If M is a null set, then not even the measurability of τ_g can be guaranteed. This follows easily from the examples 1, 2 using the well-known fact that to every real function $h: \langle a, b\rangle \to E_1$ there exists a function $f: \langle a, b\rangle \to E_1$ with the Darboux property such that $\{x \in \langle a, b\rangle; f(x) \neq h(x)\}$ is a null set (cf. [4]).

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An unfinitizability proof by means of restricted reduced power

by

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In this paper I try to make some progress in solving the problem of the unfinitizability of theories containing the arithmetic of natural numbers. This problem remained open after Ryll-Nardzewski's paper [6] proving the infinitizability of the rule of induction in elementary arithmetic. The method of the present paper consists of constructing a kind of reduced power restricted to functions and sets definable by means of n-quantifiers. The main observation (Theorem 1) is that in this case only the sentences containing n-quantifiers which are true in the basic model $\mathfrak{M}$ remain true in the reduced ultra power $\mathfrak{M}^*$. Finding a theorem which is not preserved, we get the unfinitizability proof. Dividing by a filter cut up to sets definable by n-quantifiers may be conceived as adding new "defective" objects having only n-quantifier properties. It might be presumed that these new objects preserve only n-quantifier statements.

The result obtained in this way was independently obtained also by Ryll-Nardzewski by means of the method of his old paper [6]. It is probably not the strongest one. The problem remains open for theories containing arithmetic and dealing with two kinds of objects: natural numbers and the other objects (sets, classes etc.). A partial result in this domain was obtained by A. Mostowski in [4]. The main contribution of this paper is an outline of a new method.

1. Restricted filters, functions and ultrapower. Let C be an arbitrary family of subsets of a given set M; we shall consider the following notion of ultrafilter restricted to C:

(1) $$D \in \mathrm{Uf}(C) \Longleftrightarrow \begin{array}{l} 1.\ D \subset C, \\ 2.\ \emptyset \notin D, \\ 3.\ X, Y \in D \wedge X \cap Y \in C \to X \cap Y \in D, \\ 4.\ X \in D \wedge X \subset Y \wedge Y \in C \to Y \in D, \\ 5.\ X \cup Y \in D \wedge X, Y \in C \to X \in D \vee Y \in D. \end{array}$$

In the applications M will be the set of natural numbers (also non-standard) and C the class Σ_n^0 of Kleene–Mostowski hierarchy. $C \subset P(M)$ and $<,>$ is a fixed pairing function: $M \times M \to M$. We shall consider a class of restricted functions $\mathrm{Fu}(D, C)$.

$$(2) \qquad f \in \mathrm{Fu}(D, C) \Longleftrightarrow \begin{array}{l} 1.\ f \subset M, \\ 2.\ x, y, z \in M \wedge \langle x, y\rangle \in f \wedge \langle x, z\rangle \in f \to y = z, \\ 3.\ \{x\colon \exists y \langle x, y\rangle \in f\} \in D, \\ 4.\ f \in C. \end{array}$$

Now suppose $\mathfrak{M} = \langle M, R\rangle$ to be a relational system. Having a pairing function, we can assume that $R \subset M$. We shall consider the restricted ultrapower $\beta(\mathfrak{M}, D, C)$:

$$\beta(\mathfrak{M}, D, C) = \mathfrak{M}^* = \langle M^*, R^*\rangle$$

where

$$(3) \qquad M^* = \{[f]\colon f \in \mathrm{Fu}(D, C)\}$$

and $[f] = \{g \in \mathrm{Fu}(D, C)\colon \{i\colon \exists y\ \langle i, y\rangle \in f \cap g\} \in D\}$,

$$(4) \quad R^* = \{([f], [g])\colon \{i\colon \exists y, z(\langle i, y\rangle \in f \wedge \langle i, z\rangle \in g \wedge \langle y, z\rangle \in R)\} \in D\}.$$

LEMMA 1. *If $D \in \mathrm{Uf}(C)$ and C satisfies the conditions:*

$$(5) \qquad X, Y \in C \to X \cap Y \in C,$$

$$(6) \qquad X \in C \to \{i\colon \exists y \langle i, y\rangle \in X\} \in C,$$

$$(7) \qquad \begin{cases} X \in C \to \{\langle x, y\rangle\colon \langle y, x\rangle \in X\} \in C, \\ X \in C \to \{\langle x, y\rangle\colon x \in X\} \in C, \\ X \in C \to \{\langle x, \langle y, z\rangle\rangle\colon \langle\langle x, y\rangle, z\rangle \in X\} \in C, \end{cases}$$

$$(8) \qquad R \in C$$

then the relation

$$f \simeq g \Longleftrightarrow \{i\colon \exists y \langle i, y\rangle \in f \cap g\} \in D$$

is a congruence in $\mathrm{Fu}(D, C)$, *and the relation* R^* *is correctly defined.*

Proof. Reflexivity by (2) 3. Symmetry logicaly true. Proving transitivity we need first (5) in order to prove that $\{i\colon \exists y \langle i, y\rangle \in f \cap g\} \cap \cap \{i\colon \exists y \langle i, y\rangle \in g \cap h\} \in D$ according to (1) 3. Hence by (2) 2 we have $\{i\colon \exists y \langle i, y\rangle \in f \cap g \cap h\} \in D$. Of course $\{i\colon \exists y \langle i, y\rangle \in f \cap g \cap h\} \subset \{i\colon \exists y \langle i, y\rangle \in f \cap h\}$, but to infer that the last set belongs to D we must show by (1) 4 that it belongs to C. For this we need also (5) and (6).

