \cup \bigcup \{B: B > B_0\})$ is closed in T; $B_0'' \subset B_0$, because $\{B': B < B_0\} \cup \{B: B \geqslant B_0\}$ is irreducible. Since T is normal, we can choose an open set B_0' in T such that $B_0'' \subset B_0' \subset \overline{B_0'} \subset B_0$. So B_0 is defined. Since $F^{(\alpha)} \subset U_{t_B}$, we have $(F - U_{t_B})^{(\alpha)} = 0$. Hence, by the inductive assumption, $F - U_{t_B}$ is well-situated in S. Hence $(F - U_{t_B}) \times \overline{B'}$, as a closed subset of $(F - U_{t_B}) \times T$, has an open covering $\mathcal{C}_B$, locally finite in $S \times T$, such that $\mathcal{C}_B$ refines $\mathcal{A}$ and $\bigcup \mathcal{C}_B \subset S \times B$. From the above constructions it follows that

$$\{(U_{t_B} \times B) \cap A: A \in \mathcal{A}_{t_B} \text{ and } B \in \mathcal{B}\} \cup \bigcup \{\mathcal{C}_B: B \in \mathcal{B}\}$$

is an open covering of $F \times T$, locally finite in $S \times T$, which refines $\mathcal{A}$. Hence F is well-situated in S.

If $R^{(\alpha)} = 0$ for the ordinal limit number α, then by Theorem 1.6 there is a locally finite closed covering $\{R_i: i \in I\}$ of R such that for each $i \in I$ there is a $\beta < \alpha$ such that $R_i^{(\beta)} = 0$. Hence every R_i is well-situated in S by the inductive assumption. So, by Theorem 2.2, R is also well-situated in S. The inductive proof is complete.

A collection $\{A_i: i \in I\}$ of subsets of a topological space is said to be *order locally finite*, if we can introduce a well ordering $<$ in the index set I such that for each $i \in I$ the family $\{A_j: j < i\}$ is locally finite at each point of A_i.

LEMMA 2.1 (Y. Katuta [6]). *Let $\{A_i: i \in I\}$ be an order locally finite collection of subsets of a space S and let $\{B_j: j \in J_i\}$ be a collection of subsets of A_i locally finite in S, for each $i \in I$, such that $J_i \cap J_{i'} = 0$ for $i \neq i'$. Put $J = \bigcup \{J_i: i \in I\}$. Then the collection $\{B_j: j \in J\}$ is order locally finite. In particular: a σ-locally finite collection is order locally finite.*

LEMMA 2.2 (Y. Katuta [6]). *A space S is paracompact iff any open covering of S has an order locally finite open refinement.*

As a generalization of Theorem of Y. Katuta [6] we have

THEOREM 2.4. *If a space S has two coverings*

$$\{F_i: i \in I\} \quad \text{and} \quad \{U_i: i \in I\}$$

such that each F_i is closed and well-situated in S, every U_i is open in S and $U_i \supset F_i$, and $\{U_i: i \in I\}$ is order locally finite, then $S \in \Pi$, i.e. for any paracompact space T the product $S \times T$ is paracompact.

Proof. Let T be any paracompact space and let $\mathcal{A}$ be any open covering of $S \times T$. Since every F_i is well-situated in S, then there is an open covering $\mathcal{B}_i = \{B_j: j \in J_i\}$ of $F_i \times T$, locally finite in $S \times T$, such that $\mathcal{B}_i$ refines $\mathcal{A}$ and $\bigcup \mathcal{B}_i \subset U_i \times T$. Without loss of generality we can assume that $J_i \cap J_{i'} = 0$ if $i \neq i'$. Since the open covering $\{U_i \times T: i \in I\}$ of $S \times T$ is order locally finite and $F_i \times T \subset \bigcup \mathcal{B}_i \subset U_i \times T$, then the family $\bigcup \{\mathcal{B}_i: i \in I\}$ is, by Lemma 2.1, an open order locally finite refinement of $\mathcal{A}$ and it covers $S \times T$. Hence, by Lemma 2.2, $S \times T$ is paracompact. The proof is complete.

