Countable paracompactness of inverse limits and products

by

Keiô Nagami (Matsuyama)

0. Introduction. The aim of this paper is to give detailed proofs for the author's abstract [9]. In this paper all spaces are assumed to be Hausdorff and all mappings to be continuous. The suffix i ranges always over the positive integers. In Section 1 the normality, paracompactness, hereditary normality, total normality, etc., of inverse limits are studied. The countable paracompactness is shown to play an essential role for these topological properties which are to be countably projective to inverse limits. In Section 2 we treat the hereditary normality of the product of two spaces and the normality of the product of uncountably many spaces. Our results for the first case is to be considered as corollaries of Katětov's work [4]. In the second case the countable paracompactness plays again a meaningful role.

1. Inverse limits. Let $\{X_i, \pi_j^i\}$ be an inverse limiting system of a sequence of spaces X_i with the onto projections π_j^i: $X_i \to X_j$ $(i \geqslant j)$. Let X be the inverse limit of this system and π_i: $X \to X_i$ the projections. A set of type $\pi_i^{-1}(S)$ is said a *cylindrical closed set* if S is a closed set of X_i. We consider the following condition, say:

(∗) An arbitrary countable covering of X consisting of monotonically increasing open sets can be refined by a countable covering consisting of cylindrical closed sets.

1.1. THEOREM. *If one of the following conditions is satisfied, then X satisfies the condition* (∗).

(i) *Every open set of every X_i is F_σ.*

(ii) *X is countably paracompact and every π_j^i is open.*

Proof. Let $\{G_i\}$ be a covering of X consisting of monotonically increasing open sets. Let G_i^j be the maximal open set of X_j with $\pi_j^{-1}(G_i^j) \subset G_i$. This notion for open sets of X will be carried out throughout the paper. Then $\bigcup \pi_j^{-1}(G_j^j) = X$.

If (i) is satisfied, then for each j, $G_j^j = \bigcup_{i=1}^{\infty} F_{ji}$ with F_{ji} closed in X_j. Evidently

$$\{\pi_j^{-1}(F_{ji}):\ i = 1, 2, \ldots,\ j = 1, 2, \ldots\}$$

is a covering of X consisting of cylindrical closed sets which refines $\{G_i\}$.

Assume that (ii) is satisfied. Since $\pi_j^{-1}(G_j^j) \subset \pi_{j+1}^{-1}(G_{j+1}^{j+1})$ for each j, there exists, by Ishikawa [3] or by an easy verification, a sequence of open sets U_j, $j = 1, 2, \ldots$, of X such that $\overline{U}_j \subset \pi_j^{-1}(G_j^j)$ for each j and such that $\bigcup U_j = X$. Since $\{\pi_k^{-1}(U_j^k):\ k \geqslant j,\ j = 1, 2, \ldots\}$ covers X, $\{\pi_k^{-1}(\overline{U_j^k}):\ k \geqslant j,\ j = 1, 2, \ldots\}$, say $\mathcal{F}$, is a covering by cylindrical closed sets. To see $\overline{\pi_k^{-1}(U_j^k)} \subset \pi_k^{-1}(\overline{U_j^k})$ let $x = \langle x_i \rangle$ be not in $\pi_k^{-1}(\overline{U_j^k})$. Then there exist an $s \geqslant k$ and an open neighborhood V of x_s such that $\pi_s^{-1}(V) \cap \pi_k^{-1}(U_j^k) = \emptyset$. Hence $\pi_k^s(V) \cap U_j^k = \emptyset$. Since π_k^s is open, x_k is not in $\overline{U_j^k}$, which implies that x is not in $\pi_k^{-1}(\overline{U_j^k})$. Thus $\overline{\pi_k^{-1}(U_j^k)} \subset \pi_k^{-1}(\overline{U_j^k})$. Since $\pi_k^{-1}(\overline{U_j^k}) \subset \pi_j^{-1}(G_j^j)$, $\mathcal{F}$ refines $\{G_i\}$ and the theorem is proved.