To verify the correctness of (4) we prove the implication

$$f \simeq f_1 \wedge g \simeq g_1 \wedge \{i\colon \exists y, z(\langle i, y\rangle \in f \wedge \langle i, z\rangle \in g \wedge \langle y, z\rangle \in R)\} \in D$$
$$\to \{i\colon \exists y, z(\langle i, y\rangle \in f_1 \wedge \langle i, z\rangle \in g_1 \wedge \langle y, z\rangle \in R)\} \in D.$$

This last set contains by (2) 2 the set

$$\{i\colon \exists y \langle i, y\rangle \in f \cap f_1\} \cap \{i\colon \exists z \langle i, z\rangle \in g \cap g_1\} \cap$$
$$\cap \{i\colon \exists y, z(\langle i, y\rangle \in f \wedge \langle i, z\rangle \in g \wedge \langle y, z\rangle \in R)\},$$

which belongs to D according to (5), (7) and (1) 3. But to prove the implication we need according to (1) 4 the premise

$$\{i\colon \exists y, z(\langle i, y\rangle \in f_1 \wedge \langle i, z\rangle \in g_1 \wedge \langle y, z\rangle \in R)\} \in C.$$

This we get by means of (5)–(8).

Now we shall consider a first order language containing the variables $V_1, V_2, \ldots$, the identity and the predicate R (corresponding to the relations $R(R^*)$ of our systems $\langle M, R\rangle$ ($\langle M^*, R^*\rangle$)), logical connectives and quantifiers $\bigvee \bigwedge$.

In this language there are some operators, i.e. structures $\mathcal{E}(V_i)(p_i, p_j, V_j)$ with two propositional free places p_i, p_j, one individual free variable V_j and one bound individual variable V_i such that, taking a formula $\Phi(V, t_1, \ldots, t_n)$ containing neither V_i nor V_j, we get by substitution a meaningful expression $\mathcal{E}(V_i)(\Phi, V_j)$ of the form

$$(\mathrm{E}) \qquad \mathcal{E}(V_i)\big(\Phi(V/V_i, t_1, \ldots, t_n), \Phi(V/V_j, t_1, \ldots, t_n), V_j\big),$$

in which V_i is bound and $V_j, t_1, \ldots, t_n$ are free variables. (Eg.: according to our definition, a μ — operator is the formula $\mathcal{E}(V_1)(p_1, p_2, V_2)$ of the shape $\bigwedge_{V_1}(V_1 < V_2 \to \neg p_1) \wedge p_2$.

We shall say that $\mathcal{E}(V_i)(p_i, p_j, V_j)$ is an *$\mathcal{E}$-operator on the ground of the set S of sentences* if for every formula $\Phi(V, t_1, \ldots, t_n)$ containing neither V_i nor V_j, V_k we have

$$(10) \qquad \ulcorner \mathcal{E}(V_i)(\Phi, V_j) \wedge \mathcal{E}(V_i)(\Phi, V_k) \to V_j = V_k \urcorner \in \mathrm{Cn}\, S$$

$$(11) \quad \ulcorner \bigvee_{V_i} \Phi(V/V_i, t_1, \ldots, t_n) \to \bigvee_{V_j}\big(\mathcal{E}(V_i)(\Phi, V_j)\big) \wedge \Phi(V/V_j, t_1, \ldots, t_n) \urcorner \in \mathrm{Cn}\, S.$$

The class C will be called semantically closed on the $\mathcal{E}$-operator applied to the formulas of the set X if, for every formula $\Phi \in X$,

$$(12) \qquad \{\langle z, y\rangle\colon \mathfrak{M} \models \mathcal{E}(V_i)\big(\Phi(V/V_i, z), \Phi(y, z), y\big)\} \in C\ (^1).$$

THEOREM 1. (Łoś's restricted satisfaction theorem for ultrapower). *If $D \in \mathrm{Uf}(C)$ and C satisfies conditions* (5)–(8) *and $\emptyset \in C$, Φ is a formula*

(1) This manner of writing satisfaction does not lead to confusion if we make a distinction between variables of the object language and the meta language.

of the form

$$\Phi(\ldots) = \bigvee_{V_1}\Omega(V_1\ldots)$$

and

(13) *if Φ' is subformula of $\Omega(V_1\ldots)$, then Φ' and $\neg\Phi'$ define in $\mathfrak{M}$ sets of the class C;*

moreover, if $\mathcal{E}(V_i)(p_i, p_j, V_j)$ is an $\mathcal{E}$-operator on the ground of the set $\mathrm{Tr}(\mathfrak{M})$ of sentences true in $\mathfrak{M}$, and the class C is semantically closed on the $\mathcal{E}$-operation applied to subformulas of $\Omega(V_1\ldots)$, then for every $f_1, \ldots, f_n \in \mathrm{Fu}(D, C)$ the following equivalence holds:

(14) $\mathfrak{M}^* \models \Phi([f_1], \ldots, [f_n])$

$$\Longleftrightarrow \{i\colon \exists y_1, \ldots, y_n(\langle i, y_1\rangle \in f_1 \wedge \ldots \wedge \langle i, y_n\rangle \in f_n \wedge \mathfrak{M} \models \Phi(y_1, \ldots, y_n))\} \in D.$$

The last formula will also be written shortly as

$$\{i\colon \mathfrak{M} \models \Phi(f_1(i), \ldots, f_n(i))\} \in D \quad \text{or} \quad \{i\colon \mathfrak{M} \models \Phi\} \in D\,.$$

Proof by induction with respect to subformulas of Φ: For atomic formulas directly from the definition (4) of R^* and Lemma 1 we get the equivalence

$$\mathfrak{M}^* \models R([f], [g]) \Longleftrightarrow \{i\colon \exists y, z(\langle i, y\rangle \in f \wedge \langle i, z\rangle \in g \wedge \mathfrak{M} \models R(y, z))\} \in D.$$

Also from lemma 1 and definition (3) we get

$$\mathfrak{M}^* \models [f] \models [g] \Longleftrightarrow f \simeq g \Longleftrightarrow \{i\colon \exists y \langle i, y\rangle \in f \cap g\} \in D$$

$$\Longleftrightarrow \{i\colon \exists y_1 y_2(\langle i, y_1\rangle \in f \wedge \langle i, y_2\rangle \in g \wedge \mathfrak{M} \models y_1 = y_2)\} \in D.$$

Now suppose (14) for Φ' and Φ'' subformulas of $\Omega(V_1)$; we shall prove (14) for $\Phi' \wedge \Phi''$ and $\neg\Phi'$. By the inductive hypothesis and according to the properties of the relation of satisfaction, we have the equivalences

(15) $$\mathfrak{M}^* \models (\Phi' \wedge \Phi'') \Longleftrightarrow \mathfrak{M}^* \models \Phi' \wedge \mathfrak{M} \models \Phi''$$

$$\Longleftrightarrow \{i\colon \mathfrak{M} \models \Phi'\} \in D \wedge \{i\colon \mathfrak{M} \models \Phi''\} \in D.$$

Hence if $\mathfrak{M}^* \models (\Phi'([f], [g]) \wedge \Phi''([f], [h]))$, then by (1), (5), and (2) 2

(16) $$I = \{i\colon \exists y, w, z(\langle i, y\rangle \in f \wedge \langle i, w\rangle \in g \wedge \langle i, z\rangle \in h \wedge \mathfrak{M} = \Phi(y, w) \wedge \mathfrak{M} \models \Phi''(y, z))\} \in D.$$

This means that $I = \{i\colon \mathfrak{M} \models \Phi' \wedge \Phi''\} \in D$. Conversely, if $\{i\colon \mathfrak{M} \models \Phi' \wedge \Phi''\} \in D$, then we have (16). Hence the following inclusions are evident:

$$I \subset \{i\colon \exists yw(\langle i, y\rangle \in f \wedge \langle i, w\rangle \in g \wedge \mathfrak{M} \models \Phi'(y, w))\} = \{i\colon \mathfrak{M} \models \Phi'\},$$

$$I \subset \{i\colon \exists yz(\langle i, y\rangle \in f \wedge \langle i, z\rangle \in h \wedge \mathfrak{M} \models \Phi''(y, z))\} = \{i\colon \mathfrak{M} \models \Phi''\}.$$

The right-hand sets in these inclusions belong to C according to (13) and (5)–(8). Hence by (1) 4 and (16) they belong to D. This according to (15) completes the proof for the conjunction.

For negation, we have first, by the inductive hypothesis,

(17) $$\mathfrak{M}^* \models \neg\Phi' \Longleftrightarrow I = \{i\colon \mathrm{E}y(\langle i, y\rangle \in f \wedge \mathfrak{M} \models \Phi'(y))\} \notin D.$$

Of course, by (13) and (5)–(8): $I \in C$. On the other hand, also by means of (13) and (5)–(8) we obtain

$$J = \{i\colon \exists y(\langle i, y\rangle \in f \wedge \mathfrak{M} \models \neg\Phi'(y))\} \in C.$$

Now we note that

$$I \cup J = \{i\colon \exists y \langle i, y\rangle \in f\} \in D$$

according to (2) 3 because $f \in \mathrm{Fu}(D, C)$. Hence, by (1) 5 if $I \notin D$, then $J \in D$. Of course, $I \cap J = \emptyset \notin D$; then by (13) and (1) 2 if $J \in D$, then $I \notin D$. Thus by (17) we get

$$\mathfrak{M}^* \models \neg\Phi' \Longleftrightarrow \hat{J} \in D\,,$$

which completes the proof for the negation.

