# Generalization of the notion of the Banach indicatrix

by

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In the theory of continuous real functions of bounded variation the notion of the Banach indicatrix is well known (cf. [1], pp. 374–375; [2], pp. 246–248). This notion can be generalized in the following natural way. Let $X$, $Y$ be two sets and $f$ a function from $X$ to $Y$. For $y \in Y$ we set $M_y = \{x \in X;\ f(x) = y\}$. If $M_y$ is a finite set, then we denote by $\tau_f(y)$ the number of elements of $M_y$. If $M_y$ is an infinite set, then we put $\tau_f(y) = +\infty$. So the function $\tau_f$ is defined for all $y \in Y$, its values lying in the set $Z = \{0, 1, 2, \ldots, +\infty\}$. We shall consider the set $Z$ as a subspace of the topological space

$$E_1^* = E_1 \cup \{-\infty\} \cup \{+\infty\}, \quad E_1 = (-\infty, +\infty),$$

in which the basis for topology is generated by all open intervals $I \subset E_1$ and all the intervals of the form $\langle -\infty, a), (b, +\infty\rangle$, $a, b \in E_1$. The function $\tau_f$ will be called the generalized Banach indicatrix of the function $f$.

In what follows some fundamental questions connected with the generalized Banach indicatrices will be studied.

If $X = \langle a, b\rangle$, $f\colon X \to E_1$ and $f$ is continuous, then the function $\tau_f$ is Lebesgue-measurable on $E_1$ (cf. [2], pp. 246–248). The detailed analysis of the proof of theorem 3 on pp. 246–248 in [2] shows that $\tau_f$ is a function of the second Baire class. In connection with this result the question of the measurability of the generalized Banach indicatrix arises.

It is easy to construct an example of the function $f\colon E_1 \to E_1$ with Lebesgue-nonmeasurable $\tau_f$.

EXAMPLE 1. Let $X = E_1$ and let $A \subset E_1$ be a nonmeasurable set (in the sense of Lebesgue). We put $f(x) = x$ for $x \in A$ and $f(x) = 0$ for $x \in E_1 - A$. Then the set $\{y \in E_1;\ \tau_f(y) = 1\} = A - \{0\}$ is a nonmeasurable set and so $\tau_f$ is also nonmeasurable.

In the previous example the function $f$ was nonmeasurable. Even in the case of the measurability of $f$ the measurability of $\tau_f$ is not guaranteed, as the following theorem shows.

THEOREM 1. *There exists a function $f\colon \langle 0,1\rangle \to E_1$ such that $f$ is continuous almost everywhere, the set $D_f$ of the discontinuity points of the function $f$ is a non-dense set in $\langle 0,1\rangle$ and $\tau_f$ is Lebesgue-nonmeasurable.*

Proof. Let $C\subset \langle 0,1\rangle$ be the Cantor set, let $g$ denote Cantor's well-known function, which maps the interval $\langle 0,1\rangle$ onto $\langle 0,1\rangle$. It is known that $g$ is a non-decreasing function, continuous on $\langle 0,1\rangle$ and constant on the component intervals of the set $C' = \langle 0,1\rangle - C$; further the values of $g$ on these intervals are rational (cf. [2], pp. 232–233). Let $M$ be a non-measurable subset of the interval $\langle 0,1\rangle$ consisting of irrational numbers. Then there exists a $P\subset C$ such that $g(P) = M$. We put $f(x) = g(x)$ for $x\in P\cup C'$ and $f(x) = 0$ for $x\in C-P$. Then $f$ is a function from $\langle 0,1\rangle$ into $E_1$ and obviously $\{y\in E_1;\ \tau_f(y) = 1\} = M\cup\{1\}$; therefore $\tau_f$ is non-measurable. It is easy to see that $D_f\subset C$; thus $D_f$ is a non-dense null set.

Conversely to the previous result, there are non-measurable functions whose generalized Banach indicatrices are measurable. This follows from the following example.

EXAMPLE 2. Let $A\subset\langle 0,1\rangle$ be a non-measurable set. Let $f$ denote the indicator function of $A$. Then we have $\tau_f(y) = 0$ for all $y\neq 0,1$, and so $\tau_f$ is a measurable function.

