for if not then \( p \neq \mathcal{B}f(a, b) \). So every neighborhood of \( p \) meets \( \mathcal{B}f(a, b) \). Thus there is a \( p' \neq \mathcal{B}f(a, b) \) in the bounded complementary domain of \( C \). So there is an open set \( U_p \) about \( p \) such that \( U_p \cap \mathcal{B}f(a, b) = \emptyset \). Therefore there exists an \( n \) and map \( f_n \) with corresponding partition of \( I \) into subintervals such that for some subinterval of this partition, say \( I_n \), \( f(I_n) \subset U_p \). But then from the definition of \( C \) and \( f(I_n) \), \( I_n \subset [a, b] \) which is not possible. Therefore \( \text{int} f(a, b) \) is simply connected.

**Theorem 2.** Let \( f: I \to S \) be the map for Moore's critical curve \( C \) and let \( \{E_i\} \), \( i = 1, 2, \ldots, p \), be a finite disjoint sequence of closed intervals in \( I \). Then

\[
\bigcup_{i=1}^{p} f(E_i) = f \left( \bigcup_{i=1}^{p} E_i \right)
\]

is tame.

**Proof.** By induction and repeated use of theorems such as 11.7, 11.8, 11.9, 11.10, 11.11, 11.12, Whyburn [4], p. 40, together with radial extension of homeomorphisms on boundaries of disks, it suffices to show if \( f(E_i) \) and \( f(E_j), i \neq j \), are 2-cells which meet in their boundaries (by the definition of \( f \) they do not meet in their interiors) then \( \mathcal{B}d(f(E_i) \cap f(E_j)) \) consists of at most two components. For suppose not, \( \mathcal{B}d(f(E_i) \cap f(E_j)) \) consists of open intervals, choose two of these intervals in \( \mathcal{B}d(f(E_i)) \) not accessible in \( \mathcal{B}d(f(E_j)) \) from unbounded complimentary domain of \( \mathcal{B}d(f(E_i) \cup f(E_j)) \). Choose two corresponding closed intervals in \( \mathcal{B}d(f(E_i)) \) which together with the open intervals form two simple closed curves not accessible in the same sense. These simple closed curves bound open disks not accessible as before and which do not meet \( f(E_i) \cup f(E_j) \). Now there exists an \( n, f_n \) and partition of \( I \) into 3\( n \)th intervals such that both disks contain a square in the corresponding partition of \( S \). But as before it is easily seen that this is not possible.

**References**


Involutions on solenoidal spaces

by

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1. Introduction. A weak solenoidal sequence (solenoidal sequence) of closed manifolds is an inverse limit sequence \( (X, f) \) such that each factor space \( X_n \) is a closed manifold and each bonding map \( f_n: X_n \to X_{n-1} \) is a covering map (regular covering map). The limit space \( X_\infty \) is called a weak solenoidal space (solenoidal space).

In section 5, we present a general technique for constructing weak solenoidal spaces from solenoidal spaces. Suppose that \( (X, f) \) is a solenoidal sequence such that each factor space \( X_n \) admits a free involution that commutes with the bonding maps. These involutions induce an involution on the solenoidal space \( X_\infty \); moreover, if \( X_\infty \) is the orbit space of this free involution on \( X_\infty \), then \( Y_\infty \) is a weak solenoidal space.

The importance of this technique is not only that we can construct new examples of weak solenoidal spaces, but we can obtain a deep insight into the internal structure of the spaces. Moreover, if we can construct a weak solenoidal space in a geometric manner and then show that we can obtain the same space as the orbit space of a known free involution on a solenoidal space, then we have tools to investigate both the global and local properties of the spaces.

We carry out this program in section 6, where we present a weak solenoidal space \( M_\infty = \lim (M, f) \) which has the following properties: (1) each factor space \( M_n \) is homeomorphic to the Klein bottle; (2) each bonding map \( f_n: M_n \to M_{n-1} \) is regular (although compositions of bonding maps are not regular); (3) the fundamental groups of any two path components of \( M_\infty \) are isomorphic; (4) \( M_\infty \) is not homogeneous; (5) there are exactly two different homeomorphic classes of path components, with only one path component in the first class; and (6) \( M_\infty \) is double-covered by the product of \( S^1 \) and the dyadic solenoid.