Now consider the operation of the existential quantifier. Let Ψ be a subformula of $\Omega(V_1)$. This comprises also the case $\Psi = \Omega(V_1)$. Of course, we have the equivalence

$$\mathfrak{M}^* \models \bigvee_V \Psi(V, [f]) \Longleftrightarrow \exists g \in \mathrm{Fu}(D, C)\, \mathfrak{M}^* \models \Psi([g], [f])\,.$$

Hence, by the inductive hypothesis,

(18) $$\mathfrak{M}^* \models \bigvee_V \Psi(V, [f]) \Longleftrightarrow \exists g \in \mathrm{Fu}(D, C),\quad I = \{i\colon \mathfrak{M} \models \Psi(g(i), f(i))\} \in D.$$

Consider the set

(19) $$J = \{i\colon \exists y, z(\langle i, z\rangle \in f \wedge \mathfrak{M} \models \Psi(y, z))\}\,.$$

According to (13) and (5)–(8): $J \in C$; on the other hand, the inclusion $I \subset J$ is evident. Hence, if $I \in D$, then by (1) 4 $J \in D$. Of course, $J = \{i\colon \exists z \langle i, z\rangle \in f \wedge \mathfrak{M} \models \bigvee_{V_1}\Psi(V_1, z)\} = \{i\colon \mathfrak{M} \models \bigvee_{V_1}\Psi(V_1, f(i))\}$. Conversely, if $J \in D$, then we define first a function g':

(20) $$\langle z, y\rangle \in g' \Longleftrightarrow \mathfrak{M} \models \mathcal{E}(V_1)(\Psi(V_1, z), \Psi(y, z), y)$$

and another function g:

(21) $$\langle i, y\rangle \in g \Longleftrightarrow \exists z(\langle i, z\rangle \in f \wedge \langle z, y\rangle \in g')\,.$$

According to the assumption of our theorem and formula (12), we get $g' \in C$. Hence, by (21) and (5)–(8), also

(22) $$g \in C\,.$$

From implication (11) for $S = \mathrm{Tr}(\mathfrak{M})$ we get

(23) $$\exists y\big(\mathfrak{M} \models \Psi(y, z)\big) \to \exists y\Big(\mathfrak{M} \models \mathcal{E}(V_1)\big(\Psi(V_1, z),\ \Psi(y, z), y\big) \wedge \Psi(y, z)\Big).$$

Hence

$$\exists y, z\big(\langle i, z\rangle \in f \wedge \mathfrak{M} \models \Psi(y, z)\big)$$

$$\iff \exists y, z\Big(\langle i, z\rangle \in f \wedge \mathfrak{M} \models \mathcal{E}(V_1)\big(\Psi(V_1, z),\ \Psi(y, z), y\big) \wedge \mathfrak{M} \models \Psi(y, z)\Big).$$

Thus, according to (19), (20) and (21),

(24) $$J = \big\{i:\ \exists y, z\big(\langle i, z\rangle \in f \wedge \langle i, y\rangle \in g \wedge \mathfrak{M} \models \Psi(y, z)\big)\big\}.$$

We need only to prove that $g \in \mathrm{Fu}(D, C)$. Condition (2) 1 is evident. (2) 2 follows from (10), (20) and (21). Formula (24) implies the inclusion

$$J \subset \{i:\ \exists y \langle i, y\rangle \in g\} = G.$$

Formulas (22) and (6) imply that $G \in C$. Hence, if $J \in D$, then also $G \in D$, which completes the proof that $g \in \mathrm{Fu}(D, C)$. Thus if $J \in D$, then there is a $g \in \mathrm{Fu}(D, C)$ such that the set J of the form (24) belongs to D, and this implies by (18) that $\mathfrak{M}^* \models \bigvee_V \Psi(V, [f])$.

Corollary 1. *If the assumptions of Theorem 1 are satisfied, the identity function I belongs to C, and the function $f \in \mathrm{Fu}(D, C)$ is definable in $\mathfrak{M}$ by the formula $\Phi(V_2, V_3)$ (this means that*

(25) $$\langle i, y\rangle \in f \iff \mathfrak{M} \models \Phi(i, y) \quad \textit{for every } i, y \in M),$$

then the element $[f] \in M^*$ is the value of the function defined in $\mathfrak{M}^*$ by the formula $\Phi(V_2, V_3)$ for the argument $[I]$.

This means that we ought to prove that

(i) $\mathfrak{M}^* \models \Phi(a, b) \wedge \mathfrak{M}^* \models \Phi(a, c) \to b = c$, for every $a, b, c \in M^*$,

(ii) $\mathfrak{M}^* \models \Phi([I], [f])$.

