On Urysohn's universal separable metric space*

by

Charles Joiner (Cleveland, Ohio)

1. Introduction. As one of his many accomplishments, Urysohn [1] constructed a universal separable metric space U; that is, a separable metric space U such that any separable metric space can be isometrically imbedded in U. The purpose of this paper is two fold. First we present an alternate and somewhat simpler construction of Urysohn's space. Second we show that U satisfies a strong homogeneity condition. Urysohn showed that U is homogeneous with respect to finite subsets. That is, if $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are any two subsets of U each containing n points and if $d(a_i, a_j) = d(b_i, b_j)$ whenever $1 \leqslant i, j \leqslant n$, then there is a distance preserving bijection φ from U onto U such that $\varphi(a_i) = b_i$ for $i = 1, 2, \ldots, n$. We shall prove that U is homogeneous with respect to convergent sequences. That is, if $\{x_n\}_{n\geqslant 1}$ and $\{y_n\}_{n\geqslant 1}$ are convergent sequences in U such that $d(x_i, x_j) = d(y_i, y_j)$, for all positive integers i and j, then there is a distance preserving bijection ψ from U onto U such that $\psi(x_i) = y_i$ for $i = 1, 2, 3, \ldots$

2. Construction of the space U. For each positive integer k define a bijection σ_k: $N \to Q_k$, where N is the set of positive integers and Q_k is the set of all ordered k-tuples $(r_1, r_2, \ldots, r_k)$ of positive rational numbers.

If $\sigma_k(t) = (r_1, r_2, \ldots, r_k)$, then we define $r_{\sigma_k(t),i} = r_i$ so that

$$\sigma_k(t) = (r_{\sigma_k(t),1}, r_{\sigma_k(t),2}, \ldots, r_{\sigma_k(t),k})$$

for all positive integers k and t.

We are now ready to begin constructing the space itself. First we will construct a countable metric space U_0. This will be done by induction.

Step 1. Include the point x_1 in the space.

Step 2. We add a point $y_{2,1}$ to the space with $d(y_{2,1}, x_1) = r_{\sigma_1(1),1}$. Rename the point $y_{2,1}$ to be x_2.

* This work was supported in part by NSF Grant GP-8394. The author wishes o thank Professor Ky Fan for his encouragement and guidance.

Suppose we have already carried out step p for all $p < k$. Suppose that at the end of step $k-1$ we had constructed a finite number of points $x_1, x_2, \dots, x_{n_{k-1}}$. In step k we will construct a new point for each non-empty subset of the set $\{x_1, x_2, \dots, x_{n_{k-1}}\}$ of all points chosen in previous steps. Thus we will add $2^{n_{k-1}}-1$ points to the space during step k. The total number of points to be chosen in the first k steps is

$$n_k = 2^{n_{k-1}} + n_{k-1} - 1 .$$

This recursion formula together with $n_1 = 1$ determines every n_k.

Step k. For each positive integer $p < k$ let A_p be the set of all ordered t-tuples $(a_1, a_2, \dots, a_t)$ of positive integers satisfying $1 \leqslant t \leqslant n_p$ and $1 \leqslant a_1 < a_2 < \dots < a_t \leqslant n_p$. We shall be concerned primarily with A_{k-1}. Next we order A_{k-1} lexicographically. Let $a = (a_1, a_2, \dots, a_t)$ and $a' = (a_1', a_2', \dots, a_{t'}')$ be two distinct members of A_{k-1}. We say $a < a'$ if $t < t'$ or if $t = t'$ and $a_i < a_i'$ where i is the smallest integer such that $a_i \neq a_i'$.

For each $a \in A_{k-1}$ we shall add a new point $y_{k,a}$ to our space. Suppose $a = (a_1, a_2, \dots, a_t) \in A_{k-1}$ is given and for each $a' \in A_{k-1}$, $a' < a$, we have already added the point $y_{k,a'}$ to the space. We add the point $y_{k,a}$ to the space where

$$d(y_{k,a}, x_{a_i}) = r_{\sigma_t(\eta_a^k),i} \quad \text{for} \quad i = 1, 2, \dots, t ,$$

and where η_a^k is a positive integer to be defined below.