In section 3, we give a convenient characterization of the path component of a weak solenoidal space; this characterization is a valuable tool in the succeeding sections. In the process we obtain some interesting results (in the general theory of inverse limit spaces) concerning the
problem of when an arc in the limit space is induced by arcs in the factor spaces.

M. C. McCord [3] and R. M. Schori [6] previously studied solenoidal spaces. McCord investigated the structure of solenoidal spaces and showed that they have some of the properties of the classical solenoids; in particular, they are homogeneous. Schori constructed an example of a nonhomogeneous weak solenoidal space. Our example is stronger than Schori's; in particular, no two of the factor spaces of his example were homeomorphic, and none of his bonding maps were regular. Furthermore, the homeomorphism types of any of the path components were not determined.

2. Notation. We follow the notation on inverse limit systems in [1] and restrict our discussion to inverse limit sequences (the directed set of indices is the positive integers). We let \( (X, f) \) denote the inverse limit sequence with factor spaces \( X_m \) and bonding maps \( f_m: X_m \to X_{m+1} \). The limit space, \( \lim (X, f) \), is denoted by \( X_{\infty} \). A point \( x \in X_{\infty} \) is represented by \( x = (x_1, x_2, \ldots) \), and we let \( f_n: X_{\infty} \to X_n \) be the projection \( f_n(x) = x_n \).

In this paper, all (weak) solenoidal sequences will be sequences of closed manifolds, that is, compact manifolds without boundary. We further assume that all (weak) solenoidal sequences are nontrivial, i.e., each bonding map is at least a 2-fold covering map.

If \( (X, f) \) and \( (Y, g) \) are two inverse sequences and \( \phi_1, \phi_2, \ldots \) is a sequence of maps \( \phi_m: X_m \to Y_m \) such that \( \phi_m f_m = g_m \phi_m \) (all \( m \leq n \)), then \( \phi_1, \phi_2, \ldots \) induces a map \( \phi: X_{\infty} \to Y_{\infty} \) defined by \( \phi(x) = \phi(x) \) (all \( m \leq n \)).

If \( F = \{ p_1, p_2, \ldots \} \) is a sequence of prime numbers (different from 1), the \( P \)-adic solenoid \( S_F \) is the limit of the inverse sequence \( (X, f) \), where each \( X_n = \{ z: |z| = 1 \} \) (the unit circle in the complex plane), and each bonding map \( f^{(n)}: X_{n+1} \to X_n \) is defined by \( f^{(n)}(z) = z^n \). \( S_F \) is a solenoidal space.

An involution on \( M, k: M \to M \), is a homeomorphism of period two \( (k^2 = 1) \). The orbit space of \( k \) is the quotient space \( \overline{M} (\sim k) \). An \( n \)-sphere involution is an involution without fixed points. The antipodal map of this involution is real projective \( n \)-space \( \mathbb{P}^n \), and the projection \( \pi: \mathbb{P}^n \to \mathbb{P}^1 \) is a double-covering map.

The connected sum, \( M \# M' \), of two connected, triangulated, closed \( n \)-manifolds \( M \) and \( M' \) is obtained by removing the interior of a polyhedral \( n \)-cell from each, and then matching the resulting \( (n-1) \)-sphere boundaries by a piecewise-linear homeomorphism (orientation reversing if both manifolds are oriented). The \( n \)-sphere serves as the identity element of this operation, that is, \( M \# S^n \simeq M \). If \( f \) is a map with domain \( X \) and \( A \subset X \), then \( f|A \) denotes the map which is the restriction of \( f \) to \( A \). We refer the reader to [7] for the theory and terminology of covering maps.

3. Characterization of path components of \( M_{\infty} \). The goal of this section is to characterize the path components of a weak solenoidal space \( M_{\infty} \). In order to do this, we first investigate a problem in the general theory of inverse limit spaces and prove a theorem (of considerable interest in itself) about arcs in \( M_{\infty} \).