Proof of (i). Suppose that $a = [f]$, $b = [g]$, $c = [h]$. By Theorem 1 the suppositions of (i) mean that

(26) $$A = \big\{i:\ \exists y_1, y_2\big(\langle i, y_1\rangle \in j \wedge \langle i, y_2\rangle \in g \wedge \mathfrak{M} \models \Phi(y_1, y_2)\big)\big\} \in D,$$

(27) $$B = \big\{i:\ \exists y_1', y_2\big(\langle i, y_1'\rangle \in j \wedge \langle i, y_2\rangle \in h \wedge \mathfrak{M} \models \Phi(y_1', y_2)\big)\big\} \in D.$$

Hence, by (25) (5)–(7) and (1), $A \cap B \in D$. According to (25)–(27) and using (2) 2 we get

$$A \cap B \subset \{i:\ \exists y_1 y_2(\langle i, y_1\rangle \in j \wedge \langle y_1 y_2\rangle \in f \wedge \langle i, y_2\rangle \in g \cap h)\}.$$

Hence

$$A \cap B \subset \{i:\ \exists y \langle i, y\rangle \in g \cap h\}.$$

The set $\{i:\ \exists y \langle i, y\rangle \in g \cap h\}$ belongs to C according to (5) and (6). Thus it belongs to D. This means that $g \simeq h$: $b = c$.

Proof of (ii). If $I \in C$, then, of course, $I \in \mathrm{Fu}(D, C)$ if $\mathrm{Fu}(D, C)$ is not empty. Hence, according to Theorem 1, assertion (ii) is equivalent to

$$\big\{i:\ \exists y_1 y_2\big(\langle i, y_1\rangle \in I \wedge \langle i, y_2\rangle \in f \wedge \mathfrak{M} \models \Phi(y_1, y_2)\big)\big\} \in D.$$

But this set, according to (25), is identical with $\{i:\ \exists y_2 \langle i, y_2\rangle \in f\}$. Hence it belongs to D by (2) 3 if $f \in \mathrm{Fu}(D, C)$.

2. Unfinitizability of the induction schema in arithmetical extensions of Peano's arithmetic. Let Ar be the set of theorems of Peano's elementary arithmetic. For the purpose of our argument it is convenient to imagine Ar as having the primitive notions $0, S, +, \cdot$ and also a few other additional recursive notions, of course definable in Ar by means of $0, S, +, \cdot$, which will be specified later. The axioms of Ar are:

1. $0 \neq SV$, 2. $SV = SW \to V = W$, 3. $0 + V = V$, 4. $SW + V = S(W + V)$, 5. $0 \cdot V = 0$, 6. $SW \cdot V = W \cdot V + V$, the definitions of the additional recursive notions and the schema of induction

(28) Ind. $$\big(\Phi(0) \wedge \bigwedge_V\big(\Phi(V) \to \Phi(SV)\big)\big) \to \bigwedge_V \Phi(V)$$

for every formula Φ built by means of primitive notions, connectives and quantifiers.

A theory E will be called an *arithmetical extension of* Ar iff E is a theory formalized in first order functional calculus, the set of primitive notions of E is finite and embraces the set of primitive notions of Ar, the axioms of Ar belong to the axioms of E and the axioms of E contain all instances of schema (28) for all formulas Φ built from the primitives of E.

Let S be the set of all sentences and S_n the set of all sentences having at most n-quantifiers. Of course $S = \bigcup S_n$.

The unfinitizability theorem. *If E is a consistent arithmetical extension of* Ar, *then for every $n \in N$ there is a theorem $T \in E$ such that*

$$T \notin \mathrm{Cn}(E \cap S_n).$$

(Cn means the operation of logical consequence).

Proof. Suppose that our arithmetic Ar has the primitive notions $0, S, +, \cdot, p_y^x, (x)_y$, p_y^x being the x power of the prime number p_y, and $(x)_y = \exp(x, y)$ being the biggest exponent with which the prime number p_y occurs in the development in primes of the number x.

The well known property of the Kleene–Mostowski hierarchy may be written as the following syntactical property of the extensions of Ar:

LEMMA 2. *For every odd natural number n there is a formula $\Phi(V_0, V_1, V_2)$ of the form*

(29) $$\Phi(V_0, V_1, V_2) = \bigvee_{V_3} \bigwedge_{V_4} \dots \bigvee_{V_{n+2}} \bigwedge_{V_{n+3}} \big(V_{n+3} < V_{n+2} \to \delta(V_0, \dots, V_{n+3})\big)$$

(Φ has exactly $n+1$ quantifiers with the first and the penultimate existentials, δ contains no quantifiers), which is universal for formulas $\Psi(V_1, V_2)$ of the same kind. This means that for every Ψ there is such a $p \in N$, that

(30) $$\ulcorner \bigwedge_{V_1, V_2} \big(\Psi(V_1, V_2) \iff \Phi(\boldsymbol{p}, V_1, V_2)\big) \urcorner \in E .$$

This lemma is a little stronger than the familiar theorem on the existence of a universal Σ_n^0 functional; therefore I shall skech the proof.