The measurability of $\tau_f$ can be guaranteed for all the real functions $f$ with the Darboux property defined on the topological space $X$ fulfilling some conditions. This will be proved in what follows. Let us remark that the function $f$ which maps a topological space $X$ into a topological space $Y$ is said to have the Darboux property if for each connected set $A\subset X$ the set $f(A)$ is connected in $Y$.

The following theorem generalizes the above-mentioned classical result on the measurability of $\tau_f$ to the case of a continuous function $f$.

THEOREM 2. *Let $X$ be a locally connected Hausdorff topological space with a countable basis. Let $f\colon X\to E_1$ have the Darboux property. Then the function $\tau_f$ is a function of the second Baire class.*

Proof. The proof will be realized in three steps.

1. At first the existence of a countable basis consisting of connected open sets will be proved.

Let $S = \{G_n\}_n$ be a countable basis of the space $X$. Let $\gamma\in G_n$. Following the assumptions of the theorem there exists a connected open set $V(\gamma)$ such that $\gamma\in V(\gamma)\subset H_n$. Then we have

$$G_n = \bigcup_{\gamma\in G_n} V(\gamma)\,. \tag{1}$$

Since $X$ is a space with a countable basis, there exists in view of (1) on account of the well-known theorem of Lindelöf (cf. [7], p. 131) a countable system $W_n\subset\{V(\gamma)\}$, $\gamma\in G_n$, such that $G_n\subset \bigcup_{V\in W_n} V$.



Let $W_n = \{V_{nk}\}_k$. Then the system $\{V_{nk}\}_{n,k}$ is a countable basis of the space $X$ consisting of connected sets.

2. Let $m$ be a natural number, let $B_m$ denote the set of all such $y\in E_1$ that $f$ attains the value $y$ in at least $m$ different points $x_1, x_2, \ldots, x_m\in X$. We shall prove that $B_m$ is an $F_\sigma$ set in $E_1$.

Let $T = \{H_i\}_i$ be a countable basis of the space $X$ consisting of connected sets $H_i$. Let $B^*$ denote the set of all $y\in E_1$ with the following property: For $y\in B^*$ there exists a set $H_i\in T$ such that $f(x) = y$ for every $x\in H_i$. Since $T$ is a countable system of sets, the set $B^*$ is also countable. Let us notice that for each $H_i\in T$ the set $f(H_i)$ is an interval. Denote by $B^{**}$ the set of all $y\in E_1$ with the following property: For $y\in B^{**}$ there exists a set $H_i\in T$ such that $y\in f(H_i)$ and $y$ is an end-point of the interval $f(H_i)$. The set $B^{**}$ is again a countable set, owing to the countability of $T$.

Put $C_m = B_m - (B^*\cup B^{**})$. If $y\in C_m$, then $y\in B_m$ and so the existence of $m$ different points $x_1, x_2, \ldots, x_m\in X$ such that $f(x_j) = y$ ($j = 1,2,\ldots,m$) is guaranteed. Since $X$ is a Hausdorff space, there exist $H_{k_j}\in T$ ($j = 1,2,\ldots,m$) such that $x_j\in H_{k_j}$ ($j = 1,2,\ldots,m$) and $H_{k_j}\cap H_{k_l} = \emptyset$ for $j\neq l$.

On account of the Darboux property of $f$, the sets $K_j = f(H_{k_j})$ ($j = 1,2,\ldots,m$) are intervals and $y\in K_j$ ($j = 1,2,\ldots,m$). Since $y\notin B^*$, $K_j$ is not a one-point set and since $y\notin B^{**}$, $y$ is an interior point of the set $K_j$ ($j = 1,2,\ldots,m$). Hence the set $K_y = K_1\cap K_2\cap\ldots\cap K_m$ is also an interval which is not a one-point set and $y$ is an interior point of this interval.