Before defining η_a^k notice that we have $a \in A_p$ whenever $1 \leqslant a_1 < a_2 < \dots < a_t \leqslant n_p$. Thus we may have $a \in A_p$ for one or more values of $p < k-1$. Now, if $a \in A_p$ for some $p < k-1$ then a point $y_{p+1,a}$ was chosen in step $p+1$. We want our new point $y_{k,a}$ to differ from $y_{p+1,a}$ in that $d(y_{k,a}, x_{a_i}) \neq d(y_{p+1,a}, x_{a_i})$ for at least one i, $1 \leqslant i \leqslant t$. This will be accomplished if we require $\eta_a^k \neq \eta_a^{p+1}$ where $d(y_{p+1,a}, x_{a_i}) = r_{\sigma_t(\eta_a^{p+1}),i}$ for $i = 1, 2, \dots, t$, serves to define η_a^{p+1}. Thus we let η_a^k be the smallest positive integer not used in some preceding step which gives distances compatible with those already chosen.

The distances $d(y_{k,a}, x_{a_i})$ for $i = 1, 2, \dots, t$ are the only distances involving our new point which we need to be particular about. Thus we define all other distances from previously chosen points to $y_{k,a}$ in any way that will yield a metric. It is not difficult to see that this is possible.

By induction it follows that we can add new points $y_{k,a}$ to the space for each $a \in A_{k-1}$. Suppose this has been done. We rename the points $\{y_{k,a}\}_{a \in A_{k-1}}$ to be $\{x_i\}_{i=n_{k-1}+1}^{n_k}$ where n_k is the total number of points chosen in the first k steps. Which $y_{k,a}$ becomes which x_i is unimportant. This completes step k and leaves us ready to begin step $k+1$ with points $x_1, x_2, \dots, x_{n_k}$ already chosen.

It follows by induction that we can carry out step n for every positive integer n. We define U_0 to be the metric space consisting of all points constructed in this induction. Since the number of points constructed in each step is finite, U_0 is countable. The following theorem is crucial to the study of U_0.

THEOREM 1. *Let $y_1, y_2, \dots, y_s$ be a finite number of distinct points of the space U_0. Suppose $\mu_1, \mu_2, \dots, \mu_s$ are s positive rational numbers which satisfy the inequalities*

$$(1) \qquad |\mu_i - \mu_j| \leqslant d(y_i, y_j) \leqslant \mu_i + \mu_j , \quad \textit{for} \quad 1 \leqslant i, j \leqslant s .$$

Then there exists a point x of U_0 such that $d(x, y_i) = \mu_i$ for $1 \leqslant i \leqslant s$.

Proof. Let t be the smallest integer such that $y_1, y_2, \dots, y_s$ were all chosen in the first t steps of the construction of U_0. Let $\sigma_s^{-1}(\mu_1, \mu_2, \dots, \mu_s) = n$ so that $\mu_i = r_{\sigma_s(n),i}$ for $i = 1, 2, \dots, s$.

Let $y_i = x_{a_i}$, for $i = 1, 2, \dots, s$, and let $a = (a_1, a_2, \dots, a_s)$. For each $k > t$ we constructed a point $y_{k,a}$ satisfying

$$d(y_{k,a}, x_{a_i}) = r_{\sigma_s(\eta_a^k),i} , \quad \text{for} \quad i = 1, 2, \dots, s .$$

Now by (1), n satisfies the inequalities

$$|r_{\sigma_s(n),p} - r_{\sigma_s(n),q}| \leqslant d(x_{a_p}, x_{a_q}) \leqslant r_{\sigma_s(n),p} + r_{\sigma_s(n),q}$$

for $1 \leqslant p \leqslant s$ and $1 \leqslant q \leqslant s$. This implies that $\eta_a^k = n$ for some $k > t$ since the inequalities tell us that choosing η_a^k to be n is compatible with all distances chosen in steps preceding step k. For this k we let $x = y_{k,a}$. We have

$$d(x, y_i) = d(y_{k,a}, x_{a_i}) = r_{\sigma_s(\eta_a),i} = r_{\sigma_s(n),i} = \mu_i , \quad \text{for} \quad i = 1, 2, \dots, s ,$$

as desired. □

The proof of the following theorem runs parallel to one given by Urysohn [1].

THEOREM 2. *Suppose U_0' is a countable metric space such that the distance between any two points is rational. Suppose that Theorem 1 is still true when U_0 is replaced by U_0'. Then there is a distance preserving bijection from U_0 onto U_0'.*

Proof. Let $U_0' = \{a_1, a_2, \dots\}$ and recall that $U_0 = \{x_1, x_2, \dots\}$.