COROLLARY 2.2. *If S is a paracompact space constituting the union of some countable family of its closed subspaces S_n with $S_n \in \Pi^*$, then $S \times T$ is paracompact for any paracompact space T.*

From Theorem 2.3 and Theorem 2.4 immediately follows

THEOREM 2.5. *If a paracompact space S has two coverings: an open covering $\{U_i: i \in I\}$ and a closed covering $\{F_i: i \in I\}$ such that $\{U_i: i \in I\}$ is order locally finite, $U_i \supset F_i$ for each $i \in I$ and every F_i is C-scattered, then $S \times T$ is paracompact for every paracompact space T.*

Remark. From the example of E. Michael [11] it follows that if $\{S_n: n < \omega\}$ is a family of paracompact spaces such that $\underset{n<\omega}{P} S_n \in \Pi$, then there is $n < \omega$ such that for each $k \geqslant n$ the set S_k is compact. To see this, suppose that S_n is not compact for $k_0 < k_1 < k_2 < \ldots$ Then S_{k_n} contains a countable infinite closed isolated set R_n. Since $\underset{n<\omega}{P} R_n$ is a closed subspace of $\underset{n<\omega}{P} S_{k_n}$, then $\underset{n<\omega}{P} R_n$ can be embedded as a closed subspace of $\underset{n<\omega}{P} S_n$. But $\underset{n<\omega}{P} R_n$ is homeomorphic to the space of all irrational numbers. $\underset{n<\omega}{P} R_n \notin \Pi$ by Michael's example and hence $\underset{n<\omega}{P} S_n \notin \Pi$ by (2.1.1).

PROBLEM 2.2. *Is the topological product of countably many paracompact scattered spaces paracompact?*

Now we give some corollaries to Theorem 2.5.

COROLLARY 2.3 (K. Morita [14]). *If S is paracompact and σ-locally compact, then $S \in \Pi$.*

COROLLARY 2.4 (T. Ishii [5]). *If S is a closed continuous image of a paracompact locally compact space, then $S \in \Pi$.*

To prove this, we apply Theorem 4 of [12]; the space S by hypothesis is C-scattered (its second derivative vanishing). Hence $S \in \Pi$.

COROLLARY 2.5 (M. Tsuda [26]). *If S is a closed continuous image of a paracompact perfectly normal σ-locally compact space, R, then $S \in \Pi$.*

From Theorem 4 of [12] it follows that by hypothesis S under assumptions is a union of some countable family of its closed C-scattered subspaces. Hence $S \in \Pi$. It is clear that the assumption that R is perfectly normal is superfluous; this answers a question of T. Ishii [6].

COROLLARY 2.6 (Y. Katuta [6]). *If a space S has two coverings $\{C_i: i \in I\}$ and $\{U_i: i \in I\}$ such that*

(a) *C_i is compact, U_i is open and $C_i \subset U_i$, for each $i \in I$, and*

(b) *$\{U_i: i \in I\}$ is order locally finite,*

then $S \in \Pi$.

Now we give an example of a scattered paracompact Lindelöf space S_0 which does not satisfy the assumption of the Theorem of Y. Katuta [6]. Let $p \in \beta N - N$ be a p-point, i.e. there is a clopen basis $\{U_\alpha: \alpha < \omega_1\}$ of nbds in p such that for every $\alpha < \omega_1$ and every $\beta < \alpha$ we have $U_\alpha - N \subset U_\beta - N$ and $(U_\alpha - U_{\alpha+1}) - N \neq 0$. Let us pick $p_\alpha \in (U_\alpha - U_{\alpha+1}) - N$ and put $A = \{p_\alpha: \alpha < \omega_1\}$. Take $S_0 = N \cup A \cup \{p\}$ with the topology of S_0 as the subspace of βN. Then S_0 is completely regular. S_0 has the Lindelöf property, because the complement of any basic nbd of p is a countable set. Hence S_0 is paracompact (see [2], p. 211). A is an isolated set, because $V_\alpha = S_0 \cap (U_\alpha - U_{\alpha+1})$ is a clopen nbd of p_α in S_0 such that $V_\alpha \cap A = \{p_\alpha\}$. Hence S_0 is scattered as a union of three isolated sets (see [8], p. 79, Theorem 2). Let C be any compact set in S_0. Then C is closed in βN. Hence C is finite or C has the cardinality of βN (see [2], p. 132); the last case is impossible, because the cardinality of S_0 is less than the cardinality of βN. So C is finite. Suppose that S_0 has two coverings, $\{C_i: i \in I\}$ and