Let $K_y^0$ denote the interior of the interval $K_y$. Then $K_y^0$ is an open interval containing the point $y$, and for each $z\in K_y^0$ there exists in each $H_{k_j}$ a point $x_j'$ such that $z = f(x_j')$ ($j = 1,2,\ldots,m$). Since $H_{k_j}$ ($j = 1,2,\ldots,m$) are pairwise disjoint, we have $z\in B_m$. Thus $K_y^0\subset B_m$. This gives

$$C_m = B_m - (B^*\cup B^{**}) \subset \bigcup_{y\in C_m} K_y^0 \subset B_m\,.$$

It follows that $B_m - \bigcup_{y\in C_m} K_y^0$ is a countable set, and since $\bigcup_{y\in C_m} K_y^0$ is open, $B_m$ is an $F_\sigma$ set in $E_1$.

3. We shall prove that for each open set $G\subset E_1^*$ the set $\tau_f^{-1}(G)$ is a $G_{\delta\sigma}$ set in $E_1$.

It suffices to prove that for each set $G$ of one of the following forms: (a) $G = (b, +\infty\rangle$ or $G = \langle b, +\infty\rangle$, $b\in E_1$, $b\geqslant 0$; (b) $G$ is a finite open interval, $G\subset (0, +\infty)$; (c) $G = (b, +\infty)$, $b\in E_1$, $b\geqslant 0$, the set $\tau_f^{-1}(G)$ is a $G_{\delta\sigma}$ set in $E_1$.

In case (a) we infer from the 2-nd step of the proof that $\tau_f^{-1}(G)$ is an $F_\sigma$ set in $E_1$ and it is also a $G_{\delta\sigma}$ set in $E_1$.

If $G=(a,b)$; $a,b\in E_1$, $a<b$, then

$$\tau_f^{-1}(G)=\tau_f^{-1}(\langle a,+\infty\rangle)-\tau_f^{-1}(\langle b,+\infty\rangle)\,,$$

and according to the 2-nd step of the proof the right-hand side is a difference of two $F_\sigma$ sets, and so $\tau_f^{-1}(G)$ is a $G_{\delta\sigma}$ set in $E_1$.

Finally, let $G=(b,+\infty)$, $b\in E_1, b\geqslant 0$. Since $\{+\infty\}=\bigcap_{m=0}^{\infty} I_m$, $I_m=(m,+\infty\rangle$, we have $\tau_f^{-1}(\{+\infty\})=\bigcap_{m=0}^{\infty} M_m$, where $M_m=\tau_f^{-1}(I_m)$ $(m=0,1,\ldots)$ is an $F_\sigma$ set in $E_1$. Further, according to the 2-nd step of the proof, we have $\tau_f^{-1}(\langle b,+\infty\rangle)=\bigcup_{n=0}^{\infty} F_n$, where $F_n$ $(n=0,1,\ldots)$ are closed in $E_1$. Since $\tau_f^{-1}(G)=\tau_f^{-1}(\langle b,+\infty\rangle)-\tau_f^{-1}(\{+\infty\})$, we get

$$\tau_f^{-1}(G)=\bigcup_{n=0}^{\infty}F_n-\bigcap_{m=0}^{\infty}M_m=\Big(\bigcup_{n=0}^{\infty}F_n\Big)\cap\Big(E_1-\bigcap_{m=0}^{\infty}M_m\Big)$$
$$=\bigcup_{n=0}^{\infty}F_n\cap\bigcup_{m=0}^{\infty}(E_1-M_m)=\bigcup_{n,m}F_n\cap M'_m\,,$$

where $M'_m=E_1-M_m$ $(m=0,1,\ldots)$ is a $G_\delta$ set in $E_1$. From this we see that $\tau_f^{-1}(G)$ is a $G_{\delta\sigma}$ set in $E_1$. This ends the proof.

Remark. Let $f$ be a real function with the Darboux property on a Hausdorff topological space $X$. Following the previous theorem, $\tau_f$ is a measurable function if $X$ fulfils the following conditions:

(*) $X$ has a countable basis for topology,

(**) $X$ is locally connected.

The following examples show that each of the previous conditions is essential in theorem 2.

