for if not then $p \in \mathrm{Bd} f[a, b]$. So every neighborhood of p meets $E^2 \setminus f[a, b]$. Thus there is a $p' \in E^2 \setminus f[a, b]$ in the bounded complementary domain of C. So there is an open set U_p about p such that $U_p \cap f[a, b] = \emptyset$. Therefore there exists an n and map f_n with corresponding partition of I into subintervals such that for some subinterval of this partition, say $I_n^a, f(I_n^a) \subset U_p$. But then from the definition of C and $f_n([a, b])$, $I_n^a \subset [a, b]$ which is not possible. Therefore $\mathrm{int}(f[a, b])$ is simply connected.

THEOREM 2. *Let $f\colon I \to S$ be the map for Moore's Crinkly curve C^3 and let $\{R_i\}$, $i = 1, 2, \dots, p$, be a finite disjoint sequence of closed intervals in I. Then*

$$\bigcup_{i=1}^{p} f(R_i) = f\Big(\bigcup_{i=1}^{p} R_i\Big)$$

is tame.

Proof. By induction and repeated use of theorems such as 11.7, Wilder [4], p. 31, and 4.42, Whyburn [3], p. 40, together with radial extension of homeomorphisms on boundary of disks, it suffices to show if $f(R_i)$ and $f(R_j)$, $i \neq j$, are 2-cells which meet in their boundaries (by the definition of f they do not meet in their interiors) then $\mathrm{Bd} f(R_i) \cap \cap \mathrm{Bd} f(R_j)$ consists of at most two components. For suppose not, $\mathrm{Bd} f(R_i) \setminus (\mathrm{Bd} f(R_i) \cap \mathrm{Bd} f(R_j))$ consists of open intervals, choose two of these intervals in $\mathrm{Bd} f(R_j)$ not accessible in $E^2 \setminus (f(R_i) \cup f(R_j))$ from unbounded complimentary domain of $\mathrm{Bd} f(R_i) \cup \mathrm{Bd} f(R_j)$. Choose two corresponding closed intervals in $\mathrm{Bd} f(R_i)$ which together with the open intervals form two simple closed curves not accessible in the same sense. These simple closed curves bound open disks not accessible as before and which do not meet $f(R_i) \cup f(R_j)$. Now there exists an n, f_n and partition of I into 3^{2n} intervals such that both disks contain a square in the corresponding partition of S. But as before it is easily seen that this is not possible.

References

[1] E. H. Moore, *On certain crinkly curves*, Trans. Amer. Math. Soc. 1 (1900), pp. 72–90.
[2] M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge 1939.
[3] G. T. Whyburn, *Topological analysis*, Princeton 1964.
[4] R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Coll. Pub. 37 (1949).

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Involutions on solenoidal spaces

by

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1. Introduction. A *weak solenoidal sequence* (*solenoidal sequence*) of closed manifolds is an inverse limit sequence (X, f) such that each factor space X_n is a closed manifold and each bonding map $f_m^n\colon X_n \to X_m$ is a covering map (regular covering map). The limit space X_∞ is called a *weak solenoidal space* (*solenoidal space*).

In section 5, we present a general technique for constructing weak solenoidal spaces from solenoidal spaces. Suppose that (X, f) is a solenoidal sequence such that each factor space X_n admits a free involution that commutes with the bonding maps. These involutions induce an involution on the solenoidal space X_∞; moreover, if Y_∞ is the orbit space of this free involution on X_∞, then Y_∞ is a weak solenoidal space.

The importance of this technique is not only that we can construct new examples of weak solenoidal spaces, but we can obtain a keen insight into the internal structure of the spaces. Moreover, if we can construct a weak solenoidal space in a geometric manner and then show that we can obtain the same space as the orbit space of a known free involution on a solenoidal space, then we have tools to investigate both the global and local properties of the spaces.

We carry out this program in section 6, where we present a weak solenoidal space $M_\infty = \lim(M, f)$ which has the following properties: (1) each factor space M_n is homeomorphic to the Klein bottle; (2) each bonding map f_n^{n+1} is regular (although compositions of bonding maps are not regular); (3) the fundamental groups of any two path components of M_∞ are isomorphic; (4) M_∞ is not homogeneous; (5) there are exactly two different homeomorphism classes of path components, with only one path component in the first class; and (6) M_∞ is double-covered by the product of S^1 and the dyadic solenoid.