Proof of the lemma. Consider the following notion Q of a function universal for functions which are primitive recursive in $P_1, \dots, P_k$. The axioms for Q may be the following:

$$\begin{aligned}
Q(0, x, y) &= 0 \iff P_1(x, y) ,\\
Q(0, x, y) &= 1 \iff \neg P_1(x, y) ,\\
&\dots\dots\dots\dots\dots\dots\\
Q(\boldsymbol{k}-1, x, y) &= 0 \iff P_k(x, y) ,\\
Q(\boldsymbol{k}-1, x, y) &= 1 \iff \neg P_k(x, y) ,\\
Q(\boldsymbol{k}, x, y) &= x ,\\
Q(\boldsymbol{k}+1, x, y) &= Sx ,\\
Q_1(\boldsymbol{k}+2, x, y) &= 0 ,\\
Q(\boldsymbol{k}+3, x, y) &= p_y^x ,\\
Q(\boldsymbol{k}+4, x, y) &= (x)_y ,\\
Q(\boldsymbol{k}+5, x, y) &= x \cdot y ,\\
Q(\boldsymbol{k}+6, x, y) &= x + y .
\end{aligned}$$

For $n \geqslant \boldsymbol{k}+6$:

$$Q(n+1, x, y) = \begin{cases} Q\big((n)_0, y, x\big) , & \text{when } (n)_2 = 0 ,\\ Q\big((n)_0, x, x\big) , & \text{when } (n)_2 = 1 ,\\ Q\Big((n)_0, x, Q\big((n)_1, x, y\big)\Big) , & \text{when } (n)_2 = 2 \end{cases}$$

when $(n)_2 \geqslant 3$:

$$\begin{aligned}
Q(n+1, 0, y) &= Q\big((n)_0, y, y\big) ,\\
Q(n+1, x+1, y) &= Q\big((n)_1, \langle x, y\rangle, Q(n+1, x, y)\big) . \text{[2]}
\end{aligned}$$

[2] $\langle x, y\rangle = 2^x \cdot 3^y$.

These axioms, for heuristic reasons, are written in variables n, x, y instead of the original variables V_1, V_2, V_3 of our object language.

It is well known that such an inductive definition may be reduced to a normal one. Hence Q is definable in E.

Further, we realize that the definition of Q in E may have the following shape:

$$Q(n, x, y) = z \iff \bigvee_{V_1} \bigwedge_{V_2} \big(V_2 < V_1 \to \delta'(V_1, V_2, n, x, y, z)\big)$$

where δ' does not contain any quantifier.

We do not introduce the notion Q in the system E. Q is useful only for finding the formula δ' of the above equivalence which satisfies all inductive postulates for the function Q. Having this δ' and the postulates for Q, we prove our lemma in the following steps:

1. For every name formula $\varphi(V_1, V_2)$ there is a number p such that

$$\ulcorner \bigwedge_{V_1, V_2} \big(\varphi(V_1, V_2) = Q(\boldsymbol{p}, V_1, V_2)\big) \urcorner \in E$$

(by induction with respect to φ).

2. For every sentential formula $\Delta(V_1, V_2)$ without quantifiers there is such a p that

$$\ulcorner \bigwedge_{V_1, V_2} \big(\Delta(V_1, V_2) \iff Q(\boldsymbol{p}, V_1, V_2) = 0\big) \urcorner \in E .$$

3. For every sentential formula $\Delta(V_1, V_2, \dots, V_n)$ without quantifiers there is a p such that

$$\ulcorner \bigwedge_{V_1 V_2 \dots V_n} \big(\Delta(V_1, V_2, \dots, V_n) \iff Q(\boldsymbol{p}, V_1, \langle V_2, \langle \dots V_n\rangle\rangle) = 0\big) \urcorner \in E .$$

4. Using pairing functions, we construct from δ' a formula δ without quantifiers and such that for every formula $\Delta(V_1, V_2, V_3)$ without quantifiers there is a p such that

$$\ulcorner \bigwedge V_3 \Big(\bigvee_{V_1} \bigwedge_{V_2} \big(V_2 < V_1 \to \Delta(V_1, V_2, V_3)\big) \iff \bigvee_{V_1} \bigwedge_{V_2} \big(V_2 < V_1 \to \delta(\boldsymbol{p}, V_1, V_2, V_3)\big)\Big) \urcorner \in E .$$

5. Using 3 we generalize 4 for more quantifiers.

Now, if E is consistent, then there is a model $\mathfrak{M} = \langle M, R_1, \dots \dots, R_{k+1}, a_1, \dots, a_t\rangle$ such that

(31) $$E \subset \mathrm{Tr}(\mathfrak{M}) ,$$

$R_1, \dots, R_k$ being interpretations of $P_1, \dots, P_k$ and $a_1, \dots, a_t$ being interpretations of arithmetical notions.

Let C be the class of sets Σ_n^0 in $R_1, \dots, R_k$, i.e. the class of subsets of M which are definable by formulas with at most n quantifiers, the first of them existential and the last general but limited. An exact definition of the class C may be the following. If $X \subset M$ then

(32) $X \in C \Longleftrightarrow$ there exists a formula $\Psi(V)$ with one free variable of the shape (29) and such that

$$\forall x \in M\big(x \in X \Longleftrightarrow \mathfrak{M} \models \Psi(x)\big) .$$

Let F be a proper non-principal ultrafilter over the set M. Let $D = C \cap F$. Hence $D \in \mathrm{Uf}(C)$ according to (1).