We now define the bijection $\varphi\colon U_0 \to U_0'$. Let $\varphi(x_1) = a_1$. Suppose we have already defined φ from $\{x_1', x_2', \dots, x_n'\}$ onto $\{a_1', a_2', \dots, a_n'\}$ with $\varphi(x_i') = a_i'$ and $x_1 = x_1'$, $a_1 = a_1'$. If n is even let x_{n+1}' be the first point in the sequence $\{x_1, x_2, \dots\}$ not in $\{x_1', x_2', \dots, x_n'\}$. Let $\mu_i = d(x_i', x_{n+1}')$. Then the hypotheses of Theorem 1 are satisfied for the space U_0' with $s = n$ and $y_i = a_i'$ for $i = 1, 2, \dots, n$. Thus there is a point a_{n+1}' in U_0'

such that $d(a'_i, a'_{n+1}) = \mu_i = d(x'_i, x'_{n+1})$ for all $i = 1, 2, \ldots, n$. We define $\varphi(x'_{n+1}) = a'_{n+1}$.

If n is odd we reverse the roles of U_0 and U'_0. We begin by letting a'_{n+1} be the first point in the sequence $\{a_1, a_2, a_3, \ldots\}$ not in $\{a'_1, a'_2, \ldots, a'_n\}$. Then using Theorem 1 we choose x'_{n+1} from U_0 so that $d(x'_i, x'_{n+1}) = d(a'_i, a'_{n+1})$. Then we define $\varphi(x'_{n+1}) = a'_{n+1}$.

This defines our function φ inductively. It is clear that φ preserves distances and sends U_0 onto U'_0. The latter follows from the way we alternated the roles of U_0 and U'_0 in choosing x'_{n+1} and a'_{n+1}. □

Urysohn [1] constructed a space which he also called U_0 in which the distance between any two points is rational. Urysohn showed that Theorem 1 is true when his space U_0 is used instead of ours. Thus by Theorem 2, Urysohn's space U_0 and our space U_0 are isometric.

Let us compare our construction of U_0 with that of Urysohn. Both spaces were constructed inductively with the intention of obtaining Theorem 1. When we added a point to our space we were interested in its distance to the points in some subset of the previously chosen points. When Urysohn added a point to his space he was also interested primarily in its distance to the points of some subset of the previously chosen points. However, in his case this subset is always an initial segment of the previously chosen points. That is, if the subset contains the nth point chosen and $0 < k < n$, then it also contains the kth point chosen. Since he specified other distances as well, this fact does not become obvious until later when he has finished using the distances he defined. I feel that our construction is shorter, simpler, and makes possible a simpler proof of Theorem 1 than is the case with Urysohn's construction.

We let U be the completion of U_0. Urysohn showed that U is a universal separable metric space. That is U contains an isometric image of every separable metric space.

3. Homogeneity of U with respect to convergent sequences. Urysohn [1] showed that his space was homogeneous with respect to finite subsets. That is if A and B are two finite subsets of U such that there is an isometry f from A onto B, then f can be extended to an isometry from U onto itself. Our goal is to extend this result to convergent sequences. We will need the following theorem which was proved by Urysohn and is a consequence of Theorem 1.

THEOREM 3. *Let $x_1, x_2, \ldots, x_s$ be s distinct points of U and suppose $a_1, a_2, \ldots, a_s$ are s positive real numbers which satisfy the inequalities*

$$|a_i - a_j| \leqslant d(x_i, x_j) \leqslant a_i + a_j, \quad \text{for} \quad i, j = 1, 2, \ldots, s.$$

Then there exists a point y of U such that $d(y, x_i) = a_i$ for $i = 1, 2, \ldots, s$.

This can be extended to the following result.

THEOREM 4. *Let $\{x_i\}_{i\geqslant 1}$ be a convergent sequence in Urysohn's space U. Suppose non-negative real numbers $\{\mu_i\}_{i\geqslant 1}$ are given so that*

$$(2) \qquad |\mu_i - \mu_j| \leqslant d(x_i, x_j) \leqslant \mu_i + \mu_j, \quad \text{for} \quad i, j \geqslant 1.$$

Then there is a point $p \in U$ so that $d(p, x_i) = \mu_i$ for each $i \geqslant 1$.