In section 3, we give a convenient characterization of the path component of a weak solenoidal space; this characterization is a valuable tool in the succeeding sections. In the process we obtain some interesting results (in the general theory of inverse limit spaces) concerning the problem of when an arc in the limit space is induced by arcs in the factor spaces.

M. C. McCord [3] and R. M. Schori [6] previously studied solenoidal spaces. McCord investigated the structure of solenoidal spaces and showed that they have some of the properties of the classical solenoids; in particular, they are homogeneous. Schori constructed an example of a nonhomogeneous weak solenoidal space. Our example is stronger than Schori's; in particular, no two of the factor spaces of his example were homeomorphic, and none of his bonding maps were regular. Furthermore, the homeomorphism types of any of the path components were not determined.

2. Notation. We follow the notation on inverse limit systems in [1] and restrict our discussion to inverse limit sequences (the directed set of indices is the positive integers). We let (X, f) denote the inverse limit sequence with *factor spaces* X_n and *bonding maps* $f_m^n\colon X_n \to X_m$ $(m \leqslant n)$. The limit space, $\lim(X, f)$, is denoted by X_∞. A point $x \in X_\infty$ is represented by $x = (x_1, x_2, \ldots)$, and we let $f_n\colon X_\infty \to X_n$ be the projection $f_n(x) = x_n$.

In this paper, all (weak) solenoidal sequences will be sequences of closed manifolds, that is, compact manifolds without boundary. We further assume that all (weak) solenoidal sequences are nontrivial, i.e., each bonding map is at least a 2-fold covering map.

If (X, f) and (Y, g) are two inverse sequences and $\varphi_1, \varphi_2, \ldots$ is a sequence of maps $\varphi_n\colon X_n \to Y_n$ such that $\varphi_m f_m^n = g_m^n \varphi_n$ (all $m \leqslant n$), then $\varphi_1, \varphi_2, \ldots$ induces a map $\varphi_\infty\colon X_\infty \to Y_\infty$ defined by $\varphi_\infty(x) = (\varphi_1(x), \varphi_2(x), \ldots)$.

If $P = (p_1, p_2, \ldots)$ is a sequence of prime numbers (different from 1), the P-adic solenoid Σ_P is the limit of the inverse sequence (X, f), where each $X_n = \{z\colon |z| = 1\}$ (the unit circle in the complex plane), and each bonding map $f_n^{n+1}\colon X_{n+1} \to X_n$ is defined by $f_n^{n+1}(z) = z^{p_n}$. Σ_P is a solenoidal space.

An *involution* on M, $h\colon M \to M$, is a homeomorphism of period two $(h^2 = 1_M)$. The *orbit space* of h is the quotient space $M/\{x \sim h(x)\}$. A *free involution* is an involution without fixed points. The antipodal map on the n-sphere is a free involution. The orbit space of this involution is real projective n-space P_n, and the projection $\pi\colon S^n \to P_n$ is a double-covering map.

The *connected sum*, $M \# M'$, of two connected, triangulated, closed n-manifolds M and M' is obtained by removing the interior of a polyhedral n-cell from each, and then matching the resulting $(n-1)$-sphere boundaries by a piecewise-linear homeomorphism (orientation reversing if both manifolds are oriented). The n-sphere serves as the identity element of this operation, that is, $M \# S^n \approx M$. If f is a map with domain X and $A \subset X$, then $f|A$ denotes the map which is the restriction of f to A. We refer the reader to [7] for the theory and terminology of covering maps.

3. Characterization of path components of M_∞. The goal of this section is to characterize the path components of a weak solenoidal space M_∞. In order to do this, we first investigate a problem in the general theory of inverse limit spaces and prove a theorem (of considerable interest in itself) about arcs in M_∞.