1.2. Theorem. *Let every X_i be normal. If X satisfies the condition* (∗), *then X is countably paracompact.*

Proof. Let $\{H_i\}$ be a countable open covering of X. Set $G_i = \bigcup_{j \leqslant i} H_j$. Let F_{jk}, $k \geqslant j$, be a closed set of X_k with $\pi_k^{-1}(F_{jk}) \subset \pi_k^{-1}(G_j^k)$ and with $\bigcup \{\pi_k^{-1}(F_{jk}):\ k \geqslant j,\ j = 1, 2, \ldots\} = X$. It can easily be seen that the existence of these closed sets is assured by the condition (∗). Let V_{jk} be a cozero set of X_k such that $F_{jk} \subset V_{jk} \subset G_j^k$. Since $\{\pi_k^{-1}(V_{jk}):\ k \geqslant j,\ j = 1, 2, \ldots\}$, say $\mathcal{V}$, is a countable cozero covering of X, $\mathcal{V}$ can be refined by a locally finite open covering $\mathcal{W} = \{W_{jk}:\ k \geqslant j,\ j = 1, 2, \ldots\}$ such that $W_{jk} \subset \pi_k^{-1}(V_{jk})$.

$$\{W_{jk} \cap H_i:\ i = 1, 2, \ldots, j,\ k \geqslant j,\ j = 1, 2, \ldots\}$$

is a locally finite open covering of X refining $\{H_i\}$, and the theorem is proved.

In the sequel $\mathfrak{m}$ denotes an infinite power. A space is said to have the property $L(\mathfrak{m})$ or to be $L(\mathfrak{m})$ if every open covering has a subcovering consisting of at most $\mathfrak{m}$ elements. To be $L(\aleph_0)$ is nothing else to be Lindelöf. A topological property τ is said to be *countably projective* if the following condition is satisfied: If every X_i has τ, then X has τ too.

1.3. Theorem. *If X satisfies the condition* (∗), *then the following properties are countably projective.*

(i) *Normality.*

(ii) *Paracompactness.*

(iii) $\mathfrak{m}$-*paracompactness + normality.*

(iv) *Collectionwise normality.*

(v) $L(\mathfrak{m})$ + *normality.*

Proof. To prove all cases, except the case (iv), at the same time, let $\mathfrak{U} = \{U_\alpha:\ \alpha \in A\}$ be an arbitrary open covering of X. For the case (i) or (iii) consider that A consists of a finite number of indices or of $\mathfrak{m}$ indices, respectively. Set $V_i = \bigcup \{U_\alpha^i:\ \alpha \in A\}$. Then $\pi_1^{-1}(V_1) \subset \pi_2^{-1}(V_2) \subset \ldots$, and $\bigcup \pi_i^{-1}(V_i) = X$. By the condition (∗) there exists, for each i, a closed set F_i of X_i such that $\bigcup \pi_i^{-1}(F_i) = X$. Let C_i be a cozero set of X_i with $F_i \subset C_i \subset V_i$. For the first three cases $\mathfrak{U}|\pi_i^{-1}(C_i)$ is normal, since it is refined by $\{\pi_i^{-1}(U_\alpha^i):\ \alpha \in A\}|\pi_i^{-1}(C_i)$. Since $\{\pi_i^{-1}(C_i):\ i = 1, 2, \ldots\}$ is a countable cozero covering, it is normal. Therefore by Morita [6], Theorem 1.1, $\mathfrak{U}$ is normal.

For the case (v) $\mathfrak{U}|\pi_i^{-1}(C_i)$ has a subcovering of power $\leqslant \mathfrak{m}$, by the same reason as in the above. Hence $\mathfrak{U}$ has a subcovering of power $\leqslant \mathfrak{m}$.

To prove the case (iv) let $\{K_\xi:\ \xi \in \Xi\}$ be a discrete collection of closed sets in X. Let U_i be the maximal open set of X_i such that for each $x \in \pi_i^{-1}(U_i)$ there exists an open neighborhood V of $\pi_i(x)$ such that $\pi_i^{-1}(V)$ meets at most one K_ξ. Then $\{\pi_i^{-1}(U_i):\ i = 1, 2, \ldots\}$ is a monotonically increasing open covering of X. By the condition (∗) there exists for each i a closed set F_i of X_i such that $F_i \subset U_i$ and $\{\pi_i^{-1}(F_i):\ i = 1, 2, \ldots\}$ covers X. Choose an open set H_i of X_i with $F_i \subset H_i \subset \overline{H}_i \subset U_i$ and with $(\pi_i^{i+1})^{-1}(\overline{H}_i) \subset H_{i+1}$. Set

$$\mathfrak{L}_1 = \{L_{1\xi} = \overline{\pi_1(K_\xi)} \cap \overline{H}_1:\ \xi \in \Xi\}\,.$$

Then $\mathfrak{L}_1$ is a discrete collection of closed sets of $\overline{H}_1$. Let $\{D_{1\xi}:\ \xi \in \Xi\}$ be a discrete collection of relatively open sets of $\overline{H}_1$ such that $L_{1\xi} \subset D_{1\xi}$ for each ξ. Set

$$\mathfrak{L}_2 = \{L_{2\xi} = (\overline{\pi_2(K_\xi)} \cap \overline{H}_2) \cup (\pi_1^2)^{-1}(\overline{D}_{1\xi}):\ \xi \in \Xi\}\,.$$

