A reduced free product of lattices

by

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1. The purpose of this note is to provide a generalization of the Basic Lemma of [1]. The new form is more general and easier to apply than the form given in [1].

To state the result we need some notation. For a lattice $K$ with 0 and 1 let $C(K)$ denote the set of complemented pairs, that is,

$$C(K) = \{(x, y) \mid x, y \in K, x \land y = 0, x \lor y = 1\}.$$ 

$K$ is called a lattice with no comparable complements if $(x, y), (x, z) \in C(K)$, and $y \geq z$, imply $y = z$.

Let $I_0, \lambda \in A$, be pairwise disjoint lattices with 0 and 1. Let $C$ be a set of two element subsets of $\bigcup \{I_0 \mid \lambda \in A\}$ such that if $(x, y) \in C$ then for some $\lambda, \mu \in A$, $x \in I_0, y \in I_0, x \neq 0, 1, y \neq 0, 1,$ and $\lambda \neq \mu$.

**Theorem 1.** Let $I_0, \lambda \in A$, and $C$ be given as described above. Assume that all $I_0$ are lattices with more than one element and with no comparable complements, and that $C$ satisfies the following condition:

(P) if $(x_1, y_1), (x_2, y_2) \in C, x_1, x_2 \in I_0, y_1, y_2 \in I_0, (\lambda, \nu \in A), x_1 \leq x_2, y_1 \leq y_2$, then $x_1 = x_2,$ and $y_1 = y_2$.

Then there exists a lattice $L$ with 0 and 1 satisfying the following conditions:

(i) $L$ contains all $I_0$ as $(0, 1)$-sublattices;

(ii) $L$ is generated by $\bigcup \{I_0 \mid \lambda \in A\}$;

(iii) $C(L) = \bigcup \{C(I_0) \mid \lambda \in A\} \cup C$.

A lattice $L$ satisfying Theorem 1 can be described using free products. Let $K$ be the free product of the lattices $I_0, \lambda \in A$. (Note that $K$ has neither 0 nor 1 if $A$ is infinite.) In terms of $C$ we define a congruence relation $\Theta(C)$ on $K$:

$\Theta(C)$ is the smallest congruence relation satisfying the following conditions:

(a) if $0_1$ is the zero of $I_0$, and $x \leq 0_1$, then $x = 0_1(\Theta(C))$;

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If \( A \in L_1 \) then \( A_{01} \) and \( A^{(0)} \) exist, and they are both equal to \( A \); \( A_{01} \) and \( A^{(0)} \) do not exist for \( \mu \neq \lambda \).

(ii) If \( A = B \lor C \) then \( A^{(0)} \) exists if and only if \( B^{(0)} \) and \( C^{(0)} \) both exist and in this event \( A^{(0)} = B^{(0)} \lor C^{(0)} \) (the join is in \( L_1 \), of course).

Furthermore, \( A_{01} \) exists if and only if at least one of \( B_{01} \), \( C_{01} \) exists; \( A_{01} = B_{01} \) (respectively \( C_{01} \)) if only \( B_{01} \) (respectively \( C_{01} \)) exists, and \( A_{01} = B_{01} \lor C_{01} \) if both \( B_{01} \) and \( C_{01} \) exist.

(iii) If \( A = B \land C \) then \( A_{01} \) exists if and only if \( B_{01} \) and \( C_{01} \) both exist and in this event \( A_{01} = B_{01} \land C_{01} \), \( A^{(0)} \) exists if and only if at least one of \( B^{(0)} \), \( C^{(0)} \) exists; \( A^{(0)} = B^{(0)} \) (respectively \( C^{(0)} \)) if only \( B^{(0)} \) (respectively \( C^{(0)} \)) exists, and \( A^{(0)} = B^{(0)} \land C^{(0)} \) if both \( B^{(0)} \), \( C^{(0)} \) exist.

Definition 3 (Quasi ordering on \( P(Q) \)). For any \( A, B \in P(Q) \) we define by induction on \( |A| + |B| \) the relation \( A \subset B \) to hold if and only if at least one of the conditions (1) to (6) below holds:

1. \( A = B \);
2. there is a \( \lambda \in A \) such that \( A^{(2)} = B_{01} \) exist and \( A^{(0)} \leq B_{01} \) in \( L_1 \);
3. \( A = A \lor A_1 \), where \( A_1 \subseteq B \) and \( A_1 \not\subseteq B \);
4. \( A = A \land A_1 \), where \( A_1 \subseteq B \) or \( A_1 \not\subseteq B \);
5. \( B = B \lor B_1 \), where \( A \not\subseteq B_1 \) or \( A \not\subseteq B_1 \);
6. \( B = B \land B_1 \), where \( A \not\subseteq B_1 \) and \( A \not\subseteq B_1 \).