Let $X = \lim(X, f)$ be an arbitrary inverse limit space, and let A be an arc in X. When does there exist a positive integer n such that $m > n$ implies that $f_m(A)$ is an arc? The answer is sometimes negative, as Fort and Segal [2] have shown that an arc can be represented as the inverse limit of 2-cells with onto bonding maps. The following example shows that even if each $f_n(A)$ is one-dimensional, the question has no positive answer in general. For each positive integer n, let X_n be the union of the unit interval $I = [0, 1]$ and a perpendicular segment S_n of length $1/n$ which intersects I at the midpoint of $[1/n+1, 1/n]$. Let $f_n^{n+1}\colon X_{n+1} \to X_n$ be a map which collapses S_{n+1} to a point, maps $(1/n+1, 1/n)$ onto $(1/n+1, 1/n) \cup S_n$, and is the identity elsewhere. If X_∞ is the inverse limit of (X, f), then $(0, 0, \ldots)$ and $(1, 1, \ldots)$ are the only nonseparating points of X_∞; therefore X is an arc. However, $f_n(X)$ is never an arc.

If X_∞ is a weak solenoidal space, then the question has a positive answer. We precede the proof of this statement with two lemmas; the first is well known and the second is a special case of Theorem 3 of [4].

LEMMA 1. *Let M be a compact manifold with metric d. There exists $\varepsilon > 0$ such that if α and β are paths in M and $d(\alpha, \beta) < \varepsilon$, then $\alpha \simeq \beta$.*

LEMMA 2. *Let $X_\infty = \lim(X, f)$ be an inverse limit space, let $\nu\colon I \to X_\infty$ be an embedding, and let $\varepsilon > 0$. Then there exists an integer n and a map $\psi\colon f_n \circ \nu(I) \to I$ such that $d(f_1^n| f_n \circ \nu(I),\ f_1 \circ \nu \circ \psi) < \varepsilon$.*

Proof. Represent I as the trivial inverse limit $I_\infty = \lim(I, g)$, where each $I_n = I$ and each bonding map g_n^{n+1} is the identity. Let $\nu_1 = f_1 \circ \nu \circ g_1^{-1} = f_1 \circ \nu$. As in the proof of [4], Theorem 3, there exists an integer n and a map $\psi\colon f_n \circ \nu(I) \to I_1$ such that $d(f_1^n|f_n \circ \nu(I), \nu_1 \circ \psi) < \varepsilon$. □

THEOREM 1. *Let $M_\infty = \lim(M, f)$ be a weak solenoidal space, and let A be an arc in M_∞. Then there exists a positive integer n such that $m > n$ implies that $f_m(A)$ is an arc.*

Proof. Let ν be a homeomorphism of I onto A. For each i, $\nu_i = f_i \circ \nu$ is a path in M_i.

For each i, ν_i contains no contractible loop, since a contractible loop would lift to a loop in each factor space M_k, $k > i$, and hence to a loop in M_∞. However ν_n lifts to ν, which contains no loops.

Let ε be a positive number such that if α and β are paths in M_1, then $d(\alpha, \beta) < \varepsilon$ implies that $\alpha \simeq \beta$. Lemma 2 implies that there exist an integer n and a map $\psi\colon f_n(A) \to I$ such that $d(f_1^n/f_n(A), \nu_1 \circ \psi) < \varepsilon$. Hence $d(\nu_1 \circ \psi \circ \nu_n, f_1^n \circ \nu_n) < \varepsilon$, and so $\nu_1 \circ \psi \circ \nu_n \simeq f_1^n \circ \nu_n$. Because $\nu_1 \circ \psi \circ \nu_n$

factors through I, it is homotopic to a constant, and hence $f_1^n \circ \nu_n$ is also homotopic to a constant. Since f_1^n has the homotopy lifting property, ν_n is homotopic to a constant. Therefore ν_n contains no noncontractible loops. Thus ν_n is a embedding and $f_n(A)$ an arc. Since $f_m(A)$ covers $f_n(A)$ for $m > n$, $f_m(A)$ is also an arc. □

We use Theorem 1 and the next definition to obtain a useful characterization of path components in M_∞.