From the well-known facts about hierarchy it follows that C satisfies conditions (5)–(8) of lemma 1.

Let $\mathcal{E}$ be the μ-operator:

$$\mathcal{E}(V_i)(p_i, p_2, V_2) = \bigwedge_{V_i}(V_i < V_2 \to \neg p_i) \wedge p_2 .$$

Provided the theory E contains the schema of induction (28), this μ-operator satisfies conditions (10) and (11) for $S = E$. Hence it does so also for $S = \mathrm{Tr}(\mathfrak{M})$.

Finally, it is easy to show that the class C is semantically closed on this operator applied to formulas Φ having fewer than n-quantifiers. Indeed, consider the set X defined as follows:

$$X = \big\{\langle z, y\rangle\colon\ \mathfrak{M} \models \mathcal{E}(V_i)\big(\Phi(V/V_i, z), \Phi(y, z), y\big)\big\};$$

hence

$$X = \big\{\langle z, y\rangle\colon\ \mathfrak{M} \models \Phi(y, z) \wedge \bigwedge_{V_i}\big(V_i < y \to \neg\Phi(V_i, z)\big)\big\} .$$

Thus, if Φ has fewer than n-quantifiers, then the only non-trivial case is that of Φ having $n-1$ quantifiers, the first of them being general. Hence $\neg\Phi(V_i, z)$ in the normal prenex form has $n-1$ quantifiers, the first of them being existential. But in E we have the theorem:

$$\bigwedge_{V_i}\big(V_i < V \to \bigvee_{V_1}\Psi(V_i, V_1, \ldots)\big)$$
$$\Longleftrightarrow \bigvee_{V_1}\bigwedge_{V_i}\big(V_i \to V \to \Psi\big(V_i, \exp(V_1, V_i), \ldots\big)\big) .$$

Hence this equivalence is true in $\mathfrak{M}$, and thus the formula $\bigwedge V_i(V_i < y \to \neg\Phi(V_i, z))$ is equivalent to another formula which has n quantifiers the first of them being existential. Hence the set X belongs to the class C.

For every formula $\Phi(\ldots) = \bigvee_{V_1}\Omega(V_1, \ldots)$ such that Ω has fewer a n-quantifiers the assumptions of Theorem 1 are therefore satisfied. Thus if Φ has no free variable, we infer from theorem 1 that:

(33) Φ is true in $\mathfrak{M} \Longleftrightarrow \Phi$ is true in $\mathfrak{M}^*$.

Hence the same is also true for the negation of Φ. Hence, according to the De Morgan rules, we get equivalence (33) also for dual sentences (with a dual prenex form). Thus equivalence (33) holds for all sentences

having at most n-quantifiers. Hence by (31) we have the inclusions:

(34) $$E \cap S_n \subset \mathrm{Tr}(\mathfrak{M}) \cap S_n \subset \mathrm{Tr}(\mathfrak{M}^*) .$$

There are perhaps many methods of proving that for sentences having more quantifiers this inclusion fails. Here, to construct a counter-example, I shall imitate the argument of A. Ehrenfeucht from [1]. First let us consider the formula

(35) $$\xi(V_1, V_2) = \bigvee_{V_3}\Big(\bigwedge_{V_4,V_5}\Big((V_4 \leqslant V_1 \wedge V_5 \leqslant V_1)$$
$$\to \big(\bigvee_{V_6}\Phi(V_4, V_5, V_6) \to \exp(V_3, \langle V_4, V_5\rangle) \neq 0\big) \wedge$$
$$\wedge \big(\exp(V_3, \langle V_4, V_5\rangle) \neq 0 \to \Phi\big(V_4, V_5, \exp(V_3, \langle V_4, V_5\rangle) - 1\big)\big) \wedge$$
$$\wedge \bigwedge_{V_6}\big(V_6 < \exp(V_3, \langle V_4, V_5\rangle) \dot{-} 1 \to \neg\Phi(V_4, V_5, V_6)\big)\Big) \wedge$$
$$\wedge V_2 = 1 + \sum_{V_4 \leqslant V_1}\ \sum_{V_5 \leqslant V_1} \exp(V_3, \langle V_4, V_5\rangle) \dot{-} 1\Big) .$$

The last member of the conjunction may be considered as an additional primitive notion $\pi(V_1, V_2, V_3)$, of course recursive. The intuitive sense of $\xi(V_1, V_2)$ is the following:

$$V_2 = 1 + \sum_{V_4 \leqslant V_1}\ \sum_{V_5 \leqslant V_1} \mu V\Phi(V_4, V_5, V) .$$

Now consider the sentences

(36) $$\bigwedge_{V_1}\bigvee_{V_2}\xi(V_1, V_2)$$

(37) $$\bigwedge_{V_0,V_1}\Big(\big(V_0 < V_1 \wedge \bigwedge_{V_2,V_3}\big(\Phi(V_0, V_1, V_2) \wedge \Phi(V_0, V_1, V_3) \to V_2 = V_3\big)\big)$$
$$\to \bigwedge_{V_4,V_5}\big(\big(\Phi(V_0, V_1, V_5) \wedge \xi(V_1, V_4)\big) \to V_5 < V_4\big)\Big) .$$

Sentences (36) and (37) are both theorems of E.