Then $\mathfrak{L}_2$ is a discrete collection of closed sets of $\overline{H}_2$. Let $\{D_{2\xi}:\ \xi \in \Xi\}$ be a discrete collection of relatively open sets of $\overline{H}_2$ such that $L_{2\xi} \subset D_{2\xi}$ for each ξ. Continuing these processes successively, we obtain for each i a discrete collection $\{D_{i\xi}:\ \xi \in \Xi\}$ of relatively open sets of $\overline{H}_i$ with $L_{i\xi} \subset D_{i\xi}$, where

$$L_{i\xi} = (\overline{\pi_i(K_\xi)} \cap \overline{H}_i) \cup (\pi_{i-1}^i)^{-1}(\overline{D}_{i-1\xi})\,, \qquad \xi \in \Xi\,.$$

Set

$$D_\xi = \bigcup \{\pi_i^{-1}(D_{i\xi} \cap H_i):\ i = 1, 2, \ldots\}\,, \qquad \mathfrak{D} = \{D_\xi:\ \xi \in \Xi\}\,.$$

Then $\mathfrak{D}$ is a disjoint open collection of X with $K_\xi \subset D_\xi$ for each ξ. Thus X is collectionwise normal and the theorem is proved.

It is to be noted that all of (i)–(v) are not countably projective in general even if π_j^i are open. This can be seen by Michael's hereditarily paracompact and Lindelöf space X in [5] as follows. Let Y be the irrationals. Then $X \times Y$ is not normal. Since Y is completely metrizable,

Y is the inverse limit of countable discrete spaces Y_i. Then $X\times Y$ is the inverse limit of $X\times Y_i$, where each $X\times Y_i$ satisfies hereditary paracompactness and Lindelöf.

1.4. COROLLARY. *Let each X_i be normal and each π_j^i open. Then the following conditions are equivalent.*

(i) *X is countably paracompact.*

(ii) *X is countably paracompact and normal.*

(iii) *X satisfies the condition* (*).

1.5. COROLLARY. *The perfect normality is countably projective. The perfect normality combined with any one of the following properties is countably projective.*

(i) *Paracompactness.*

(ii) *$\mathfrak{m}$-paracompactness.*

(iii) *Collectionwise normality.*

(iv) *$L(\mathfrak{m})$.*

It is to be noted that Cook–Fitzpatrick [2], Theorem 2, proved the first half of this corollary.

1.6. COROLLARY. *Let $\prod_{i=1}^{\infty} P_i$ be countably paracompact. If $\prod_{i\leqslant n} P_i\ (n=1,2,\ldots)$ satisfy any one of the conditions* (i)–(v) *in Theorem* 1.3, *then $\prod_{i=1}^{\infty} P_i$ satisfies it as well.*

1.7. THEOREM. *Let each X_i be a normal space with $\dim X_i\leqslant n$. Let X be countably paracompact and each π_j^i open. Then X is a normal space with $\dim X\leqslant n$.*

Proof. By Corollary 1.4 X is normal. To prove $\dim X\leqslant n$ let $\mathfrak{U}=\{U_a\colon a\in A\}$ be an arbitrary finite open covering of X. Set $V_i=\bigcup\{U_a^i\colon a\in A\}$. Then by Corollary 1.4 there exists for each i a closed set F_i of X_i such that $F_i\subset V_i$ and $\{\pi_i^{-1}(F_i)\colon i=1,2,\ldots\}$ covers X. Let C_i be a cozero set of X_i such that $F_i\subset C_i\subset\overline{C}_i\subset V_i$ and $(\pi_i^{i+1})^{-1}(\overline{C}_i)\subset C_{i+1}$. Consider $\{U_a^2\colon a\in A\}|C_2$. Since $\dim C_2\leqslant n$, there exists for each a an open set W_{2a} of C_2 such that $\operatorname{ord}\{W_{2a}\colon a\in A\}\leqslant n+1$ and $\bigcup\{W_{2a}\colon a\in A\}=C_2$. Consider

$$\{(\pi_2^3)^{-1}(W_{2a})\cup\big(U_a^3-(\pi_1^3)^{-1}(\overline{C}_1)\big)\colon a\in A\}|C_3\,.$$