$\{U_i\colon i \in I\}$, satisfying (a) and (b) from Corollary 2.6. Since every C_i is finite, then each $C_i - \{p\}$ is compact. Now the coverings $\{C_i - \{p\}\colon i \in I\}$ and $\{U_i - \{p\}\colon i \in I\}$ satisfy the conditions (a) and (b) from Corollary 2.6 for the space $S = S_0 - \{p\}$. Hence, by Corollary 2.6, S is in particular paracompact. So S is collectionwise normal. But this is a contradiction, because $\{\{p_\alpha\}\colon \alpha < \omega_1\}$ is a discrete family of closed sets in S and there is no discrete family of open sets $\{B_\alpha\colon \alpha < \omega_1\}$ such that $p_\alpha \in B_\alpha$. Hence S_0 cannot satisfy the condition of the Theorem of Y. Katuta [6].

PROBLEM 2.3. *Does the class of all paracompact C-scattered spaces coincide with the class Π^*?*

The product space of two spaces satisfying the conditions (a) and (b) from Corollary 2.6 for non-cofinal ordered systems $\langle I, \leqslant\rangle$ and $\langle J, \leqslant\rangle$ need not satisfy the conditions (a) and (b) for some $\langle K, \leqslant\rangle$. To check this it suffices to take $Q \times L$, where Q is the space of all rationals and L is the subspace of $\{\alpha\colon \alpha \leqslant \omega_1\}$ consisting of all nonlimit countable ordinals and of the greatest element ω_1.

3. Total and absolute paracompactness. In this section we consider covering properties much stronger than paracompactness, but still weaker than compactness. Totally paracompact metric spaces were studied by A. Lelek [9] and [10]. This section is a generalization of Theorem 3 in my earlier paper [24]. Corollary 3.3 in a slightly weaker form was proved also by H. Kok. A proof that the space of all rationals is not absolutely paracompact is contained in a joint paper [7] of H. Kok and the author. The last paper is a natural continuation of this section.

A space S is said to be *totally paracompact* if every open basis of S contains a locally finite covering of S. A family $\mathcal{B}$ of open sets in S is said to be an *outer basis for* $R \subset S$ if for every $p \in R$ and every open nbd U of p in S there is a $B \in \mathcal{B}$ such that $p \in B \subset U$. A subspace R of S is said to be *totally paracompact relative to* S if every open outer basis of R in S contains a covering of R which is locally finite in S. Finally, a space R is said to be *absolutely paracompact* if R is totally paracompact relative to every paracompact space S such that R is embedded in S as a closed subspace.

Putting $R = S$, we see that every absolutely paracompact space is totally paracompact. Clearly, every compact space is absolutely paracompact. Moreover, we have

THEOREM 3.1. *If R is a paracompact C-scattered space, then R is absolutely paracompact.*

Proof. Let S be a paracompact space such that R is a closed subspace of S and let $\mathcal{B}$ be an outer basis for R in S. We will prove by transfinite induction on α that if $R^{(\alpha)} = 0$, then R is absolutely paracompact.

If $R^{(0)} = 0$, then $R = 0$. Hence the theorem is true.

