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On some problems of Borsuk

by

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1. Introduction. In [1] Borsuk introduced the concept of the index of r-proximity of two topological spaces. The purpose of this article is to answer three problems concerning this index posed by Borsuk in [2], p. 208. Theorem 8 provides the answers to all three problems.

2. Main result.

DEFINITION 1. A natural number n is said to be the *index of r-proximity* of two spaces X and Y provided there is a system of $n+1$, but not less, spaces $X_0, X_1, \ldots, X_n$ with $X_0 = X$, $X_n = Y$ and such that X_i and X_{i+1} are r-neighbors for $i = 0, 1, \ldots, n-1$.

A space X is a *r-image* of a space Z, denoted $X \leqslant_r Z$, if there are maps $r\colon Z \to X$ and $i\colon X \to Z$ such that $ri = \mathrm{id}_X$, the identity on X. $X \not\leqslant_r Z$ means not $X \leqslant_r Z$; $X <_r Z$ means that both $X \leqslant_r Z$ and $Z \not\leqslant_r X$. X is an r-neighbor of Z if there is no space that is strictly r-between X and Z; i.e. there is no space Y with $X <_r Y <_r Z$.

See [2] for definitions.

DEFINITION 2. A Hausdorff space X is a *Peano space* if and only if X is the continuous image of the unit interval, $I = [0, 1]$.

LEMMA 3. *If X is a Peano space, then there exists a map $h\colon (0, 1] \to X \times I$ such that $\overline{h((0,1])} = h((0,1]) \,\dot\cup\, (X \times 0)$, (disjoint). Also $h|_{[t,1]}$ is an embedding for each $0 < t < 1$.*

Proof. Let $f\colon I \to X$ with $f(I) = X$ be given. Define $g\colon (0, 1] \to I \times I$ by $g(t) = \left(\frac{1}{2}\sin\frac{1}{t} + \frac{1}{2}, t\right)$. Let $p\colon I \times I \to I$ be the projection $p(x, y) = x$ and set $\varphi = pg$. Finally, define $h\colon (0, 1] \to X \times I$ by $h(t) = (f\varphi(t), t)$. Then h is continuous and satisfies the conditions given.

$X \times 0$ *is contained in* $\overline{h((0,1])}$. For each $(x, 0) \in X \times 0$ there is a t in I such that $f(t) = x$ and a sequence of points $\{t_n\}$ in $(0, 1]$ such that $\{t_n\}$ converges to zero and $\varphi(t_n) = t$ for each n. Then $h(t_n) = (f\varphi(t_n), t_n) = (x, t_n)$ and so $\{h(t_n)\}$ converges to $(x, 0)$. That is, $(x, 0) \in \overline{h((0,1])}$.

$\overline{h((0,1])}$ *is contained in* $h((0,1]) \cup (X \times 0)$. Let z be a limit point of $h((0,1])$ and choose a sequence of points $\{t_n\}$ in $(0,1]$ such that $\{h(t_n)\}$ converges to z. If $t_n \geqslant \varepsilon$ for some $\varepsilon > 0$, then by compactness we can assume $\{t_n\}$ converges to some t in $[\varepsilon, 1]$ and so $z = h(t)$, which is in $h((0,1])$. Otherwise, we may assume $\{t_n\}$ converges to zero. Since X is compact metric, there is a subsequence $\{f\varphi(t_{n_k})\}$ of $\{f\varphi(t_n)\}$ that converges to some point $x \in X$. Then $\{h(t_{n_k})\} = \{(f\varphi(t_{n_k}), t_{n_k})\}$ converges to $(x, 0)$ and so $z = (x, 0)$, which is in $X \times 0$. Since $h((0,1])$ and $X \times 0$ are disjoint this establishes the first part of the lemma.

Since $[t,1]$ is compact and $X \times 0$ is Hausdorff $h|\widehat{[t,1]}$ is a closed map and thus easily an embedding for $0 < t < 1$.

DEFINITION 4. Let X and Y be Peano spaces and h, h' defined as in Lemma 3 for X and Y respectively. Define $W = \overline{h((0,1])} \bigcup_f \overline{h'((0,1])} = \overline{h((0,1])}$ attached to the space $\overline{h'((0,1])}$ by the map f that sends $h(1)$ to $h'(1)$.

LEMMA 5. *If X and Y are non-degenerate, then* $h((0,1])$ *and W are not locally connected.*

Proof. We will prove the lemma for $\overline{h((0,1])}$, the proof for W being similar. By contradiction, assume $\overline{h((0,1])}$ is locally connected. Let $(x, 0) \in X \times 0$ and choose a sequence $\{t_n\}$ in $(0,1]$ such that $h(t_n)$ converges to $(x, 0)$. Let N be a closed neighborhood of $(x, 0)$ that does not contain all of $X \times 0$. Let V be a connected neighborhood of $(x, 0)$ contained in N and fix k so that $h(t_n) \in V$ for $n \geqslant k$. Then by the connectedness of V, $h((0, t_k])$ is contained in V and hence $\overline{h((0, t_k])}$ is contained in N. By the proof of Lemma 3, it follows that $X \times 0 \subset \overline{h((0, t_k])} \subset N$. This contradicts the choice of N and Lemma 5 is established.