There exists for each a an open set W'_{3a} of C_3 such that

$$W'_{3a}\subset(\pi_2^3)^{-1}(W_{2a})\cup\big(U_a^3-(\pi_1^3)^{-1}(\overline{C}_1)\big)\,,$$

$$\operatorname{ord}\{W'_{3a}\colon a\in A\}\leqslant n+1\,,$$

$$\bigcup\{W'_{3a}\colon a\in A\}=C_3\,.$$

Set

$$W_{3a}=W'_{3a}\cup\big((\pi_2^3)^{-1}(W_{2a})\cap(\pi_1^3)^{-1}(C_1)\big)\,.$$

Then

$$W_{3a}\subset U_a^3\,,$$

$$\operatorname{ord}\{W_{3a}\colon a\in A\}\leqslant n+1\,,$$

$$\bigcup\{W_{3a}\colon a\in A\}=C_3\,,$$

$$W_{3a}\cap(\pi_1^3)^{-1}(C_1)=(\pi_2^3)^{-1}(W_{2a})\cap(\pi_1^3)^{-1}(C_1)\,.$$

Continuing these processes we obtain for each i an open set W_{ia} of X_i such that

$$W_{ia}\subset U_a^i\,,$$

$$\operatorname{ord}\{W_{ia}\colon a\in A\}\leqslant n+1\,,$$

$$\bigcup\{W_{ia}\colon a\in A\}=C_i\,,$$

$$W_{ia}\cap(\pi_{i-2}^i)^{-1}(C_{i-2})=(\pi_{i-1}^i)^{-1}(W_{i-1a})\cap(\pi_{i-2}^i)^{-1}(C_{i-2})\,.$$

Set

$$W_a=\bigcup\{\pi_i^{-1}(W_{ia})\colon i=2,3,\ldots\}\,.$$

Then W_a is an open set of X with $W_a\subset U_a$ such that $\operatorname{ord}\{W_a\colon a\in A\}\leqslant n+1$ and $\bigcup\{W_a\colon a\in A\}=X$. Thus we obtain $\dim X\leqslant n$ and the theorem is proved.

1.8. THEOREM. *Let each X_i be hereditarily normal or hereditarily paracompact. Let each π_j^i be open. If a subset G' of X is countably paracompact and G' is dense in some open set, say G, then G' is respectively normal or paracompact.*

Proof. Let $\mathfrak{U}=\{U'_a\colon a\in A\}$ be an arbitrary finite or infinite relatively open covering of G' according respectively to the case when each X_i is hereditarily normal or when each X_i is hereditarily paracompact. Let us prove the normality of $\mathfrak{U}$. Let U_a be an open set of X with $U_a\cap G'=U'_a$ and $U_a\subset G$. Set $V_i=\bigcup\{U_a^i\colon a\in A\}$. Then $G'\subset\bigcup\pi_i^{-1}(V_i)\subset G$ and $\pi_i^{-1}(V_i)\subset\pi_{i+1}^{-1}(V_{i+1})$. Applying Ishikawa [3] to the countably paracompact set G' there exists for each i a relatively open set D'_i of G' such that $[D'_i]\subset\pi_i^{-1}(V_i)\cap G'$ and $\bigcup D'_i=G'$, where $[D'_i]$ indicates the relative closure of D'_i. Let D_i be an open set of X such that $D_i\cap G'=D'_i$ and $\overline{D_i}\subset\pi_i^{-1}(V_i)$. Then $G'\subset\bigcup\pi_i^{-1}(D_i^i)\subset\bigcup\pi_i^{-1}(V_i)$ and $D_i^i\subset V_i$. Set $B_i=\overline{D_i^i}\cap(\overline{V_i}-V_i)$. To prove $\pi_i^{-1}(B_i)\cap G'=\emptyset$ assume the contrary. Pick a point $x=\langle x_j\rangle$ from $\pi_i^{-1}(B_i)\cap G'$. Let j be an arbitrary natural number with $i\leqslant j$ and $U(x_j)$ an arbitrary open neighborhood of x_j. Set $U(x_i)=\pi_i^j\big(U(x_j)\big)$. Then $U(x_i)$ is an open neighborhood of x_i since π_i^j is open. Set $V=U(x_i)\cap D_i^i$. Then V is open and non-empty. Set $U=\pi_i^{-1}(V)\cap\pi_j^{-1}\big(U(x_j)\big)$. Then U is open and non-empty in G. Thus