Set \( A \preceq B \) if \( A \subset B \) and \( B \subseteq A \).

Theorem 3 (The structure theorems of free products of lattices [3]).

(i) The relation \( \preceq \) is a quasi-order (that is, \( \preceq \) is reflexive and transitive) and thus \( \preceq \) is an equivalence relation.

(ii) Given \( A \in P(Q) \) let \( \langle A \rangle \) denote the equivalence class of \( A \) under \( \preceq \), and let \( L = \{ \langle A \rangle : A \in P(Q) \} \). Define the binary relation \( \prec \) on \( L \) by \( \langle A \rangle \prec \langle B \rangle \) if and only if \( A \subset B \). Then \( \prec \) is a partial order on \( L \) with respect to which \( L \) is a lattice. Moreover, \( \langle A \rangle \lor \langle B \rangle = \langle A \lor B \rangle \) and \( \langle A \rangle \land \langle B \rangle = \langle A \land B \rangle \).

(iii) For each \( \lambda \in A \) the mapping \( \psi_{\lambda} : L_1 \to L_1 \), given by \( \psi_{\lambda}(\alpha) = \langle \alpha \rangle \), is a \( 1 \)-1 lattice homomorphism, and \( \langle \psi_{\lambda} \rangle \lambda \in A \rangle = L_1 \) is the free product of the family \( \{ L_1 : \lambda \in A \} \).

(iv) For each \( \lambda \in A \) and \( \langle A \rangle \in P(Q) \), \( A_{01} \) exists if and only if \( \{ \alpha \in L_1 : \langle \alpha \rangle \neq \langle A \rangle \} \not= \emptyset \), and in this event \( A_{01} = \lor \{ \alpha \in L_1 : \langle \alpha \rangle \neq \langle A \rangle \} \), and dually for \( A^{(0)} \). Therefore, if both \( A_{01} \) and \( A^{(0)} \) exist, then \( A_{01} \leq A^{(0)} \).

(v) For \( \lambda, \mu \in A \) and \( A \in P(Q) \), if both \( A_{01} \) and \( A^{(0)} \) exist, then \( \lambda = \mu \).

3. In this section let \( L_1, A \in L_1 \) and \( C \) be given as in Theorem 1. We denote by \( \emptyset \) and \( L_1 \) the zero and unit of \( L_1 \). Set \( Q = \{ \emptyset, L_1 \} \lambda \in A \rangle \) as in [2]. The following definition contains the idea of the proof of Theorem 1.
DEFINITION 4. A subset $R(Q)$ of $P(Q)$ is defined by induction on the length of the polynomial:

(i) if $A \in Q$, then $A \in L_0$ for exactly one $\lambda \in A; A \in R(Q)$ iff $A$ is not $0_1$ or $1_1$;

(ii) if $A = B \cup C$, then $A \in R(Q)$ iff $B, C \in R(Q)$ and the following two conditions are satisfied:

(iiia) $B \subseteq A$, for no $\lambda \in A$;

(iiib) $C \subseteq B$, for no $(x, y) \in C$;

(iiic) if $A = B \times C$, then $A \in R(Q)$ iff $B, C \in R(Q)$ and the following two conditions are satisfied:

(iiiia) $A \subseteq B$, for no $\lambda \in A$;

(iiiib) $C \subseteq A$, for no $(x, y) \in C$.

Now we are ready to construct $L$:

$$L = \{0, 1\} \cup \{\langle A \rangle \mid A \in R(Q)\},$$

partially ordered by

$$0 < \langle A \rangle < 1 \quad \text{for all } A \in R(Q),$$

$$\langle A \rangle \leq \langle B \rangle \quad \text{if } A \subseteq B.$$

In other words, $L = \{0, 1\}$ is a subset of the free product; the partial ordering on $L = \{0, 1\}$ is the same as on the free product. Thus $L$ is obviously a partially ordered set.

To show that $L$ is a lattice, take $X, Y \in L$; we have to find $X \vee Y$. If $X$ or $Y$ or both $0, 1$ this is obvious. So let $X \notin \{0, 1\}$; then $X = \langle B \rangle, Y = \langle C \rangle, B, C \in R(Q)$. We claim that $X \vee Y = \langle A \rangle$ if $A$ satisfies (iiia) and (iiib), and $X \vee Y = 1_1$ otherwise. This follows from the observation that if $A, A_1 \in P(Q), A \subseteq A_1$, and $A$ violates (iiia) or (iiib), then so does $A_1$. The dual argument now proves that $L$ is a lattice.