Proof. We will construct a sequence $\{p_i\}_{i\geqslant 1}$ which converges to p. Choose p_1 so that $d(p_1, x_1) = \mu_1$. Let

$$\nu_1 = \sup_{i\geqslant 1} |d(p_1, x_i) - \mu_i|.$$

Since $\{x_i\}_{i\geqslant 1}$ converges we know ν_1 is finite. We define $n_1 = 1$.

We continue by induction. Suppose $p_1, p_2, \ldots, p_{k-1}$ and $n_1, n_2, \ldots, n_{k-1}$ have been chosen so that each p_t is a point of U, each n_t is a positive integer, and the following conditions are satisfied:

$$n_1 < n_2 < \ldots < n_{k-1};$$

$$\nu_t = \sup_{i\geqslant 1} |d(p_t, x_i) - \mu_i|, \quad \text{for} \quad 1 \leqslant t \leqslant k-1;$$

$$d(p_t, x_i) = \mu_i, \quad \text{for} \quad 1 \leqslant i \leqslant n_t;$$

$$d(p_t, p_s) = \nu_t, \quad \text{whenever} \quad t < s;$$

and

$$\nu_t \leqslant \nu_{t-1}/2, \quad \text{for} \quad 2 \leqslant t \leqslant k-1.$$

If some $\nu_t = 0$ then let $p = p_t$ and by the definition of ν_t we are finished. Thus we may assume $\nu_t > 0$ for each positive integer t.

Let y be any point of U. Then for any positive integers i and j we have

$$d(y, x_i) \leqslant d(y, x_j) + d(x_i, x_j) \leqslant d(y, x_j) + \mu_i + \mu_j.$$

Thus,

$$d(y, x_i) - \mu_i \leqslant d(y, x_j) + \mu_j.$$

From (2) we have

$$\mu_i - \mu_j \leqslant d(x_i, x_j) \leqslant d(y, x_j) + d(y, x_i).$$

Thus,

$$\mu_i - d(y, x_i) \leqslant d(y, x_j) + \mu_j.$$

Thus we have

$$|d(y, x_i) - \mu_i| \leqslant d(y, x_j) + \mu_j$$

for any positive integers i and j.

By choosing $y = p_t$, $1 \leqslant t \leqslant k-1$, we have

$$(3) \qquad \nu_t = \sup_{i\geqslant 1} |d(p_t, x_i) - \mu_i| \leqslant \inf_{j\geqslant 1} \big(d(p_t, x_j) + \mu_j\big)$$

for $1 \leqslant t \leqslant k-1$.

Choose $n_k > n_{k-1}$ so that $i, j \geqslant n_k$ implies

$$|\mu_i - \mu_j| \leqslant d(x_i, x_j) \leqslant \nu_{k-1}/4 .$$

Choose p_k so that $d(p_k, x_i) = \mu_i$, for $i \leqslant n_k$, and $d(p_k, p_t) = \nu_t$, for $t < k$.

We will next show that such a choice of p_k is possible. By Theorem 3, we need only show:

(a) $|\mu_i - \mu_j| \leqslant d(x_i, x_j) \leqslant \mu_i + \mu_j$, for $1 \leqslant i, j \leqslant n_k$;

(b) $|\nu_t - \nu_s| \leqslant d(p_t, p_s) \leqslant \nu_t + \nu_s$, for $1 \leqslant s, t \leqslant k-1$;

and

(c) $|\nu_t - \mu_i| \leqslant d(p_t, x_i) \leqslant \nu_t + \mu_i$, for $1 \leqslant t \leqslant k-1$ and $1 \leqslant i \leqslant n_k$.

(a) follows from (2).

Proof of (b). Without loss of generality we assume $t < s$ so that $d(p_t, p_s) = \nu_t > \nu_s$. Then (b) becomes $|\nu_t - \nu_s| \leqslant \nu_t \leqslant \nu_t + \nu_s$. Since $0 < \nu_s < \nu_t$ this is true.

Proof of (c). By the definition of ν_t we know

$$\nu_t \geqslant |d(p_t, x_i) - \mu_i| \geqslant d(p_t, x_i) - \mu_i , \quad \text{for any } i .$$

Thus, $d(p_t, x_i) \leqslant \nu_t + \mu_i$ as desired.

If $\nu_t \geqslant \mu_i$, then by (3), $\nu_t \leqslant d(p_t, x_i) + \mu_i$ or $\nu_t - \mu_i \leqslant d(p_t, x_i)$. Since $\nu_t \geqslant \mu_i$ we have $|\nu_t - \mu_i| \leqslant d(p_t, x_i)$ as desired.