DEFINITION. Let $M_\infty = \lim(M, f)$ be a weak solenoidal space, and let $z = (z_1, z_2, \ldots)$ belong to M_∞. Let $y \in M_i$ and let $\nu_i\colon (I, 0, 1) \to (M_i, z_i, y)$ be an embedding of I in M_i. For each $j > i$, ν_i lifts to an embedding $\nu_j\colon (I, 0, 1) \to \big(M_j, z_j, \nu_j(1)\big)$. The sequence $\{\nu_i(1)\}_{j=i}^{\infty}$ is called an *endpoint sequence induced by* z *and* y.

THEOREM 2. *Let $M_\infty = \lim(M, f)$ be a weak solenoidal space. Let $z = (z_1, z_2, \ldots) \in M_\infty$, and let K be the path component of M_∞ which contains z. Then $K = \{(x_1, x_2, \ldots) \in M_\infty$: for some i, $x_i, x_{i+1}, \ldots$ is an endpoint sequence determined by z and $x_i\}$.*

Proof. Suppose $x = (x_1, x_2, \ldots)$ is a point of M_∞ such that x_i, $x_{i+1}, \ldots$ is an endpoint sequence determined by z and x_i. Then there exists an embedding $\nu_i\colon (I, 0, 1) \to (M_i, z_i, x_i)$ which lifts, for $j > i$, to an embedding $\nu_j\colon (I, 0, 1) \to (M_j, z_j, x_j)$ and hence to a path $\nu\colon (I, 0, 1) \to (M_\infty, z, x)$. Therefore x belongs to K.

On the other hand, let $x = (x_1, x_2, \ldots)$ be a point of K, and let A be an arc from z to x. Let $\nu\colon (I, 0, 1) \to (A, z, x)$ be a homeomorphism. For each positive integer i, $\nu_i = f_i \circ \nu$ is a path in M_i from z_i to x_i. By Theorem 1 we can find an integer n such that $\nu_n(I)$ is an arc. Hence $x_n, x_{n+1}, \ldots$ is an endpoint sequence determined by x_n and z. □

4. Path component models. Weak solenoidal spaces have a very complicated structure in general, making it difficult to distinguish between homeomorphism classes of path components. It is often useful to construct "untangled" models of the path components. In certain cases, these models will enable us to distinguish between homeomorphism classes of the path components.

Let (M, f) be a weak solenoidal sequence and let $b \in M_\infty$. Denote by K_b the path component of M_∞ containing the point $b = (b_1, b_2, \ldots)$. Let b_i be a basepoint for M_i, and consider the descending chain of subgroups of $\pi_1(M_1, b_1)$,

$$\pi_1(M_1, b_1) \supset (f_1^2)_* \pi_1(M_2, b_2) \supset (f_2^3)_* \pi_1(M_3, b_3) \supset \ldots$$

Let $\varrho_1\colon (\overline{K}_b, \overline{b}) \to (M_1, b_1)$ be the covering space determined by the subgroup $\bigcap_n (f_1^n)_* \pi_1(M_n, b_n)$ of $\pi_1(M_1, b_1)$ (see Theorem 2.5.13 of [7]). For each n $(n \geq 1)$ there is a unique covering map $\varrho_{n+1}\colon (\overline{K}_b, \overline{b}) \to (M_{n+1}, b_{n+1})$ such that $f_n^{n+1} \varrho_{n+1} = \varrho_n$. The sequence $\varrho_1, \varrho_2, \ldots$ induces the map $\varrho\colon (\overline{K}_b, \overline{b}) \to (M_\infty, b)$, where $\varrho(x) = \big(\varrho_1(x), \varrho_2(x), \ldots\big)$.

$\overline{K}_b$ will be our model for the path component K_b. In this connection, it is convenient to consider a second topology on K_b. Let $\mathcal{B}$ be a basis for the topology of K_b induced by that of M_∞. Let lc-$\mathcal{B}$ be the set of path component of elements of $\mathcal{B}$. Then lc-$\mathcal{B}$ forms a basis for a topology for K_b. We call this the lc-*topology* (local connectivity topology) for K_b. Notice that the lc-topology is independent of the choice of the basis $\mathcal{B}$.