Let \( X = \lim (X, f) \) be an arbitrary inverse limit space, and let \( A \) be an arc in \( X \). When there exists a positive integer \( n \) such that \( n > m \) implies that \( f_m(A) \) is an arc, the answer is sometimes negative, as Fort and Segal [2] have shown that an arc can be represented as the inverse limit of 2-cells with bonding maps. The following example shows that even if each \( f_m(A) \) is one-dimensional, the question has no positive answer in general. For each positive integer \( n \), let \( X_n \) be the union of the unit interval \( I = [0, 1] \) and a perpendicular segment \( S_n \) of length \( 1/n \) which intersects \( I \) at the midpoint of \( [1/n+1, 1/n] \). Let \( f^{(n)}: X_{n+1} \to X_n \) be a map which collapses \( S_{n+1} \) to a point, maps \( (1/n+1, 1/n) \subset S_n \) and is the identity elsewhere. If \( X_{\infty} \) is the inverse limit of \( (X, f) \), then \( (0, 0, \ldots) \) and \( (1, 1, \ldots) \) are the only nonseparating points of \( X_{\infty} \); therefore \( X \) is an arc. However, \( f^{(n)}(X) \) is never an arc.

If \( X_{\infty} \) is a weak solenoidal space, then the question has a positive answer. We precede the proof of this statement with two lemmas; the first is well known and the second is a special case of Theorem 3 of [4].

**Lemma 1.** Let \( M \) be a compact manifold with metric \( d \). There exists \( \epsilon > 0 \) such that if \( a \) and \( b \) are paths in \( M \) and \( d(a, b) < \epsilon \), then \( a \simeq b \).

**Lemma 2.** Let \( X_{\infty} = \lim (X, f) \) be an inverse limit space, let \( v: I \to X_{\infty} \) be an embedding, and let \( \epsilon > 0 \). Then there exists an integer \( n \) and a map \( \psi: f_n \simeq v(I) \to I \) such that \( d(v(I), v(I)) = f_n \simeq v(I) \), \( f_n \simeq v(I) \to I \), and \( d(v(I), v(I)) < \epsilon \).

**Proof.** Represent \( I \) as the trivial inverse limit \( I = \lim (I, f) \), where each \( I_n = I \) and each bonding map \( g^{(n)}: I_n \to I_n \) is the identity. Let \( n = f_n \simeq v(I) \to I \), \( f_n \simeq v(I) \to I \), and \( d(v(I), v(I)) < \epsilon \).

**Theorem 1.** Let \( M_{\infty} = \lim (M, f) \) be a weak solenoidal space, and let \( A \) be an arc in \( M_{\infty} \). Then there exists a positive integer \( n \) such that \( n \geq n \) implies that \( f_m(A) \) is an arc.

**Proof.** Let \( v \) be a homeomorphism of \( I \) onto \( A \). For each \( i, v_i = f_i \simeq v \) is a path in \( M_i \).

For each \( i, v_i \) contains no contractible loop, since a contractible loop would lift to a loop in each factor space \( M_i \), \( k \geq i \), and hence to a loop in \( M_{\infty} \). However, \( v_n \) lifts to \( v \), which contains no loops.

Let \( x \) be a positive number such that if \( a \) and \( b \) are paths in \( M_i \), then \( d(a, b) < \epsilon \) implies that \( a \simeq b \). Lemma 2 implies that there exist an integer \( n \) and a map \( \psi: f_n(A) \to I \) such that \( d(f_n(A), f_n(A)) < \epsilon \). Hence \( d(v_0, v_0) < \epsilon \), and so \( v_0 \simeq v_0 \simeq f_n(A) \). Because \( v_0 \simeq v_0 \),
factors through $I$, it is homotopic to a constant, and hence $f_{h_{n}} \circ \nu_{h_{n}}$ is also homotopic to a constant. Since $f_{h_{n}}$ has the homotopy lifting property, $\nu_{h_{n}}$ is homotopic to a constant. Therefore $\nu_{h_{n}}$ contains no noncontractible loops. Thus $\nu_{h_{n}}$ is an embedding and $f_{a}(A)$ an arc. Since $f_{a}(A)$ covers $f_{a}(A)$ for $m > n$, $f_{a}(A)$ is also an arc. □

We use Theorem 1 and the next definition to obtain a useful characterization of path components in $M_{\infty}$.

**Definition.** Let $M_{\infty} = \lim (M, f)$ be a weak solenoidal space, and let $x = (x_{1}, x_{2}, \ldots)$ belong to $M_{\infty}$. Let $y \in M_{1}$ and let $\nu: (I, 0, 1) \rightarrow (M_{1}, x_{1}, y)$ be an embedding of $I$ in $M_{1}$. For each $j > i$, $x_{j}$ lifts to an embedding $\nu_{j}: (I, 0, 1) \rightarrow (M_{j}, x_{j}, y(1))$. The sequence $(\nu_{n}(1))_{n=1}^{\infty}$ is called an endpoint sequence induced by $x$ and $y$.