$U \cap G' \neq \emptyset$ and hence $\pi_j^{-1}(U(x_j)) \cap \pi_i^{-1}(D_i^i) \cap G' \neq \emptyset$. Therefore $x \in [\pi_i^{-1}(D_i^i) \cap G'] \subset [D_i \cap G'] = [D_i']$. This implies that $[D_i']$ is not contained in $\pi_i^{-1}(V_i) \cap G'$, which is a contradiction. Let C_i be a cozero set of $X_i - B_i$ with $\overline{D_i^i} - B_i \subset C_i \subset V_i$. Then $\pi_i^{-1}(C_i)$ is a cozero set of $X - \pi_i^{-1}(B_i)$. Set $\pi_i^{-1}(C_i) \cap G' = S_i$. Then S_i is a cozero set of G' such that $\bigcup S_i = G'$. Since $\{\pi_i^{-1}(U_a^i)\colon a \in A\}$ is normal,

$$\{\pi_i^{-1}(U_a^i)\colon a \in A\}|S_i$$

is a normal relatively open covering of S_i. Therefore

$$\bigcup\{\{\pi_i^{-1}(U_a^i)\colon a \in A\}|S_i\colon i = 1, 2, \ldots\}$$

is a normal relatively open covering of G', by Morita [6], Theorem 1.1, which refines $\mathfrak{U}$. The proof is completed.

1.9. THEOREM. *Let each π_j^i be open. If each open set of X is countably paracompact, then the following properties are countably projective.*

(i) *Hereditary normality.*

(ii) *Hereditary paracompactness.*

(iii) *Total normality.*

Proof. Since the cases (i) and (ii) are evident from Theorem 1.8, we prove the last case. Let G be an arbitrary open set of X. Then as we have done in the preceding proof, there exists for each i an open set C_i of X_i such that $\pi_i^{-1}(C_i)$ is cozero in G and $\{\pi_i^{-1}(C_i)\colon i = 1, 2, \ldots\}$ covers G. Let $\{C_{ia}\colon a \in A_i\}$ be a locally finite (in C_i) covering of C_i such that each C_{ia} is cozero in X_i. Let D_i be a cozero set of G such that $\overline{D}_i \cap G \subset \pi_i^{-1}(C_i)$, $\bigcup D_i = G$ and $\{D_i\colon i = 1, 2, \ldots\}$ is locally finite in G. Set $T_{ia} = \pi_i^{-1}(C_{ia}) \cap D_i$. Then T_{ia} is cozero in X by Nagami [8], Lemma 5. $\{T_{ia}\colon a \in A_i, i = 1, 2, \ldots\}$ is locally finite in G and covers G. Thus X is totally normal and the theorem is proved.

1.10. COROLLARY. *Let each open set of $\prod_{i=1}^{\infty} P_i$ be countably paracompact. If for each n, $\prod_{i=1}^{n} P_i$ is hereditarily normal, hereditarily paracompact or totally normal, then $\prod_{i=1}^{\infty} P_i$ is respectively hereditarily normal, hereditarily paracompact or totally normal.*

2. Products. A mapping $f\colon Q \to R$ is said to be *quasi-perfect* if f is closed and every point-inverse under f is countably compact. According to Morita [6], Theorem 6.1, a space Q is said an M-space if it is the inverse image of a metric space under a quasi-perfect mapping. It is to be noted that Arhangel'skiĭ [1], Definition 5, got almost the same idea. The following Theorem 2.2 and 2.5 are to be considered as corollaries of Katětov [4], Corollary 1, as stated in the following:

2.1. LEMMA. *Let the product $P \times Q$ be hereditarily normal. If Q contains a countable non-discrete set, then P is perfectly normal.*

2.2. THEOREM. *Let Q be an M-space and the product $P \times Q$ be hereditarily normal. Then either P is perfectly normal or Q is discrete.*

Proof. Let f be a quasi-perfect mapping of Q onto a metric space R. First consider the case when R is discrete. Then $\{f^{-1}(z)\colon z \in R\}$ is a discrete collection. If each $f^{-1}(z)$ is discrete, then Q becomes discrete. If there exists a non-discrete point-inverse, say $f^{-1}(z_0)$, then $f^{-1}(z_0)$ contains a countably infinite subset Q_0 with $\overline{Q}_0 - Q_0 \neq \emptyset$ by the countable compactness of $f^{-1}(z_0)$. Hence P is perfectly normal by Lemma 2.1.