If $R^{(\alpha+1)} = 0$, then $R^{(\alpha)}$ is locally compact and closed in R. Hence there is a locally finite family $\{C_i\colon i \in I\}$ in S of compact sets such that $R^{(\alpha)} = \bigcup\{C_i\colon i \in I\}$. Since S is paracompact, there is a locally finite family $\{U_i\colon i \in I\}$ in S of open sets in S such that $C_i \subset U_i$ for each $i \in I$. Since every C_i is compact, there is a finite subfamily $\mathcal{B}_i \subset \mathcal{B}$ such that $C_i \subset \bigcup \mathcal{B}_i \subset U_i$. Clearly, $\mathcal{B}_0 = \bigcup\{\mathcal{B}_i\colon i \in I\}$ is a locally finite family in S, and it covers $R^{(\alpha)}$ and $\mathcal{B}_0 \subset \mathcal{B}$. Now, $R - \bigcup \mathcal{B}_0$ is a closed subspace of R and $(R - \bigcup \mathcal{B}_0)^{(\alpha)} = 0$, because $R - \bigcup \mathcal{B}_0 \subset R - R^{(\alpha)}$. So $R - \bigcup \mathcal{B}_0$ is absolutely paracompact by the inductive assumption. Since $\mathcal{B}$ is an outer basis also for $R - \bigcup \mathcal{B}_0$, $\mathcal{B}$ contains a covering $\mathcal{A}_0$ of $R - \bigcup \mathcal{B}_0$ locally finite in S. Clearly, $\mathcal{A}_0 \cup \mathcal{B}_0$ is a covering of R, locally finite in S contained in $\mathcal{B}$.

If $R^{(\alpha)} = 0$ for the ordinal limit number α, then by Theorem 1.6 there is a closed covering $\{F_i\colon i \in I\}$ of R, locally finite in R (and hence also in S, because R is closed in S) and such that $R = \bigcup\{F_i\colon i \in I\}$ and for each $i \in I$ there is a $\beta < \alpha$ such that $F_i^{(\beta)} = 0$. So, by the inductive assumption, every F_i is absolutely paracompact. Let $\{U_i\colon i \in I\}$ be a locally finite open family in S such that $U_i \supset F_i$ for every $i \in I$. Let i be fixed; since $\mathcal{B}$ is an outer basis also for F_i, there is a $\mathcal{B}_i \subset \mathcal{B}$ which is locally finite in S and such that $F_i \subset \bigcup \mathcal{B}_i \subset U_i$. From the above construction it follows that $\mathcal{A} = \bigcup\{\mathcal{B}_i\colon i \in I\}$ is a locally finite family in S, $\mathcal{A}$ covers R and $\mathcal{A} \subset \mathcal{B}$.

The inductive proof is complete.

COROLLARY 3.1. *If R is paracompact and locally compact, then R is absolutely paracompact.*

COROLLARY 3.2 (Theorem 3 in [24]). *Each paracompact scattered space is totally paracompact.*

PROBLEM 3.1. *Is R C-scattered if R is absolutely paracompact?*

PROBLEM 3.2. *Is R totally paracompact if R is σ-compact? In particular, is R totally paracompact if R is countable?*

COROLLARY 3.3. *Each paracompact C-scattered space is totally paracompact.*

PROBLEM 3.3. *Is $S \times T$ totally paracompact if S is totally paracompact and T is compact? More generally: let $T = f(S)$ and f be a perfect mapping. Is S then totally paracompact iff T is totally paracompact?*

References

[1] J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. 23 (1944), pp. 65–76.

[2] R. Engelking, *Outline of General Topology*, Amsterdam 1968.