Example I. Let $X$ be a set, $\overline{\overline{X}}>c$ ($\overline{\overline{M}}$ denotes the cardinal number of the set $M$, $C$ being the power of the continuum). Let $S=2^X$ (the power set of $X$). The space $X$ with the topology $S$ fulfils the condition (**) but it does not fulfil the condition (*). Let $A\subset E_1$ be a non-measurable set. Since $\overline{\overline{A}}\leqslant c$, there exists a subset $X_1$ of the set $X$ such that the sets $X_1$ and $A$ are equivalent. Let $g$ denote a one-to-one mapping from $X_1$ onto $A$. Let $t_0\in E_1-A$. We put $f(x)=g(x)$ for $x\in X_1$ and $f(x)=t_0$ for $x\in X-X_1$. Since the system of all connected subsets of the space $X$ coincides with the system of all one-point sets, every function from $X$ to $E_1$ has the Darboux property and so $f$ also has this property. Further, $\{y\in E_1;\ \tau_f(y)=1\}=A$; therefore $\tau_f$ is non-measurable.

Example II. Put $X=C\subset\langle 0,1\rangle$, $C$ denoting the Cantor set. We consider the space $X$ with the usual Euclidean topology. Then $X$ fulfils the condition (*), but it does not fulfil the condition (**). Let $A$ and $t_0$ have the same meaning as in example I. Put $C'=C\cap\langle 0,\frac{1}{2}\rangle$. Since $\overline{\overline{A}}\leqslant c$ and $\overline{\overline{C'}}=c$, there exists a subset $C_1$ of $C'$ such that the sets $C_1$ and $A$ are equivalent. Let $g$ denote a one-to-one mapping from $C_1$ onto $A$. Put $f(x)=g(x)$ for $x\in C_1$ and $f(x)=t_0$ for $x\in C-C_1$. It can be checked in the same way as in example I that $f$ has the Darboux property while $\tau_f$ is not measurable.





It follows from theorem 2 that for each function $f\colon\langle 0,1\rangle\to E_1$ with the Darboux property the function $\tau_f$ is Lebesgue-measurable. Since many non-measurable functions belong to the class of all Darbouxian functions $f\colon\langle 0,1\rangle\to E_1$ (cf. [5]; [6], example 33, p. 97–98), we obtain a class of non-measurable functions $f\colon\langle 0,1\rangle\to E_1$ with measurable generalized Banach indicatrices $\tau_f$.

From the non-negativity of $\tau_f$ for each real function $f$ follows the existence of the integral $\int_{-\infty}^{\infty}\tau_f(y)\,dy$ in the case of measurability of $\tau_f$. Of course, this integral can have the value $+\infty$. Especially for the real functions $f$ with the Darboux property, defined on a locally connected Hausdorff topological space with a countable basis this integral exists.

Theorem 3. *Let $X$ be a locally connected Hausdorff topological space with a countable basis. Let $f\colon X\to E_1$ have the Darboux property and be discontinuous at least one point of the space $X$. Then we have* $\int_{-\infty}^{\infty}\tau_f(y)\,dy=+\infty$.

Proof. Let $f$ be discontinuous at the point $x_0\in X$. Let $T=\{G\}$ be a countable basis of $X$ consisting of connected sets. Such a basis exists according to the 1-st step of the proof of theorem 2.

Let $x_0\in G$ $(G\in T)$. We consider the set $G$ as a subspace (with the relative topology) of the space $X$. Then $G$ is a locally connected Hausdorff topological space. According to the Darboux property of the function $f$, the set $f(G)=Y$ is an interval which cannot be a one-point set. Namely, if the set $Y$ were a one-point set, then $f$ would be constant on the set $G$ and thus it would be continuous at $x_0$, contrary to the assumption.

Denote by $Y_0$ $(Y_1)$ the set of all $y\in Y$ for which the set $\{x\in G;\ f(x)=y\}$ is finite (closed in $G$). Since $G$ is a Hausdorff space, every finite subset of $G$ is closed in $G$ and thus we have

$$Y_0\subset Y_1\,. \tag{2}$$

In paper [3] J. S. Lipiński proved that if $X^*$ is a locally connected space, then the function $h\colon X^*\to E_1$ is continuous on $X^*$ if and only if $h$ has the Darboux property and simultaneously the following property $(G)$; there exists a set $H\subset h(X^*)$ dense in $h(X^*)$ such that for each $y\in H$ the set $h^{-1}(\{y\})$ is closed in $X^*$. Since $f$ is not continuous on $G$, we infer from the above mentioned result of Lipiński that the set $Y_1$ is not dense in $Y$. According to (2) not even the set $Y_0$ is dense in $Y$. Hence there exists an interval $I \subset Y$ such that $I \cap Y_0 = \emptyset$. Then for each $y \in I$ we have $\tau_f(y) = +\infty$, and thus in view of the non-negativity of $\tau_f$ we get $\int_{-\infty}^{\infty} \tau_f(y)\,dy \geqslant \int_I \tau_f(y)\,dy = +\infty$. This ends the proof.