If $\nu_t < \mu_i$ then using the definition of ν_t we have,

$$\nu_t \geqslant |d(p_t, x_i) - \mu_i| = |\mu_i - d(p_t, x_i)| .$$

Thus, $\nu_t \geqslant \mu_i - d(p_t, x_i)$ or $\mu_i - \nu_t \leqslant d(p_t, x_i)$. Since $\nu_t < \mu_i$ we have $|\mu_i - \nu_t| \leqslant d(p_t, x_i)$ as desired. Hence (c) is true.

Having verified (a), (b), and (c) we know it is possible to choose p_k satisfying the desired properties.

Next we define

$$\nu_k = \sup_{i \geqslant 1} |d(p_k, x_i) - \mu_i| .$$

To complete the induction we need only show $\nu_k \leqslant \nu_{k-1}/2$. Recall that we chose n_k so large that

$$|\mu_i - \mu_j| \leqslant d(x_i, x_j) \leqslant \nu_{k-1}/4 \quad \text{whenever} \quad i, j \geqslant n_k .$$

Since $i \leqslant n_k$ implies $d(p_k, x_i) = \mu_i$ we have

$$\nu_k = \sup_{i > n_k} |d(p_k, x_i) - \mu_i| .$$

Fix $i > n_k$. Assume first that $d(p_k, x_i) \geqslant \mu_i$. Then,

$$|d(p_k, x_i) - \mu_i| \leqslant |d(p_k, x_{n_k}) + d(x_{n_k}, x_i) - \mu_i| = |\mu_{n_k} + d(x_{n_k}, x_i) - \mu_i|$$
$$\leqslant |\mu_{n_k} - \mu_i| + |d(x_{n_k}, x_i)| \leqslant \nu_{k-1}/4 + \nu_{k-1}/4 = \nu_{k-1}/2 .$$

Now assume that $d(p_k, x_i) < \mu_i$. We have,

$$|d(p_k, x_i) - \mu_i| = \mu_i - d(p_k, x_i) \leqslant [\mu_{n_k} + |\mu_i - \mu_{n_k}|] - [d(p_k, x_{n_k}) - d(x_{n_k}, x_i)]$$
$$\leqslant [\mu_{n_k} + \nu_{k-1}/4] - [\mu_{n_k} - \nu_{k-1}/4] = \nu_{k-1}/2 .$$

Thus in either case $|d(p_k, x_i) - \mu_i| \leqslant \nu_{k-1}/2$ and we have

$$\nu_k = \sup_{i > n_k} |d(p_k, x_i) - \mu_i| \leqslant \nu_{k-1}/2$$

as desired. This completes the induction and shows that we can choose the sequence $\{p_i\}_{i \geqslant 1}$ as indicated.

Since $\lim_{i \to \infty} \nu_i = 0$ and $d(p_t, p_s) = \nu_t$ for $t < s$ we know that $\{p_i\}_{i \geqslant 1}$ is a Cauchy sequence. Define $p = \lim_{i \to \infty} p_i$.

Fix $k \geqslant 1$. Since $d(p_i, x_k) = \mu_k$ for all i large enough so that $n_i \geqslant k$ it follows easily that $d(p, x_k) = \mu_k$, for all $k \geqslant 1$. □

Definition. A metric space E is said to be *homogeneous with respect to convergent sequences* if given any two convergent sequences $\{x_n\}_{n \geqslant 1}$ and $\{y_n\}_{n \geqslant 1}$ in E satisfying $d(x_i, x_j) = d(y_i, y_j)$ for all positive integers i and j, then there is an isometric transformation φ from E onto E such that $\varphi(x_i) = y_i$ for all positive integers i.

Theorem 5. *Urysohn's space U is homogeneous with respect to convergent sequences.*