LEMMA 6. *The only retracts of* $\overline{h((0,1])}$ *are*

(i) *retracts of* $X \times 0$,

(ii) *a closed interval or point*,

(iii) *homeomorphs of* $\overline{h((0,1])}$.

Proof. Let Z be a retract of $\overline{h((0,1])}$. Then Z is compact and connected. If $Z \subset X \times 0$, then Z is also a retract $X \times 0$. If $Z \subset h((0,1])$, then $Z \subset h([\varepsilon, 1])$ for some $\varepsilon > 0$, by the compactness of Z. Since $h([\varepsilon, 1])$ is homeomorphic to $[\varepsilon, 1]$, Z is (homeomorphic to) an interval or a point. If Z intersects both $X \times 0$ and $h((0,1])$, let $s = \sup\{t\colon h(t) \in Z\}$. Since Z is closed and connected it follows that $Z = \overline{h((0, s])}$, which is homeomorphic to $\overline{h((0,1])}$.

LEMMA 7. *The only retracts of* $W = \overline{h((0,1])} \cup_f \overline{h'((0,1])}$ *are*

(i) *retracts of* $X \times 0$ *and* $Y \times 0$,

(ii) *a closed interval or point*,

(iii) *homeomorphs of* $\overline{h((0,1])}$ *and* $\overline{h'((0,1])}$,

(iv) *homeomorphs of* W.

The proof of Lemma 7 is similar to that of Lemma 6.

THEOREM 8. *If X and Y are non-degenerate Peano spaces of different r-types, then the index of r-proximity of X and Y is not greater than 4.*

Proof. It suffices to show that

$$X <_r \overline{h((0,1])} <_r \overline{h((0,1])} \cup_f \overline{h'((0,1])} >_r \overline{h'((0,1])} >_r Y$$

is a system of r-neighbors.

X is a left r-neighbor of $\overline{h((0,1])}$. $X \leqslant_r \overline{h((0,1])}$. Define $r\colon \overline{h((0,1])} \to X$ by $r(x, t) = x$. The map r is continuous since it is simply the restriction to $\overline{h((0,1])}$ of the projection map of $X \times I$ onto X. The embedding $i\colon X \to X \times 0$ defined by $x \to (x, 0)$ satisfies $ri = \mathrm{id}_X$. $X \not\geqslant_r \overline{h((0,1])}$. Any r-image of X is locally connected, whereas $\overline{h((0,1])}$ is not locally connected, by Lemma 3. That there is no space that is strictly r-between X and $\overline{h((0,1])}$ follows from Lemma 6. The same argument shows that Y is a left r-neighbor of $\overline{h'((0,1])}$.

$\overline{h((0,1])}$ *is a left r-neighbor of* $W = \overline{h((0,1])} \cup_f \overline{h'((0,1])}$. $\overline{h((0,1])} \leqslant_r W$. This is clear. $\overline{h((0,1])} \not\geqslant_r W$. Assume $\overline{h((0,1])} \geqslant_r W$. Then by Lemma 6, W must be homeomorphic to $\overline{h((0,1])}$, since the other spaces of Lemma 6 are locally connected while W is not. Let $\theta\colon W \to \overline{h((0,1])}$ be a homeomorphism and let $p = \theta(h(1))$. (We consider $h(1) = h'(1)$ as a point of W.) The complement $W - h(1)$ has two components, $\overline{h((0,1])} - h(1)$ and $\overline{h'((0,1])} - h(1)$. Then $p \neq h(1)$ and $p \notin X \times 0$ since then $W - h(1)$ is homeomorphic to $\overline{h((0,1])} - p$ which is connected. Therefore $p = h(t)$ for some t in $(0,1)$. But then $\overline{h((t,1])}$ must be homeomorphic to one of the components of $W - h(1)$. But neither of these components is locally connected, whereas $h((t,1])$ is locally connected. Hence $\overline{h((0,1])} \not\geqslant_r W$. That there is no space that is strictly r-between $\overline{h((0,1])}$ and W follows from Lemma 7. By the same argument $\overline{h'((0,1])}$ is also a left r-neighbor of W. This completes the proof.

DEFINITION 9. A space X is a *continuum* if and only if X is compact, connected and metric.

THEOREM 10 (Hahn–Mazurkiewicz). *X is a Peano space if and only if X is a locally connected continuum.*

Because of Theorem 10, the class of spaces to which Theorem 8 applies is quite extensive.

3. Three problems of Borsuk. Theorem 8 provides the following answers to the problems posed in [2], p. 208.

PROBLEM 1. The index of r-proximity of the torus, $S^1 \times S^1$, and the 2-sphere is finite (in fact no greater than four), since they are both Peano spaces.

PROBLEM 2. The index of two ANR sets with different dimensions greater than 1 is not necessarily infinite. For example any two compact, connected topological manifolds with different dimensions greater that 1 have index no greater than four. In particular, take an m-ball and an n-ball, with $1 \leqslant m < n$.

PROBLEM 3. The index of r-proximity of two ANR spaces X, Y is not necessarily $\geqslant \sum_{0}^{\infty} |p_k(X) - p_k(Y)|$ where $p_k(Z)$ is the k-dimensional Betti number of Z. For example take X to be a 2-sphere and Y to be a 2-sphere with three or more handles.

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