Let us consider the interval $X = \langle a, b\rangle$, $a \subset b$, with the usual Euclidean topology. Then $X$ fulfils the assumptions of theorem 2, and thus $\tau_f$ is a measurable function for each real function $f\colon \langle a, b\rangle \to E_1$ which has the Darboux property. This implies the existence of the integral $\int_{-\infty}^{\infty} \tau_f(y)\,dy$. If $f$ is continuous, then we have the equality

$$\bigvee_a^b (f) = \int_{-\infty}^{\infty} \tau_f(y)\,dy \tag{3}$$

between the variation $\bigvee_a^b (f)$ of $f$ and the above-mentioned integral (cf. [1], p. 374–375; [2], p. 246–248). We shall prove that (3) holds for every function $f\colon \langle a, b\rangle \to E_1$, which has the Darboux property.

At first an auxiliary result will be proved.

Lemma 1. *Let $f\colon \langle a, b\rangle \to E_1$ have the Darboux property and be discontinuous at a point of the interval $\langle a, b\rangle$. Then $\bigvee_a^b (f) = +\infty$.*

Proof. Let $f$ be discontinuous on the right at $x_0$, $a \leqslant x_0 < b$. Since $f$ has the Darboux property, the following inequality must be true:

$$l = \liminf_{x \to x_0^+} f(x) < \limsup_{x \to x_0^+} f(x) = L\,. \tag{4}$$

Let us choose two real numbers $\alpha, \beta$ such that $l < \alpha < \beta < L$. It easily follows from (4) that there exist real numbers $x_n, y_n \in \langle a, b\rangle$ such that $f(x_n) \leqslant \alpha$, $f(y_n) \geqslant \beta$ $(n = 1, 2, \dots)$ and $x_1 > y_1 > x_2 > y_2 > \dots > x_0$. Obviously,

$$\bigvee_a^b (f) \geqslant \sum_{i=1}^{n} |f(x_i) - f(y_i)| \geqslant n(\beta - \alpha) \qquad (n = 1, 2, \dots)\,,$$

hence $\bigvee_a^b (f) = +\infty$.

Theorem 4. *Let $f\colon \langle a, b\rangle \to E_1$ have the Darboux property. Then*

$$\bigvee_a^b (f) = \int_{-\infty}^{\infty} \tau_f(y)\,dy\,. \tag{5}$$

Proof. In the case of the continuity of the function $f$ on $\langle a, b\rangle$ the assertion of theorem is true (cf. [1], p. 374–375; [2], p. 246–248).

If $f$ is not continuous on $\langle a, b\rangle$, then the equality follows from theorem 3 and lemma 1.

The assumption of the Darboux property of the function $f$ is essential in theorem 4. This is shown by the following example.

Example 3. Let $f(x) = 0$ for $0 < x \leqslant 1$ and $f(0) = 1$. Obviously $f$ does not have the Darboux property. Further, $\tau_f(0) = +\infty$, $\tau_f(1) = 1$ and $\tau_f(y) = 0$ for $y \neq 0, 1$. So we have $\int_{-\infty}^{\infty} \tau_f(y)\,dy = 0$ but $\bigvee_a^b (f) = 1$.

The following example shows that the assumption of the Darboux property of the function $f$ is not necessary for the validity of the equality (5).