LEMMA 3. *The map $\varrho\colon (\overline{K}_b, \overline{b}) \to (M_\infty, b)$ is one-to-one and onto the path component K_b. If K_b is given the* lc-*topology, then ϱ is a homeomorphism.*

Proof. Let $x \in K_b$. Theorem 2 shows that there is an arc A from b to x and an index n such that $f_n(A)$ is an arc from b_n to x_n. Lift $f_n(A)$ to an arc $\overline{A}$ beginning at $\overline{b}$. Let $\overline{x}$ be the endpoint of $\overline{A}$. Then $\varrho(\overline{x}) = x$.

Now let ν be a path in $\overline{K}_b$ between two points, say $\overline{x}$ and $\overline{y}$, such that $\varrho(\overline{x}) = \varrho(\overline{y})$.

Since $(\varrho_1)_* \pi_1(\overline{K}_b, \overline{b}) = \bigcap_n (f_1^n)_* \pi_1(M_n, b_n)$, we must have $\overline{x} = \overline{y}$.

Hence ϱ is one-to-one. Finally, ϱ is a map if K_b is given the lc-topology. To see that ϱ is open, it is sufficient to notice that a small open cell C in $\overline{K}_b$ is mapped onto a component of $f_1^{-1}\big(\varrho_1(C)\big)$. □

COROLLARY. *If the path components K_a and K_b are homeomorphic, then $\overline{K}_a$ is homeomorphic to $\overline{K}_b$.*

COROLLARY. *$\varrho_*\colon \pi_1(\overline{K}_b, \overline{b}) \to \pi_1(K_b, b)$ is an isomorphism.*

Proof. Since ϱ is a fibration with unique path-lifting and multiplicity 1, this follows from [7], Theorem 2.3.9.

COROLLARY. *Each path component of M is dense in M_∞.*

Proof. Represent $\overline{K}_b$ as the trivial inverse limit $K_\infty = \lim(\overline{K}_b, g)$, where each bonding map is the identity. The sequence $\varrho_1, \varrho_2, \ldots$ of maps induces the map $\varrho_\infty = \varrho\colon K_\infty \to M_\infty$. Since each $\varrho_n \circ g_n = \varrho_n$ is surjective, $\varrho_\infty(K_\infty) = \varrho(\overline{K}_b) = K$ is dense in M_∞ ([5], Theorem 2.5, p. 430).

5. Involutions on weak solenoidal spaces. The purpose of this section is to introduce a technique for constructing examples of weak solenoidal spaces. We use these results in section 6.

Let (A, f) be a (weak) solenoidal sequence such that there is an involution $a_i\colon A_i \to A_i$, for each i, with a nonempty set of isolated fixed points. We require that the involutions commute with the bonding maps; that is, for each i, $a_i f_i^{i+1} = f_i^{i+1} a_{i+1}$. Let B be any closed manifold admitting an involution B with isolated fixed points (possibly fixed point free). Then $a_i \times \beta$ is an involution on $A_i \times B$, and we have the commutative diagram

$$\begin{array}{ccc} A_i \times B & \xleftarrow{f_i^{i+1} \times 1} & A_{i+1} \times B \\ \big\downarrow{\scriptstyle a_i \times \beta} & & \big\downarrow{\scriptstyle a_{i+1} \times \beta} \\ A_i \times B & \xleftarrow{f_i^{i+1} \times 1} & A_{i+1} \times B \end{array}$$

Let C_i denote the orbit space of $a_i \times \beta$ and let $p_i\colon A_i \times B \to C_i$ denote the projection map. Since f_i^{i+1} maps each fiber of p_{i+1} onto a single fiber of p_i, it is easily seen that the covering map $f_i^{i+1} \times 1$ induces a unique covering map $g_i^{i+1}\colon C_{i+1} \to C_i$ such that $p_i(f_i^{i+1} \times 1) = g_i^{i+1} p_{i+1}$.

Let $A_\infty = \lim(A, f)$ and $C_\infty = \lim(C, g)$. The sequence $(a_i \times \beta)$ of involutions induces the involution $a_\infty \times \beta$ on the (weak) solenoidal space $A_\infty \times B$. Clearly the orbit space of $a_\infty \times \beta$ is C_∞, and the sequence $p_1, p_2, \ldots$ of double-covering maps induces the projection $p_\infty\colon A_\infty \times B \to C_\infty$, where $p_\infty(a_\infty \times \beta) = p_\infty$.