Let us consider the case when R is not discrete. Then there exists a sequence of points $z_0, z_1, z_2, \ldots$ in R such that $z_i \neq z_j$ whenever $i \neq j$ and z_0 is the limit point of $z_1, z_2, \ldots$ Pick a point y_i from $f^{-1}(z_i)$ for $i = 1, 2, \ldots$ Let Q_1 be the closure of $\{y_1, y_2, \ldots\}$. If $Q_1 \cap f^{-1}(z_0) \neq \emptyset$, then $\{y_1, y_2, \ldots\}$ is not discrete. Hence by Lemma 2.1 P has to be perfectly normal. If $Q_1 \cap f^{-1}(z_0) = \emptyset$, $Q - Q_1$ is an open neighborhood of $f^{-1}(z_0)$. By the closedness of f, $Q - Q_1$ contains $f^{-1}(z_i)$, $i = 1, 2, \ldots$, equifinally, which is a contradiction. Thus the theorem is proved.

2.3. COROLLARY. *Let P be a paracompact M-space. If $P \times P \times P$ is hereditarily normal, then P is metric.*

Proof. When P is not discrete, $P \times P$ is perfectly normal. Hence P is metric by Okuyama [10], Theorem 1.

This is an analogy to Katětov [4], Corollary 2. Let $\mathcal{F}$ be a covering of a space Q and y a point of Q. Let $C(y, \mathcal{F})$ denote the intersection of all elements F of $\mathcal{F}$ with $y \in F$. Let $\{\mathcal{F}_i\}$ be a sequence of locally finite closed coverings of Q satisfying the condition:

If $K_1 \supset K_2 \supset \ldots$ is a sequence of non-empty closed sets of Q such that $K_i \subset C(y, \mathcal{F}_i)$ for some point y and for each i, then $\bigcap K_i \neq \emptyset$.

Such a sequence $\{\mathcal{F}_i\}$ is said a Σ-net of Q. According to Nagami [7], Definition 1.1, a space having a Σ-net is said a Σ-space. This notion is a generalization of M-spaces. Let $C(y) = \bigcap C(y, \mathcal{F}_i)$. It is countably compact.

2.4. LEMMA (Nagami [7], Lemma 1.4). *Let Q be a Σ-space. Then Q has a Σ-net $\{\mathcal{F}_i\}$ which satisfies the following:*

(i) *Every $\mathcal{F}_i$ is finitely multiplicative.*

(ii) $\mathcal{F}_i = \{F(a_1 \ldots a_i)\colon a_1, \ldots, a_i \in \Omega\}$.

(iii) *Every $F(a_1 \ldots a_i)$ is the sum of all $F(a_1 \ldots a_i a_{i+1})$, $a_{i+1} \in \Omega$.*

(iv) *For every $y \in Q$ there exists a sequence $a_1, a_2, \ldots \in \Omega$ such that if $C(y) \subset U$ with U open, then $C(y) \subset F(a_1 \ldots a_i) \subset U$ for some i.*

2.5. Theorem. *Let the product $P \times Q$ be hereditarily normal and Q a Σ-space. Then either P is perfectly normal or Q is a countable sum of closed discrete sets.*

Proof. Let $\{\mathcal{F}_i\}$ be a Σ-net of Q as stated in Lemma 2.4. If there exists a point y in Q such that $C(y) \neq C(y, \mathcal{F}_i)$ for any i, pick a point y_i from $C(y, \mathcal{F}_i) - C(y)$ for each i. Set $Q_0 = \{y_1, y_2, \ldots\}$. Since $\bar{Q}_0 - Q_0 \neq \emptyset$, P is perfectly normal by Lemma 2.1.

Consider the case when, for each point y in Q, there exists an i with $C(y) = C(y, \mathcal{F}_i)$ and $C(y)$ is finite. Let Q_i be the sum of all elements $F(a_1 \ldots a_i) \in \mathcal{F}_i$ such that, for some point y, $C(y) = C(y, \mathcal{F}_i) = F(a_1 \ldots a_i)$. Then $Q = \bigcup Q_i$ and each Q_i is a closed discrete set of points.

Consider finally the case when there exists a point y_0 in Q such that $C(y_0)$ contains infinite points. Since $C(y_0)$ is countably compact, $C(y_0)$ contains a countable set which is not discrete. Hence by Lemma 2.1 P is perfectly normal. The proof is finished.