Example 4. Let $g$ be a function which maps every interval $I \subset \langle 0, 1\rangle$ onto the interval $\langle -1, 1\rangle$. Such a function can be constructed, e.g. by means of a subseries of a divergent series $\sum_1^{\infty} a_n$, $\sum_{n;a_n \geqslant 0} a_n = \sum_{n;a_n<0} |a_n| = +\infty$, $a_n \to 0$ (cf. [8]). Put $f(x) = g(x)$ if $g(x)$ is irrational and $f(x) = 0$ in the opposite case. Obviously $f$ has not the Darboux property (even in an interval $I \subset \langle 0, 1\rangle$); further, $\bigvee_a^b (f) = +\infty$ and since for each irrational value $y \in \langle -1, 1\rangle$ we have $\tau_f(y) = +\infty$, the value of the integral $\int_{-\infty}^{\infty} \tau_f(y)\,dy$ is also $+\infty$.

Let $f\colon \langle a, b\rangle \to E_1$ have the Darboux property. We shall investigate the question what influence the change of values of the function $f$ has on the value of the integral $\int_{-\infty}^{\infty} \tau_f(y)\,dy$.

Let $g\colon \langle a, b\rangle \to E_1$ and let us put $M = \{x \in \langle a, b\rangle;\ f(x) \neq g(x)\}$. If $M$ is a countable set, then obviously the set $\{y \in E_1;\ \tau_f(y) \neq \tau_g(y)\}$ is also countable. Hence $\tau_g$ is a measurable function and

$$\int_{-\infty}^{\infty} \tau_f(y)\,dy = \int_{-\infty}^{\infty} \tau_g(y)\,dy\,.$$

But for the function $g$ the equality $\bigvee_a^b (g) = \int_{-\infty}^{\infty} \tau_g(y)\,dy$ need not be true. This is shown by the following

Example 5. Let $f(x) = x$ on $\langle 0, 1\rangle$. Let us put $g(x) = f(x)$ on $(0, 1\rangle$ and $g(0) = 1$. Then we have $\int_{-\infty}^{\infty} \tau_f(y)\,dy = \int_{-\infty}^{\infty} \tau_g(y)\,dy$ but $\bigvee_0^1 (g) = 2$, $\bigvee_0^1 (f) = 1$.

If $M$ is a null set, then not even the measurability of $\tau_g$ can be guaranteed. This follows easily from the examples 1, 2 using the well-known fact that to every real function $h: \langle a, b\rangle \to E_1$ there exists a function $f: \langle a, b\rangle \to E_1$ with the Darboux property such that $\{x \in \langle a, b\rangle;\ f(x) \neq h(x)\}$ is a null set (cf. [4]).

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# An unfinitizability proof by means of restricted reduced power

by

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In this paper I try to make some progress in solving the problem of the unfinitizability of theories containing the arithmetic of natural numbers. This problem remained open after Ryll-Nardzewski's paper [6] proving the infinitizability of the rule of induction in elementary arithmetic. The method of the present paper consists of constructing a kind of reduced power restricted to functions and sets definable by means of $n$-quantifiers. The main observation (Theorem 1) is that in this case only the sentences containing $n$-quantifiers which are true in the basic model $\mathfrak{M}$ remain true in the reduced ultra power $\mathfrak{M}^*$. Finding a theorem which is not preserved, we get the unfinitizability proof. Dividing by a filter cut up to sets definable by $n$-quantifiers may be conceived as adding new "defective" objects having only $n$-quantifier properties. It might be presumed that these new objects preserve only $n$-quantifier statements.

The result obtained in this way was independently obtained also by Ryll-Nardzewski by means of the method of his old paper [6]. It is probably not the strongest one. The problem remains open for theories containing arithmetic and dealing with two kinds of objects: natural numbers and the other objects (sets, classes etc.). A partial result in this domain was obtained by A. Mostowski in [4]. The main contribution of this paper is an outline of a new method.

**1. Restricted filters, functions and ultrapower.** Let $C$ be an arbitrary family of subsets of a given set $M$; we shall consider the following notion of ultrafilter restricted to $C$:

(1) $D \in \mathrm{Uf}(C) \iff$ 1. $D \subset C$,
2. $\emptyset \notin D$,
3. $X, Y \in D \wedge X \cap Y \in C \to X \cap Y \in D$,
4. $X \in D \wedge X \subset Y \wedge Y \in C \to Y \in D$,
5. $X \cup Y \in D \wedge X, Y \in C \to X \in D \vee Y \in D$.