**Theorem 2.** Let $M_{\infty} = \lim (M, f)$ be a weak solenoidal space. Let $x = (x_{1}, x_{2}, \ldots) \in M_{\infty}$, and let $K$ be the path component of $M_{\infty}$ which contains $x$. Then $K = \{(x_{1}, x_{2}, \ldots) \in M_{\infty} : \text{for some } i, x_{i}, x_{i+1}, \ldots \text{ is an endpoint sequence determined by } x \text{ and } x_{i}\}$.

**Proof.** Suppose $x = (x_{1}, x_{2}, \ldots)$ is a point of $M_{\infty}$ such that $x_{i}, x_{i+1}, \ldots$ is an endpoint sequence determined by $x$ and $x_{i}$. Then there exists an embedding $\nu_{j}: (I, 0, 1) \rightarrow (M_{j}, x_{j}, y(j))$ which lifts, for $j > i$, to an embedding $\nu_{j}: (I, 0, 1) \rightarrow (M_{j}, x_{j}, y(j))$, and hence to a path $\nu: (I, 0, 1) \rightarrow (M_{\infty}, x, y)$. Therefore $x$ belongs to $K$.

On the other hand, let $x = (x_{1}, x_{2}, \ldots)$ be a point of $K$, and let $A$ be an arc from $x$ to $x$. Let $\nu: (I, 0, 1) \rightarrow (M_{\infty}, x, y)$ be a homeomorphism. For each positive integer $i$, $x_{i} = \nu^{-1}(y(i))$ is a path in $M_{i}$ from $x_{i}$ to $x_{i+1}$. By Theorem 1, we can find an integer $n_{0}$ such that $\nu_{n}(I)$ is an arc. Hence $x_{n_{0}}, x_{n_{0}+1}, \ldots$ is an endpoint sequence determined by $x_{n_{0}}$ and $x_{n_{0}+1}$. □

**4. Path component models.** Weak solenoidal spaces have a very complicated structure in general, making it difficult to distinguish between homeomorphism classes of path components. It is often useful to construct "untangled" models of the path components. In certain cases, these models will enable us to distinguish between homeomorphism classes of the path components.

Let $(M, f)$ be a weak solenoidal sequence and let $b \in M_{\infty}$. Denote by $K_{b}$ the path component of $M_{\infty}$ containing the point $b = (b_{1}, b_{2}, \ldots)$. Let $b_{0}$ be a basepoint for $M_{1}$, and consider the descending chain of subgroups of $\pi_{1}(M_{1}, b_{0})$:

$$\pi_{1}(M_{1}, b_{0}) \supset (f_{1})_{*} \pi_{1}(M_{2}, b_{2}) \supset (f_{2})_{*} \pi_{1}(M_{3}, b_{3}) \supset \cdots$$

Let $\varphi_{1}: (K_{b}, \delta) \rightarrow (M_{1}, b_{1})$ be the covering space determined by the subgroup $(f_{1})_{*} \pi_{1}(M_{2}, b_{2})$ of $\pi_{1}(M_{1}, b_{0})$ (see Theorem 2.5.13 of [7]). For each $n (n \geq 1)$ there is a unique covering map $\varphi_{n}: (K_{b}, \delta) \rightarrow (M_{n+1}, b_{n+1})$ such that $(f_{n})_{*} \varphi_{n+1} = \varphi_{n}$. The sequence $\varphi_{1}, \varphi_{2}, \ldots$ induces the map $\varphi: (K_{b}, \delta) \rightarrow (M_{n}, b_{n})$, where $x = (\varphi_{1}(x), \varphi_{2}(x), \ldots)$.

$K_{b}$ will be our model for the path component $K_{b}$. In this connection, it is convenient to consider a second topology on $K_{b}$. Let $\mathfrak{B}$ be a basis for the topology of $K_{b}$ induced by that of $M_{\infty}$. Let $\mathfrak{B}_{\text{lc}}$ be the set of path components of elements of $\mathfrak{B}$. Then $\mathfrak{B}_{\text{lc}}$ forms a basis for a topology for $K_{b}$. We call this the lc-topology (local connectivity topology) for $K_{b}$.

Notice that the lc-topology is independent of the choice of the basis $\mathfrak{B}$.