The proof of (36) proceeds by induction. The number V_3 constitutes a two-dimentional diagram of the values of the function $\mu V\Phi(V_4, V_5, V)$ for all $V_4, V_5 < V_1$. Hence V_3^* for V_1+1 is obtained from V_3 (for V_1) by adding to the diagram the new values

$$V_3^* = V_3 \cdot \prod_{V_4 \leqslant V_1} p_{\langle V_4, V_1+1\rangle}^{\mu V\Phi(V_4, V_1+1, V)} \cdot \prod_{V_5 \leqslant V_1+1} p_{\langle V_1+1, V_5\rangle}^{\mu V\Phi(V_1+1, V_5, V)} ,$$

where

$$\mu V\Phi(V \ldots) = \begin{cases} \mu V\Phi(V \ldots) + 1 & \text{if for some } V\Phi(V \ldots) , \\ 0 & \text{if not} . \end{cases}$$

The number V_2 is recursively defined by V_3.

The proof of (37) in E presents no difficulty.

Now we shall show that the conjunction of (36) and (37) is not true in $\mathfrak{M}^*$. Suppose that both are true in $\mathfrak{M}^*$. Take the element $[I] \in M^*$. From (36) we get for some $b = [f] \in M^*$

$$\xi([I], [f]) . \tag{38}$$

For $f \in C$, according to (32), (30) and (31), there exists a natural number $p \in N$ such that

$$\forall_{x,y} \in M(\langle x, y\rangle \in f \Longleftrightarrow \mathfrak{M} \models \Phi(p, x, y)) . \tag{39}$$

Consider the constant function $p(x) = p$, where p is the standard number p of the model $\mathfrak{M}^*$. Of course $[p] \in M^*$, and

$$\mathfrak{M}^* \models [p] < [I] , \tag{40}$$

because $[I]$ is a non-standard element of M^*. On the other hand, from (39) and the corollary 1 we get

$$\mathfrak{M}^* \models \bigwedge_{V_1,V_2,V_3}(\Phi(\boldsymbol{p}, V_1, V_2) \wedge \Phi(\boldsymbol{p}, V_1, V_3) \rightarrow V_2 = V_3) \tag{41}$$

and

$$\mathfrak{M}^* \models \Phi(\boldsymbol{p}, [I], [f]) . \tag{42}$$

Of course, for $p \in N$ and every formula Ψ the following equivalence holds:

$$\mathfrak{M}^* \models \Psi([p] \ldots) \Longleftrightarrow \mathfrak{M}^* \models \Psi(\boldsymbol{p} \ldots) . \tag{43}$$

From (43) and (41), (42) we get

$$\mathfrak{M}^* \models \bigwedge_{V_2 V_3}((\Phi[p], [I], V_2) \wedge \Phi([p], [I], V_3) \rightarrow V_2 = V_3) \tag{44}$$

$$\mathfrak{M}^* \models \Phi([p], [I], [f]) . \tag{45}$$

Now, supposing that (37) is true in $\mathfrak{M}^*$, consider the valuation: $[p]$ for V_0, $[I]$ for V_1, and $[f]$ for both V_4 and V_5 in the second part of the implication. Hence by (40), (44), (45) and (38) we get a false conclusion:

$$\mathfrak{M}^* \models [f] < [f] .$$

Let T be the conjunction of (36) and (37). Hence $T \notin \mathrm{Tr}(\mathfrak{M}^*)$. From this and (34) we conclude that $T \notin \mathrm{Cn}(E \cap S_n)$.

Of course, if E with definable notions is not finitizable, then E with any other choice of primitive notions is not finitizable either.

Strictly speaking, we deal here with a property a little stronger than unfinitizability as it is familiarly understood in the literature. Having a pairing function as a primitive arithmetical notion, we get a theory Ar or T in which the set S_n of sentences with n quantifiers is essentially infinite. (There are infinitely many non-equivalent arithmetical sentences with n quantifiers Cnf [5]).

Instead of adding a pairing function as a primitive notion, we can change the definition of S_n:

S_n is the set of all sentences which have at most n block of similar quantifiers.

Every block may have arbitrarily many quantifiers of the same kind (all existential or all general). Hence Ar may be limited to the primitives 0 $S +\cdot$ and axioms 1-6 and Ind. schema. The unfinitizability theorem remains true for this new notion of S_n for Ar as described above and for every theory T containing Ar containing some new notions $P_1, \ldots, P_k$ and the Ind. schema (28) for all formulas. This result is stronger that the similar ones published in the papers [6], [3], [2].

This method can be applied also to proofs of unfinitizability (in this stronger sense) for other theories admitting models with an $\mathcal{E}$-operator. (E.g. II order arithmetic, and set theory.)

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