For $a \in L_0, a \neq 0_1, 1_1$, identify $a$ with $0_1$; identify $0_1$ with 0 and 1 with 1. This makes $L_{1_1} = \{0, 1\}$-sublattice of $L$. (\(\langle a \rangle = \langle 0 \rangle\) implies $a = b$ by (1) of Definition 3; the identification preserves meets and joins in view of the discussion in the previous paragraph.) Thus (i) of Theorem 1 has been verified. (ii) of Theorem 1 is obvious.

Finally, we verify (iii) of Theorem 1. It follows from (i) of Theorem 1 that $C(L_0) \supseteq C(L)$. Let $(x, y) \in C(L)$, and (iiiia) of Definition 4 yield $x \vee y = 1_1$ and $x \wedge y = 0_1$ in $L$, hence $(x, y) \in C(L_0)$. This proves (iii) of Theorem 1.

To prove the converse, let $X, Y \in C(L_0)$, then $X \vee Y = 0, X \wedge Y = 1$. We can assume that $X, Y \neq \{0, 1\}$. Hence $X = \langle A \rangle, Y = \langle B \rangle, A, B \in R(Q)$. Therefore $A \vee B$ violates (iia) or (iiib) and $A \wedge B$ violates (iiiia) or (iiiib) of Definition 4. The four cases that arise are handled separately.
Hence if \( A, B \in \mathcal{R}(\mathcal{O}) \), \( \langle A \rangle \neq \langle B \rangle \), then \( \langle A \rangle \neq \langle B \rangle (\mathcal{O}_\Phi) \). Since \( \mathcal{O}(C) \subseteq \Phi \) is obvious, we conclude that \( \langle A \rangle \neq \langle B \rangle (\mathcal{O}(C)) \), showing that every congruence class modulo \( \mathcal{O}(C) \) other than \([0]_{\mathcal{O}(C)} \) and \([1]_{\mathcal{O}(C)} \) contains exactly one element of \( \mathcal{L} - \{0, 1\} \).

5. The concept of \((0, 1)\)-free product of lattices is the same as that of free product of lattices, except that it is applied only to lattices with 0 and 1, and homomorphism is replaced by \((0, 1)\)-homomorphism. Let us make two observations. First, the construction of \( \mathcal{L} \) in \( \S 3 \) and the proof that \( \mathcal{L} \) satisfies (i) and (ii) of Theorem 1 made no use of the assumptions of Theorem 1. Second, the proof of \( K(\mathcal{O}(C)) \cong \mathcal{L} \) in \( \S 4 \) is independent of the assumptions of Theorem 1. Hence this isomorphism holds for \( \mathcal{L}_1 \) arbitrary and \( \mathcal{C} = \emptyset \), showing that \( \mathcal{L} \) is the \((0, 1)\)-free product of the \( \mathcal{L}_1 \lambda \in \mathcal{A} \). Since the word problem in \( \mathcal{L} \) is solved we conclude:

**Theorem 4.** The word problem of \((0, 1)\)-free product of lattices \( \mathcal{L}_1 \lambda \in \mathcal{A} \), \( \lambda \in \mathcal{A} \), is solved relative to the \( \mathcal{L}_1 \lambda \in \mathcal{A} \).

This result is not new, as it can also be concluded from a result of [5].

Next we specialize Theorem 1 to \( \mathcal{C} = \emptyset \); this result appears to be new.

**Theorem 5.** Let \( \mathcal{L}_1 \lambda \in \mathcal{A} \), be lattices with 0 and 1, \( 0 \neq 1 \), and with no comparable complements. Let \( \mathcal{L} \) be the \((0, 1)\)-free product of the \( \mathcal{L}_1 \lambda \in \mathcal{A} \). For \( a, b \in \mathcal{L} \), \( a \) is a complement of \( b \) iff for some \( \lambda \in \mathcal{A} \), \( a, b \in \mathcal{L}_1 \lambda \), and \( a \) is a complement of \( b \) in \( \mathcal{L}_1 \lambda \).

As a further application we prove the following result of R. P. Dilworth [2]:

**Theorem 6.** Every lattice \( \mathcal{M} \) can be embedded in a uniquely complemented lattice.

**Proof.** Let \( \mathcal{A} = (0, 1, 2, \ldots) \); let \( \mathcal{L}_1 \) be \( \mathcal{M} \) with a new zero and unit. Let \( \mathcal{X}_i \), \( i = 1, 2, \ldots \), be pairwise disjoint infinite sets, \( |\mathcal{X}_i| = \max(n_i, |\mathcal{M}|) \); let \( \mathcal{L}_1 \) be the lattice freely generated by \( \mathcal{X}_i \) with zero and unit added. Since \( |\mathcal{X}_i| \geq |\mathcal{M}| \) we can define a function \( f_i: \mathcal{M} \to \mathcal{X}_i \), which is one-to-one, set \( \mathcal{C}_i = \langle \langle x, y \rangle \mid y = f_i(x), x \in \mathcal{M} \rangle \). Let \( \mathcal{M}_i \), be the \( \mathcal{C}_i \)-reduced free product of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Assuming then \( \mathcal{M}_i \) has been defined, let \( f_i \) be a one-to-one map from the non-complemented elements of \( \mathcal{M}_i \) into \( \mathcal{X}_{i+1}, \mathcal{C}_{i+1} = \langle \langle x, y \rangle \mid x \in \mathcal{M}_i, y = f_i(x) \rangle \), and let \( \mathcal{M}_{i+1} \) be the \( \mathcal{C}_{i+1} \)-reduced free product of \( \mathcal{M}_i \) and \( \mathcal{M}_{i+1} \). Then \( \mathcal{M}_i \subseteq \mathcal{M}_i \subseteq \mathcal{M}_i \subseteq \mathcal{M}_i \); the lattice \( \mathcal{L} = \bigcup \langle \mathcal{M}_i \rangle \), \( i = 1, 2, \ldots \), is the uniquely complemented lattice containing \( \mathcal{M} \).

The generalizations of Theorem 6 given in [1] can also be proved in a similar fashion. The present proof of Theorem 6 is equivalent to the proof given in [1].

Finally, we give an application of Theorem 3 which is crucial in some applications that are given in [4]:

**Theorem 7.** Let \( \mathcal{L}_1 \lambda \in \mathcal{A}, \mathcal{C} \) and \( \mathcal{L}_1 \lambda \in \mathcal{A} \), \( \lambda \in \mathcal{A} \) be given as in Theorem 1. For every \( \lambda \in \mathcal{A} \), let \( \varphi_1 \) be a \((0, 1)\)-homomorphism of \( \mathcal{L}_1 \) into \( \mathcal{L}_1 \) such that if \( (x, y) \in \mathcal{C}, x \in \mathcal{L}_1 \), \( y \in \mathcal{L}_1 \), then \( (\varphi_1(x), \varphi_1(y)) \in \mathcal{C} \). Let \( \mathcal{L} \) be the \( \mathcal{C}_1 \)-reduced free product of the \( \mathcal{L}_1 \lambda \in \mathcal{A} \), and \( \mathcal{D} \) the \( \mathcal{C}_2 \)-reduced free product of the \( \mathcal{L}_1 \lambda \in \mathcal{A} \). Then there exists a \((0, 1)\)-homomorphism \( \varphi \) of \( \mathcal{L} \) into \( \mathcal{D} \) such that \( \varphi \) restricted to \( \mathcal{L}_1 \) is \( \varphi_1 \), for all \( \lambda \in \mathcal{A} \).

**Proof.** Let \( \mathcal{K} \) be the free product of the \( \mathcal{L}_1 \lambda \in \mathcal{A} \), and the \( \mathcal{L}_1 \lambda \in \mathcal{A} \), respectively. Since \( \varphi_1 \) maps \( \mathcal{L}_1 \) into \( \mathcal{L}_1 \subseteq \mathcal{C} \), by the free product property, there exists a homomorphism \( \psi \) of \( \mathcal{K} \) into \( \mathcal{K} \), such that \( \psi \) restricted to \( \mathcal{L}_1 \) is \( \varphi_1 \) for all \( \lambda \in \mathcal{A} \). Set \( \mathcal{L} = \mathcal{K}/\mathcal{O}(\mathcal{C}) \), and \( \mathcal{D} = \mathcal{K}/\mathcal{O}(\mathcal{C}) \), and let \( a \) and \( d \) denote the natural homomorphisms. Then \( \psi^d: \mathcal{K} \to \mathcal{D} \) is a homomorphism; let \( \Theta \) be the congruence relation of \( \mathcal{K} \) induced by \( \psi^d \). We claim that \( \mathcal{O}(\Theta) \subseteq \Theta \). This follows from the assumption that \( (x, y) \in \mathcal{C} \) implies \( (\varphi_1(x), \varphi_1(y)) \in \mathcal{C} \). The computation is based on \( (a)^d \) of the definition of \( \mathcal{O}(\Theta) \), the details are left to the reader. Hence there is a natural homomorphism \( \varphi \) from \( \mathcal{L} = \mathcal{K}/\mathcal{O}(\mathcal{C}) \) into \( \mathcal{K}/\mathcal{O}(\Theta) \). Since \( a \) is the identity map on \( \mathcal{L}_1 \), \( d \) is the identity map on \( \mathcal{L}_1 \), and \( \psi \) restricted to \( \mathcal{L}_1 \) is \( \varphi_1 \), the relation \( ap = pd \) implies \( \varphi \) restricted to \( \mathcal{L}_1 \) is \( \varphi_1 \), completing the proof of Theorem 7.

**References**


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