Proof. Let $\{x_n\}_{n \geqslant 0}$ and $\{y_n\}_{n \geqslant 0}$ be two convergent sequences such that

$$d(x_i, x_j) = d(y_i, y_j) , \quad \text{for} \quad i, j \geqslant 0 .$$

Let $\{a_1, a_2, a_3, \ldots\}$ be a countable dense subset of U. Suppose points $x_{-1}, x_{-2}, \ldots, x_{-n}$ and $y_{-1}, y_{-2}, \ldots, y_{-n}$ have been chosen so that

$$d(x_i, x_j) = d(y_i, y_j) \quad \text{for} \quad i, j \geqslant -n .$$

If n is even let $x_{-(n+1)}$ be the first member in the sequence $\{a_1, a_2, \ldots\}$ not in $\{x_k\}_{k \geqslant -n}$. Let $\mu_i = d(x_{-(n+1)}, x_i)$ for all $i \geqslant -n$. These μ_i satisfy the hypothesis of Theorem 4 for the convergent sequence $\{y_k\}_{k \geqslant -n}$. Thus by Theorem 4, we choose $y_{-(n+1)}$ so that

$$d(y_{-(n+1)}, y_i) = \mu_i = d(x_{-(n+1)}, x_i) , \quad \text{for} \quad i \geqslant -n .$$

If n is odd we reverse the roles of $\{x_i\}_{i \geqslant -n}$ and $\{y_i\}_{i \geqslant -n}$. Thus we let $y_{-(n+1)}$ be the first point in the sequence $\{a_1, a_2, \ldots\}$ not in $\{y_i\}_{i \geqslant -n}$. Then we use Theorem 4 to choose $x_{-(n+1)}$ so that

$$d(x_{-(n+1)}, x_i) = d(y_{-(n+1)}, y_i) , \quad \text{for} \quad i \geqslant -n .$$

We continue this construction and obtain sequences $\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ and $\{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$ so that

$$d(x_i, x_j) = d(y_i, y_j) ,$$

for all integers i and j, and so that

$$\{a_1, a_2, \ldots\} \subset \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\},$$

and

$$\{a_1, a_2, \ldots\} \subset \{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}.$$

Hence each of the two sets $\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ and $\{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$ is dense in U. We now define $\varphi\colon U \to U$ by

$$\varphi(x_i) = y_i, \quad \text{for any integer } i.$$

On the other points of U we define φ by

$$\varphi(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} \varphi(x_{n_i})$$

where $\{x_{n_i}\}_{i\geqslant 1}$ is any Cauchy sequence taken from $\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$. It is easy to see that φ is one to one, maps U onto U, and preserves distances. □

Urysohn [1] gave an example of two bounded isometric subsets A and B of U with the property that no isometry from A onto B can be extended to an isometry from U onto itself. By considering countable dense subsets of A and B we can show that neither Theorem 4 nor Theorem 5 can be extended to arbitrary bounded sequences.

References

[1] P. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. 51 (1927), pp. 43–64 74–90.

UNIVERSITY OF CALIFORNIA
Santa Barbara, California

Reçu par la Rédaction le 28. 4. 1970

C-scattered and paracompact spaces *

by

Rastislav Telgársky (Wrocław)

0. Introduction. The main problem considered in this paper is the problem of the topological product of paracompact spaces (Section 2). C-scattered spaces, which play an important rôle in this problem, are studied in Section 1 and some strong covering properties of C-scattered paracompact spaces are proved in Section 3. The results from this paper were partially announced in [25].

Each topological space considered in this paper is assumed to be completely regular.

The problem in a general setting reads as follows: what kind of separation and covering properties are preserved by the Cartesian product of finitely many spaces?

The Cartesian product of two normal (even paracompact) spaces need not be normal ([11], [19]). But, as J. Dieudonné [1] proved, the product $S\times T$ of a paracompact space S and a compact space T is always paracompact and hence normal. An excellent result of H. Tamano [23] reads: a completely regular space S is paracompact iff $S\times\beta S$ is normal. K. Morita [13] proved that if S is paracompact and such that each point has a nbd basis of the cardinality $\leqslant m$ and T is an m-compact normal space, then $S\times T$ is normal. This phenomenon appeared earlier also in the product of two $\aleph_0$-compact spaces, as is explicitly stated in the following Theorem of C. Ryll-Nardzewski [17]: if S is $\aleph_0$-compact and such that each point has a nbd basis of the cardinality $\leqslant m$ and T is m-compact, then $S\times T$ is $\aleph_0$-compact. An assumption concerning the cardinality of a basis plays an essential role also in the Product Theorem of K. Morita [13]: S is a normal m-paracompact space iff $S\times[0,1]^m$ is

* This is an essential part of my doctoral dissertation finished in June 1969 written under the supervision of Professor Czesław Ryll-Nardzewski. This work was supported by the Educational Ministry of the ČSSR during my doctoral studies at the Mathematical Institute of the Polish Academy of Sciences, 1967–1970.