**Lemma 3.** The map $\varphi: (K_{b}, \delta) \rightarrow (M_{\infty}, b)$ is one-to-one and onto the path component $K_{b}$. If $K_{b}$ is given the lc-topology, then $\varphi$ is a homeomorphism.

**Proof.** Let $x \in K_{b}$. Theorem 2 shows that there is an arc $A$ from $b$ to $x$ and an index $n$ such that $f_{n}(A)$ is an arc from $b_{n}$ to $x_{n}$. Lift $f_{n}(A)$ to an arc $\tilde{A}$ beginning at $\tilde{b}$. Let $\tilde{x}$ be the endpoint of $\tilde{A}$. Then $\varphi(\tilde{x}) = x$.

Now let $y$ be a path in $K_{b}$ between two points, say $\tilde{x}$ and $\tilde{y}$, such that $\varphi(\tilde{x}) = y$. Since $(\varphi_{1}, \varphi_{2}, \ldots) \subset (K_{b}, \delta) = \bigcap (f_{n})_{*} \pi_{1}(M_{n}, b_{n})$, we must have $\tilde{x} = \tilde{y}$.

Hence $\varphi$ is one-to-one. Finally, $\varphi$ is a map if $K_{b}$ is given the lc-topology. To see that $\varphi$ is open, it is sufficient to notice that a small open cell $C$ in $K_{b}$ is mapped onto a component of $f_{n}(C)$.

**Corollary.** If the path components $K_{a}$ and $K_{b}$ are homeomorphic, then $K_{b}$ is homeomorphic to $K_{a}$.

**Corollary.** If $\pi_{1}(K_{b}, \delta) \rightarrow \pi_{1}(K_{a}, \delta)$ is an isomorphism.

**Proof.** Since $\varphi$ is a fibration with unique path-lifting and multiplicity 1, this follows from [7], Theorem 2.3.9.

**Corollary.** Each path component of $M_{\infty}$ is dense in $M_{\infty}$.

**Proof.** Represent $K_{b}$ as the trivial inverse limit $M_{\infty} = \lim (K_{b}, \psi)$, where each bonding map is the identity. The sequence $\psi_{1}, \psi_{2}, \ldots$ of maps induces the map $\varphi_{0} = \psi: K_{b} \rightarrow M_{\infty}$. Since each $\varphi_{n} = \varphi_{n}$ is surjective, $\varphi_{0}(K_{b}) = \varphi(K_{b}) = K$ is dense in $M_{\infty}$ ([7], Theorem 2.5, p. 430).

5. **Involutions on weak solenoidal spaces.** The purpose of this section is to introduce a technique for constructing examples of weak solenoidal spaces. We use these results in section 6.

Let $(A, f)$ be a (weak) solenoidal sequence such that there is an involution $ac: A_{1} \rightarrow A_{1}$, for each $i$, with a nonempty set of isolated fixed points. We require that the involutions commute with the bonding maps; that is, for each $i$, $af_{i} = f_{i} a_{i}$, for each $i$, with a nonempty set of isolated fixed points (possibly fixed point
free). Then \( a_1 \times \beta \) is an involution on \( A_1 \times B \), and we have the commutative diagram

\[
\begin{array}{ccc}
A_1 \times B & \overset{f \times 1}{\longrightarrow} & A_1 \times B \\
\downarrow \text{id} \times \beta & & \downarrow \text{id} \times \beta \\
A_1 \times B & \overset{f \times 1}{\longrightarrow} & A_1 \times B
\end{array}
\]

Let \( C_1 \) denote the orbit space of \( a_1 \times \beta \) and let \( p_1: A_1 \times B \to C_1 \) denote the projection map. Since \( f \times 1 \) maps each fiber of \( p_1 \) onto a single fiber of \( p_1 \), it is easily seen that the covering map \( f \times 1 \) induces a unique covering map \( g \times 1 : C_1 \to C_1 \) such that \( p_1 \circ (f \times 1) = g \circ p_1 \).

Let \( A_m = \lim(A_1, f) \) and \( C_m = \lim(C_1, g) \). The sequence \( (a_1 \times \beta) \) of involutions induces the involution \( a_m \times \beta \) on the (weak) solenoidal space \( A_m \times B \). Clearly the orbit space of \( a_m \times \beta \) is \( C_m \), and the sequence \( p_1, p_2, \ldots \) of double-covering maps induces the projection \( p_m : A_m \times B \to C_m \), where \( p_m \circ (a_m \times \beta) = p_m \).