The next result follows directly from the preceding discussion.

THEOREM 3. *The (weak) solenoidal space $A_\infty \times B$ admits an involution $a_\infty \times \beta$ whose orbit space is the weak solenoidal space C_∞. Furthermore, $a_\infty \times \beta$ fixes a path component $K \times B$ of $A_\infty \times B$ if and only if K is a path component fixed by a_∞.*

Remark. In particular, if $b \in A_\infty$ is a fixed point of a_∞ then $a_\infty(K_b) = K_b$.

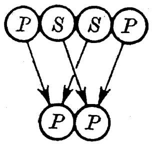
When we restrict our attention to the case where the A_i are circles, we can describe more precisely the action of $a_\infty \times \beta$ on the path components.

LEMMA 4. *Let $A_i = \{z\colon |z| = 1\}$ (the unit circle in the complex plane), $f_i^{i+1}(z) = z^2$, and $a_i(z) = \bar{z}$ ($\bar{z}$ denotes the conjugate of z) for each i. Then exactly one path component of A_∞ is fixed by a_∞, namely that path component containing the single fixed point $\bar{a} = (1, 1, \ldots)$ of a_∞.*

Proof. We need only to observe that any other path component K of A_∞ is mapped onto a distinct path component by a_∞, that is $a_\infty(K) \cap K = \emptyset$ if $\bar{a} \notin K$. Let K be a path component such that, for some $x \in K$, $a_\infty(x) = y$ also belongs to K. Then by Theorem 2, for n sufficiently large, $y_n, y_{n+1}, \ldots$ is an endpoint sequence determined by x and y_n. Let ω_n be an arc with endpoints x_n and y_n, and let ω_{n+k} be the lifting of ω_n with endpoints x_{n+k} and y_{n+k}. Since each a_i is a reflection, it is not difficult to see that $\omega_{n+k} \cup a_{n+k}(\omega_{n+k})$ must contain the point $1 \in A_{n+k}$ for each $k \geqslant 1$. If ω_∞ denotes the arc from x to y determined by the sequence (ω_i), then $\omega_\infty \cup a_\infty(\omega_\infty)$ must contain the point $\bar{a} = (1, 1, \ldots)$. Therefore K must be the unique path component containing $\bar{a}$. □

6. Examples. In this section we apply the preceding results to construct the weak solenoidal space promised in the introduction. We also indicate the construction of similar examples for the higher dimensions.

First we describe the unique manner in which $P_n \# P_n$ covers itself k times (for any integer $k > 1$) [8]. Observe that $P_n \# P_n$ is homeomorphic to the sum $P(k) = P_n \# S^n \# \ldots \# S^n \# P_n$, where S^n occurs $k-1$ times as a summand.



Let $g\colon P(k) \to P_n \# P_n$ be the k-fold covering map in which the sphere summands S^n of $P(k)$ alternately double-cover the P_n summands of the base space, in which the first P_n of $P(k)$ covers the left half of $P_n \# P_n$, and in which the last P_n of $P(k)$ covers the left (right) half of $P_n \# P_n$ if k is even (odd).

Consider the free involution $\alpha \times \beta\colon S^1 \times S^n \to S^1 \times S^n$, where $\alpha(z) = \bar{z}$ and β is the antipodal map on S^n. Then the orbit space of $\alpha \times \beta$ is homeomorphic to $P_n \# P_n$. Let $p\colon S^1 \times S^n \to P_n \# P_n$ denote the projection.

Let $f\colon S^1 \to S^1$ be the k-fold covering map $f(z) = z^k$, and let $g\colon P_n \# P_n \to P_n \# P_n$ be the k-fold covering map described above. Then one can check that the following diagram is commutative:

$$\begin{array}{ccc} S^1 \times S^n & \xleftarrow{f \times 1} & S^1 \times S^n \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ P_n \# P_n & \xleftarrow{g} & P_n \# P_n \end{array}$$

EXAMPLE 1. Let $M_\infty = \lim(M, g)$ be the weak solenoidal space where each $M_i = P_2 \# P_2$ and each $g_i^{i+1}\colon M_{i+1} \to M_i$ is the regular 2-fold covering map g described above.