2.6. Lemma. *Let N_λ, $\lambda \in M$, be copies of the discrete space of positive integers, where the power of the index set M is uncountable. Then the product $\prod N_\lambda$ is not countably paracompact.*

Proof. Set $T = \prod N_\lambda$. Let k be a positive integer and A_k the set of points $x = \langle x_\lambda \rangle$ of T such that $n \neq k$ implies the number of λ with $x_\lambda = n$ is at most 1. Then $\{A_k : k = 1, 2, \ldots\}$ forms a discrete collection of closed sets. Set $G_k = T - \bigcup \{A_i : i \neq k\}$. Then $\bigcup G_k = T$. If we assume that T is countably paracompact, then we can get a locally finite open covering $\{H_k\}$ of T such that $H_k \subset G_k$ for each k. It is clear that $A_k \subset H_k$ for each k. Pick a point $x^1 = \langle x^1_\lambda \rangle$ from A_1. Since $x^1 \in H_1$, there exists a finite set $a_1 = \{\lambda_1, \ldots, \lambda_{n(1)}\} \subset M$ with $U(x^1 : a_1) \subset H_1$, where $U(x^1 : a_1)$ is the set of all points $x = \langle x_\lambda \rangle$ with $x^1_\lambda = x_\lambda$ for each $\lambda \in a_1$. Let $x^2 = \langle x^2_\lambda \rangle$ be the point such that $x^2_\lambda = i$ if $\lambda = \lambda_i$ $(i = 1, \ldots, n(1))$ and $x^2_\lambda = 2$ if $\lambda \in M - a_1$. Since $x^2 \in A_2 \subset H_2$, there exists a finite set $a_2 = \{\lambda_1, \ldots, \lambda_{n(2)}\} \subset M$ such that $n(2) > n(1)$, $U(x^1 : a_1) \cap U(x^2 : a_2) = \emptyset$ and $U(x^2 : a_2) \subset H_2$. Continuing this process successively, we get a sequence $n(1) < n(2) < \ldots$ of positive integers, a sequence $a_i = \{\lambda_1, \ldots, \lambda_{n(i)}\}$, $i = 1, 2, \ldots$, of subsets of M and a sequence $x^1, x^2, \ldots$ of points of T such that

$$x^i_\lambda = j \quad \text{if} \quad \lambda = \lambda_j, \quad \lambda_j \in a_{i-1},$$

$$x^i_\lambda = i \quad \text{if} \quad \lambda \in M - a_{i-1},$$

$$U(x^i : a_i) \subset H_i,$$

$$U(x^i : a_i) \cap U(x^j : a_j) = \emptyset \quad \text{if} \quad i \neq j.$$

Let $p = \langle p_\lambda \rangle$ be the point of T such that $p_\lambda = j$ if $\lambda = \lambda_j$, $j = 1, 2, \ldots$, and $p_\lambda = 1$ if $\lambda \in M - \bigcup a_i$. Let β be a finite set of M such that $U(p : \beta)$ meets at most a finite number of H_i's. We assume here without loss of generality that $\beta = a_k \cup \gamma$ for some k, where $\gamma \subset M - \bigcup a_i$. For each $i > k$ consider the point $p^i = \langle p^i_\lambda \rangle$ such that

$$p^i_\lambda = j, \quad \text{if} \quad \lambda = \lambda_j, \quad \lambda_j \in a_{i-1},$$

$$p^i_\lambda = i, \quad \text{if} \quad \lambda = \lambda_j, \quad \lambda_j \in a_i - a_{i-1},$$

$$p^i_\lambda = 1, \quad \text{if} \quad \lambda \in M - a_i.$$

Then $p^i \in U(p : \beta) \cap U(x^i : a_i) \subset U(p : \beta) \cap H_i$, which implies that $U(p : \beta)$ meets H_i for each $i > k$. This contradiction yields that T is not countably paracompact. The proof is finished.

2.7. Theorem. *Let X_λ, $\lambda \in \Lambda$, be non-empty paracompact Σ-spaces and $X = \prod X_\lambda$. Then the following conditions are equivalent.*

(i) *X is normal.*

(ii) *X is countably paracompact.*

(iii) *X is paracompact.*

(iv) *X is a paracompact Σ-space.*

Proof. Assume that there exists an uncountable subset M of Λ such that X_λ is not compact for any $\lambda \in M$. Then for each $\lambda \in M$, X_λ contains a copy of N, say N_λ, as a closed subset. Hence X cannot be countably paracompact by Lemma 2.6. By the same reason X cannot be normal as was shown by Stone [11], Theorem 3. Therefore if either (i) or (ii) is satisfied, then X is the product of Y and Z, where Y is the countable product of paracompact Σ-spaces and Z is a compact space. Since the countable product of paracompact Σ-spaces is again a paracompact Σ-space by Nagami [7], Theorem 3.13, Y is a paracompact Σ-space. Since a compact space is a Σ-space, X itself is a paracompact Σ-space. The remaining implications are trivial and the proof is finished.