The next result follows directly from the preceding discussion.

**Theorem 3.** The weak solenoidal space \( A_m \times B \) admits an involution \( a_m \times \beta \) whose orbit space is the weak solenoidal space \( C_m \). Furthermore, \( a_m \times \beta \) fixes a path component \( K \times B \) of \( A_m \times B \) if and only if \( K \) is a path component fixed by \( a_m \).

**Remark.** In particular, if \( b \in A_m \times B \) is a fixed point of \( a_m \), then \( a_m(b) = b \).

When we restrict our attention to the case where the \( A_m \) are circles, we can describe more precisely the action of \( a_m \times \beta \) on the path components.

**Lemma 4.** Let \( A_1 = \{ z : |z| = 1 \} \) (the unit circle in the complex plane), \( f \times 1(z) = z^2 \), and \( a_1(z) = \bar{z} \) (\( \bar{z} \) denotes the conjugate of \( z \)) for each \( z \). Then exactly one path component of \( A_m \times B \) is fixed by \( a_m \), namely that path component containing the single fixed point \( \bar{z} = (1, 1, 1, \ldots) \) of \( a_m \).

**Proof.** We need only to observe that any other path component \( K \) of \( A_m \) is mapped onto a distinct path component by \( a_m \), that is \( a_m(K) \cap K = \emptyset \) for \( \bar{z} \notin K \). Let \( K \) be a path component such that, for some \( x \in K \), \( a_m(x) = y \) also belongs to \( K \). Then by Theorem 2, for \( n \) sufficiently large, \( y_n, y_{n+1}, \ldots \) is an endpoint sequence determined by \( x \) and \( y_n \). Let \( a_m \) be an arc with endpoints \( x_n \) and \( y_n \), and let \( a_n \) be the lifting of \( a_m \) with endpoints \( x_{n+k} \) and \( y_{n+k} \). Since each \( a_n \) is a reflection, it is not difficult to see that \( a_n \cup a_m \cup a_n \) must contain the point \( \bar{z} \notin K \) for each \( k \geq 1 \). If \( a_m \) denotes the arc from \( x \) to \( y \) determined by the sequence \( (a_m) \), then \( a_m \cup a_n \cup a_m \) must contain the point \( \bar{z}_1 = (1, 1, 1, \ldots) \). Therefore \( K \) must be the unique path component containing \( \bar{z}_1 \).

**6. Examples.** In this section we apply the preceding results to construct the weak solenoidal space promised in the introduction. We also indicate the construction of similar examples for the higher dimensions.

First we describe the unique manner in which \( P_k \neq P_n \) covers itself \( k \) times (for any integer \( k > 1 \)) \([8]\). Observe that \( P_k \neq P_n \) is homeomorphic to the sum \( P(k) = P_k \neq S^1 \neq \cdots \neq S^1 \neq P_n \), where \( S^1 \) occurs \( k-1 \) times as a summand.

Let \( g : P(k) \to P_n \neq P_n \) be the \( k \)-fold covering map in which the sphere summands \( S^1 \) of \( P(k) \) alternately double-cover the \( P_n \) summands of the base space, in which the first \( P_n \) of \( P(k) \) covers the left half of \( P_k \neq P_n \), and in which the last \( P_n \) of \( P(k) \) covers the right (left) half of \( P_k \neq P_n \) if \( k \) is even (odd).

Consider the free involution \( a \times \beta : S^1 \times S^1 \to S^1 \times S^1 \), where \( a(z) = \bar{z} \) and \( \beta \) is the antipodal map on \( S^1 \). Then the orbit space of \( a \times \beta \) is homeomorphic to \( P_n \neq P_n \). Let \( p : S^1 \times S^1 \to P_n \neq P_n \) denote the projection.

Let \( f : S^1 \to S^1 \) be the \( k \)-fold covering map \( f(z) = z^k \), and let \( g : P_n \neq P_n \to P_n \neq P_n \) be the \( k \)-fold covering map described above. Then one can check that the following diagram is commutative:

\[
\begin{array}{ccc}
S^1 \times S^1 & \overset{f \times 1}{\longrightarrow} & S^1 \times S^1 \\
\downarrow p & & \downarrow p \\
P_n \neq P_n & \overset{g}{\longrightarrow} & P_n \neq P_n
\end{array}
\]

**Example 1.** Let \( M_0 = \lim(M_0, g) \) be the weak solenoidal space where each \( M_1 = P_1 \neq P_1 \) and each \( f \times 1 : M_1 \to M_1 \) is the regular \( k \)-fold covering map \( g \) described above.