[3] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. 8 (1960), pp. 747–750.
[4] F. Hausdorff, *Mengenlehre*, Berlin 1935.
[5] T. Ishii, *On product spaces and product mappings*, J. Math. Soc. Japan 18 (1966), pp. 166–181.
[6] Y. Katuta, *A theorem on paracompactness of product spaces*, Proc. Japan Acad. 43 (1967), pp. 615–618.
[7] H. Kok and R. Telgársky, *The space of rationals is not absolutely paracompact* (this volume).
[8] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[9] A. Lelek, *On totally paracompact metric spaces*, Proc. Amer. Math. Soc. 19 (1968), pp. 168–170.
[10] — *Some cover properties of spaces*, Fund. Math. 64 (1969), pp. 209–218.
[11] E. Michael, *The product of normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), pp. 375–376.
[12] K. Morita, *On closed mappings*, Proc. Japan Acad. 32 (1956), pp. 539–543.
[13] — *Paracompactness and product spaces*, Fund. Math. 50 (1961), pp. 223–236.
[14] — *On the product of paracompact spaces*, Proc. Japan Acad. 39 (1963), pp. 559–563.
[15] — *Products of normal spaces with metric spaces*, Math. Annalen 154 (1964), pp. 365–382.
[16] K. Nagami, *A note on Hausdorff spaces with the star-finite property II*, Proc. Japan Acad. 37 (1961), pp. 189–192.
[17] C. Ryll-Nardzewski, *A remark on the cartesian product of two compact spaces*, Bull. Acad. Polon. Sci. 6 (1954), pp. 265–266.
[18] Z. Semadeni, *Sur les ensembles clairsemés*, Rozprawy Mat. 19 (1959).
[19] R. H. Sorgenfrey, *On the topological product of paracompact spaces*, Bull. Amer. Math. Soc. 53 (1947), pp. 631–632.
[20] A. H. Stone, *Absolute F_σ-spaces*, Proc. Amer. Math. Soc. 13 (1962), pp. 495–499.
[21] — *Kernel constructions and Borel sets*, Trans. Amer. Math. Soc. 107 (1963), pp. 58–70; Errata, p. 558.
[22] J. Suzuki, *Paracompactness of product spaces*, Proc. Japan Acad. 45 (1969), pp. 457–460.
[23] H. Tamano, *On compactifications*, J. Math. Kyoto Univ. 1 (1962), pp. 162–193.
[24] R. Telgársky, *Total paracompactness and paracompact dispersed spaces*, Bull. Acad. Polon. Sci. 16 (1968), pp. 567–572.
[25] — *On the product of paracompact spaces*, Bull. Acad. Polon. Sci. 17 (1969), pp. 533–536.
[26] M. Tsuda, *On the normality of certain product spaces*, Proc. Japan Acad. 40 (1964), pp. 465–467.

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The space of rationals is not absolutely paracompact

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A Hausdorff space X is said to be *paracompact* (*metacompact*) if every open covering of X has a locally finite (point-finite) open refinement.

A Hausdorff space X is said to be *totally paracompact* (*totally metacompact*) if every open base of X contains a locally finite (point-finite) covering of X.

A family $\mathcal{B}$ of open sets in X is said to be an *outer base* of $Y \subset X$ if, for each $y \in Y$ and each open set G in X, such that $y \in G$ there exists a $B \in \mathcal{B}$ such that $y \in B \subset G$.

We call a subset Y of X *totally paracompact with respect to* X if every outer base of Y in X contains a locally finite (with respect to X) covering of Y.

It is easy to prove that if Y is totally paracompact with respect to $X \supset Y$, then Y is a totally paracompact subspace of X. A paracompact space X is said to be *absolutely paracompact* if, for every paracompact space Y such that X is embedded into Y as a closed subspace, X is totally paracompact with respect to Y.

For results on totally paracompact spaces we refer to [2] and [5].

In this paper we will prove that a space of E. Michael [3] is not totally metacompact and that the space of the rationals is not absolutely paracompact. It is known that the space of the rationals is totally paracompact, and that the space of the irrationals is paracompact but not totally paracompact (cf. [2]).

Let ω^ω denote the Baire space of sequences of non-negative integers. It is well known that ω^ω is homeomorphic to the space of all irrational numbers (cf. [4], p. 143).

Let

$$D = \{f \in \omega^\omega \mid \exists n: \forall k \geqslant n: f(k) = 0\}.$$

Then D is dense in ω^ω and D is homeomorphic to the space of rationals.