THEOREM 4. *The nonhomogeneous weak solenoidal space M_∞ has the following properties:*

(1) *The fundamental group of each path component is isomorphic to Z;*

(2) *There are exactly two distinct homeomorphism classes of path components, and one class contains only a single path component;*

(3) *M_∞ is double-covered by the homogeneous space $\Sigma_2 \times S^1$, where Σ_2 is the dyadic solenoid.*

Proof. We have the commutative diagram

$$\begin{array}{ccccc} S^1\times S^1 & \xleftarrow{f_1^2\times 1} & S^1\times S^1 \leftarrow \ldots \leftarrow & \Sigma_2\times S^1 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & \downarrow{\scriptstyle p_\infty} \\ P_2\# P_2 & \xleftarrow{g_1^2} & P_2\# P_2 \leftarrow \ldots \leftarrow & M_\infty \end{array}$$

where $f_i^{i+1}(z)=z^2$ for $z\in S^1$ and p_i is the projection of $S^1\times S^1$ onto the orbit space of the free involution $a_i\times\beta\colon S^1\times S^1\to S^1\times S^1$ defined by $a_i(z)=\bar{z}$ and $\beta(z)=-z$. The sequence $p_1, p_2, \ldots$ of double-covering projections induces a double-covering map p_∞. Moreover, if a_∞ is induced by $a_1, a_2, \ldots$, then the free involution $a_\infty\times\beta$ is a nontrivial covering transformation of p_∞.

According to Theorem 3, $a_\infty\times\beta$ fixes one and only one path component of $\Sigma_2\times S^1$, say K_0. This means that p_∞ restricted to any of the remaining path components is a homeomorphism. Therefore, every path component K of $M_\infty/p_\infty(K_0)$ is homeomorphic under the lc-topology to a path component of $\Sigma_2\times S^1$. The model for each of these path components is then the same as the model for the path components of $\Sigma_2\times S^1$, namely $R\times S^1$ (R denotes the real line). Clearly $\pi_1(K)\cong Z$, for each of these path components.

Now consider the exceptional path component K_0. Let $\bar{a}$ denote the fixed point of a_∞. Then $a_\infty\times\beta(\bar{a}\times S^1)=\bar{a}\times S^1$. The set $\bar{a}\times S^1$ separates K_0 into two path components, and p_∞ restricted to either one of these is a homeomorphism. Hence the model for the path component $p_\infty(K_0)$ is $R\times S^1/\{(t,z)\sim(-t,\beta(z))\}$. Therefore $p_\infty(K_0)$ is not homeomorphic to any other path component of M_∞; however $\pi_1(p_\infty(K_0))\cong Z$. □

Example 2. Consider the weak solenoidal space $M_\infty=\lim(M,g)$, where each $M_i=P_n\# P_n$ and each $g_i^{i+1}\colon M_{i+1}\to M_i$ is a regular 2-fold covering projection. Then M_∞ is an n-dimensional example having all the properties of example 2 except that the fundamental group of the exceptional path component is of order two while the other path components are simply connected.

Remark. Other examples can be constructed similarly by letting each g_i^{i+1} be a k_i-fold covering map. Then M_∞ would be double-covered by $\Sigma_P\times S^1$, where $P=\{k_1, k_2, \ldots\}$. Moreover, in some cases M_∞ will have more than one exceptional path component.

References

[1] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[2] M. K. Fort and J. Segal, *Local connectedness of inverse limit spaces*, Duke Math. J. 28 (1961), pp. 253–260.

[3] M. C. McCord, *Inverse limit sequences with covering maps*, Trans. Amer. Math. Soc. 114 (1965), pp. 197–209.

[4] J. Mioduszewski, *Mappings of inverse limits*, Colloq. Math. 10 (1963), pp. 39–44.

[5] J. Dugundji, *Topology*, 1966.

[6] R. M. Schori, *Inverse limits and homogeneity*, Trans. Amer. Math. Soc. 124 (1966), pp. 533–539.

[7] E. H. Spanier, *Algebraic Topology*, 1966.

[8] J. L. Tollefson, *On 3-manifolds that cover themselves*, Mich. Math. J. 16 (1969), pp. 103–109.

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