**Theorem 4.** The nonhomogeneous weak solenoidal space \( M_0 \) has the following properties:

1. The fundamental group of each path component is isomorphic to \( Z \).
2. There are exactly two distinct homomorphism classes of path components, and one class contains only a single path component;
3. \( M_0 \) is double-covered by the homogeneous space \( S^1 \times S^2 \), where \( S^1 \) is the dyadic solenoid.
Proof. We have the commutative diagram

\[
\begin{array}{cccc}
S^4 \times S^4 & \overset{f_{4 \times 4}}{\longrightarrow} & S^4 \times S^4 & \cdots & \overset{\Sigma_4 \times S^4}{\longrightarrow} \\
\gamma_1 & & \gamma_1 & & \vdots \\
\gamma_0 & & \gamma_0 & & \gamma_0 \\
\gamma_0 \oplus \gamma_0 & \overset{g_{4 \times 4}}{\longrightarrow} & \gamma_0 \oplus \gamma_0 & \cdots & \overset{\Sigma_4 \oplus S^4}{\longrightarrow} \\
\end{array}
\]

where \( f_{4 \times 4}(z) = z^4 \) for \( z \in S^4 \) and \( \gamma_1 \) is the projection of \( S^4 \times S^4 \) onto the orbit space of the free involution \( \alpha \times \beta: S^4 \times S^4 \to S^4 \times S^4 \) defined by \( \alpha(\bar{z}) = \bar{z} \) and \( \beta(z) = -z \). The sequence \( \gamma_0, \gamma_1, \ldots \) of double-covering projections induces a double-covering map \( p_\infty \). Moreover, if \( \alpha_\infty \) is induced by \( \alpha_1, \alpha_2, \ldots \), then the free involution \( \alpha_\infty \times \beta \) is a nontrivial covering transformation of \( p_\infty \).

According to Theorem 3, \( \alpha_\infty \times \beta \) fixes one and only one path component of \( \Sigma_4 \times S^4 \), say \( K_4 \). This means that \( p_\infty \) restricted to any of the remaining path components is a homeomorphism. Therefore, every path component \( K \) of \( M_\infty[p_\infty(K_4)] \) is homeomorphic under the \( \infty \)-topology to a path component of \( \Sigma_4 \times S^4 \). The model for each of these path components is then the same as the model for the path components of \( \Sigma_4 \times S^4 \), namely \( R \times S^4 \) (\( R \) denotes the real line). Clearly \( \pi_1(K) \cong Z \), for each of these path components.

Now consider the exceptional path component \( K_4 \). Let \( a \) denote the fixed point of \( \alpha_\infty \). Then \( \alpha_\infty \times \beta(\bar{a} \times S^4) = \bar{a} \times S^4 \). The set \( \bar{a} \times S^4 \) separates \( K_4 \) into two path components, and \( p_\infty \) restricted to either one of these is a homeomorphism. Hence the model for the path component \( p_\infty(K_4) \) is \( R \times S^4/[t, z] \sim (-t, -\beta(z)] \). Therefore \( p_\infty(K_4) \) is not homeomorphic to any other path component of \( M_\infty \); however \( \pi_1[p_\infty(K_4)] \cong Z \).

Example 2. Consider the weak solenoidal space \( M_\infty = \lim_\leftarrow (M, g) \), where each \( M_\infty = P_\infty \neq P_\infty \) and each \( g_t^{\infty}; \) \( M_\infty \to M_\infty \) is a regular 2-fold covering projection. Then \( M_\infty \) is an n-dimensional example having all the properties of Example 2 except that the fundamental group of the exceptional path component is of order two while the other path components are simply connected.

Remark. Other examples can be constructed similarly by letting each \( g_t^{\infty} \) be a \( k_t \)-fold covering map. Then \( M_\infty \) would be double-covered by \( \Sigma_4 \times S^4 \), where \( P = \{x_t, \bar{x}_t, \ldots \} \). Moreover, in some cases \( M_\infty \) will have more than one exceptional path component.

References