A Σ-space Q is said a σ-space if each $C(y)$ is a single point. If Q is a paracompact Σ-space and $Q \times Q$ is perfectly normal, then Q is a σ-space by Nagami [7], Theorem 3.15. Hence we get at once the following triple product theorem for paracompact Σ-spaces.

2.9. Corollary. *If Q is a paracompact Σ-space and $Q \times Q \times Q$ is hereditarily normal, then Q is a σ-space.*

Let H be the product of uncountably many closed unit intervals. The author does not know whether each countably paracompact open set of H is normal or not.

References

[1] A. Arhangel'skiĭ, *On a class of spaces containing all metric and all locally bicompact spaces*, Dokl. Akad. Nauk SSSR 151 (1963), pp. 751–754; Soviet Math. Dokl. 4 (1963), pp. 1051–1055.

[2] H. Cook and B. Fitzpatrick, Jr., *Inverse limits of perfectly normal spaces*, Proc. Amer. Math. Soc. 19 (1968), pp. 189–192.
[3] F. Ishikawa, *On countably paracompact spaces*, Proc. Japan Acad. 31 (1955), pp. 686–689.
[4] M. Katětov, *Complete normality of Cartesian products*, Fund. Math. 35 (1948), pp. 271–274.
[5] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), pp. 375–376.
[6] K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), pp. 365–382.
[7] K. Nagami, *Σ-spaces*, Fund. Math. 65 (1968), pp. 169–192.
[8] — *Hereditary properties of product spaces*, Tohoku Math. J. 21 (1969), pp. 39–46.
[9] — *A note on the normality of inverse limits and products*, Proceeding of International Symposium on topology held at Herceg Novi in Yugoslavia, 1968, pp. 261–264.
[10] A. Okuyama, *On metrizability of M-spaces*, Proc. Japan Acad. 40 (1964), pp. 176–179.
[11] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), pp. 977–982.

Reçu par la Rédaction le 28. 8. 1970

On some problems of Borsuk

by

E. Barton (Urbana, Ill.)

1. Introduction. In [1] Borsuk introduced the concept of the index of r-proximity of two topological spaces. The purpose of this article is to answer three problems concerning this index posed by Borsuk in [2], p. 208. Theorem 8 provides the answers to all three problems.

2. Main result.

DEFINITION 1. A natural number n is said to be the *index of r-proximity* of two spaces X and Y provided there is a system of $n+1$, but not less, spaces $X_0, X_1, \ldots, X_n$ with $X_0 = X$, $X_n = Y$ and such that X_i and X_{i+1} are r-neighbors for $i = 0, 1, \ldots, n-1$.

A space X is a *r-image* of a space Z, denoted $X \leqslant_r Z$, if there are maps $r\colon Z \to X$ and $i\colon X \to Z$ such that $ri = \mathrm{id}_X$, the identity on X. $X \not\leqslant_r Z$ means not $X \leqslant_r Z$; $X <_r Z$ means that both $X \leqslant_r Z$ and $Z \not\leqslant_r X$. X is an r-neighbor of Z if there is no space that is strictly r-between X and Z; i.e. there is no space Y with $X <_r Y <_r Z$.

See [2] for definitions.

DEFINITION 2. A Hausdorff space X is a *Peano space* if and only if X is the continuous image of the unit interval, $I = [0,1]$.

LEMMA 3. *If X is a Peano space, then there exists a map $h\colon (0,1] \to X \times I$ such that $\overline{h((0,1])} = h((0,1]) \mathbin{\dot\cup} (X \times 0)$, (disjoint). Also $h|_{[t,1]}$ is an embedding for each $0 < t < 1$.*

Proof. Let $f\colon I \to X$ with $f(I) = X$ be given. Define $g\colon (0,1] \to I \times I$ by $g(t) = \left(\frac{1}{2}\sin\frac{1}{t} + \frac{1}{2}, t\right)$. Let $p\colon I \times I \to I$ be the projection $p(x, y) = x$ and set $\varphi = pg$. Finally, define $h\colon (0,1] \to X \times I$ by $h(t) = \big(f\varphi(t), t\big)$. Then h is continuous and satisfies the conditions given.

$X \times 0$ is contained in $\overline{h((0,1])}$. For each $(x, 0) \in X \times 0$ there is a t in I such that $f(t) = x$ and a sequence of points $\{t_n\}$ in $(0,1]$ such that $\{t_n\}$ converges to zero and $\varphi(t_n) = t$ for each n. Then $h(t_n) = \big(f\varphi(t_n), t_n\big) = (x, t_n)$ and so $\{h(t_n)\}$ converges to $(x, 0)$. That is, $(x, 0) \in \overline{h((0,1])}$.