Some properties of real functions of two variables and some consequences

by

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1. Introduction. It is interesting to note that some of the well-known results for a real function of a single real variable come as a consequence of more general theorems for real functions of two variables. A classical theorem of G. C. Young [6] is that if f is an arbitrary real function of a real variable, then the set of points for which the upper derivate of f on one side is less than the lower derivate of f on the other side is countable. This theorem comes as a special case of a theorem of H. Blumberg [1], which states that if f is a real function defined in the open half-plane on one side of a line L and if θ_1 and θ_2 are two directions, then for any point $p \in L$, except a countable set, the upper limit of f at p in the direction θ_1 is not exceeded by the lower limit of f in the direction θ_2. A second instance of this sort is the theorem of C. J. Neugebauer [5]. Neugebauer proved that if f is a continuous real-valued function of a single real variable, then, except a set of the first category, the two upper derivates are equal and the two lower derivates are equal. A. M. Bruckner and C. Goffman [2] proved that this result is also a special case of a more general theorem for functions of two variables, viz. if f is a continuous real function defined on the open half-plane on one side of a line L and if θ is a direction, then for any point $p \in L$, except a set of the first category, the total cluster set of f at p is the same as the cluster set of f at p in the direction θ. It is also proved in [2] that if f is a continuous real function defined on the open half-plane on one side of a line L and if θ_1 and θ_2 are directions, then for any point $p \in L$, except a set of the first category, the approximate upper limit of f at p in the direction θ_1 is not exceeded by the approximate lower limit of f at p in the direction θ_2. Just as the theorem of Young comes as a special case of the theorem of Blumberg, we can conclude from this last theorem of Bruckner and Goffman that for a continuous function f of a single variable the set of points where the approximate upper derivate of f on one side is less than the approximate lower derivate of f on the other side is of the first category.

Now it is natural to ask whether the above exceptional sets of the first category are also of measure zero. Since it is remarked in [5] that for a continuous function f of a single variable the set of points where two upper derivates or the two lower derivates are not equal is not necessarily of measure zero, we conclude that the exceptional set which appeared in the former theorem of Bruckner and Goffman is not necessarily of measure zero. It is found in this paper that the exceptional set which occurred in the later theorem of Bruckner and Goffman is of measure zero, and thus the analogue of the theorem of Blumberg is completed by showing that if f is continuous in the open half-plane on one side of a line L and if θ_1 and θ_2 are any two directions, then for any point $p \in L$, except a set of measure zero and of the first category, the approximate upper limit of f at p in the direction θ_1 is not exceeded by the approximate lower limit of f at p in the direction θ_2. Finally, some other results for functions of a single variable are established with the help of analogous theorems.

2. Terminology and notation. In this section we explain some of the terms used to prove our results. $|A|$ will denote the Lebesgue measure of the measurable set A. The function f is taken to be defined in the open half-plane above a line L which, in particular, may be taken as the X-axis. The open half-plane above the line L is H. The points on the line L, viz $(x, 0)$, will be denoted simply by x, while any other point in H is denoted by p.

If θ is a direction, then $L_\theta(x)$ is the half-ray in H in the direction θ terminating at x; and $L_\theta(x, h)$ is the open line segment in H in the direction θ of length h and having x as one of its end-points.

Let $S \subset H$ and suppose that $S \cap L_\theta(x)$ is measurable. Then the upper and the lower densities of S at x in the direction θ are denoted by $\bar{d}(S; x; \theta)$ and $\underline{d}(S; x; \theta)$ respectively. That is,

$$\bar{d}(S; x; \theta) = \limsup_{h\to 0} |S \cap L_\theta(x, h)|/h$$

and

$$\underline{d}(S; x; \theta) = \liminf_{h\to 0} |S \cap L_\theta(x, h)|/h .$$

Let f: $H \to R$ and assume that f is measurable along the direction θ. Then the approximate upper limit of f at x in the direction θ is the lower bound of all numbers y ($\pm\infty$ admitted) for which the upper density of the set

$$\{p\colon\ p \in H;\ f(p) > y\}$$

at x in the direction θ is zero, and it is denoted by $\limsup_{p\to x,\theta} \mathrm{ap} f(p)$. That is,

$$\limsup_{p\to x,\theta} \mathrm{ap} f(p) = \inf\{y\colon\ \bar{d}(\{p\colon\ p \in H;\ f(p) > y\};\ x;\ \theta) = 0\} .$$

Similarly,

$$\liminf_{p\to x,\theta} \mathrm{ap} f(p) = \sup\{y\colon\ \bar{d}(\{p\colon\ p \in H;\ f(p) < y\};\ x;\ \theta) = 0\} .$$

The upper and the lower limits of f at x in the direction θ are the upper and the lower limits of the function $f/L_\theta(x)$ at x and are denoted by $\limsup_{p\to x,\theta} f(p)$ and $\liminf_{p\to x,\theta} f(p)$ respectively, where $f/L_\theta(x)$ denotes the restriction of f to $L_\theta(x)$.

3. Main results.

LEMMA 1. *Let f: $H \to R$ be continuous and let θ be a direction. Then the set*

$$\{x\colon\ f(p) \leqslant \gamma \quad \text{for all } p \in L_\theta(x, h)\}$$

is a closed set for fixed values of γ, θ and h.

Proof. Since f is continuous, every limit point of the set is a member of the set and hence the lemma is proved.

LEMMA 2. *Let f: $H \to R$ be continuous and let θ be a direction. Let $m(x, h, \theta, \gamma) = |\{p\colon\ p \in L_\theta(x, h); f(p) \geqslant \gamma\}|$. Then the set*

$$\{x\colon\ m(x, h, \theta, \gamma) < \lambda h\}$$

is an open set for fixed values of h, θ, γ and λ and hence the function $m(x, h, \theta, \gamma)$ is measurable for fixed values of h, θ and γ.

Proof. We shall prove the lemma for $\gamma = 0$; for $\gamma \neq 0$ we have only to consider the function $f(p) - \gamma$. For any arbitrary constant λ we have to show that the set

$$E = \{x\colon\ m(x, h, \theta, 0) < \lambda h\}$$

is open. Since $0 \leqslant m(x, h, \theta, 0) \leqslant h$, if $\lambda > 1$ then E is the set of all points $x \in L$ and if $\lambda \leqslant 0$ then E is the null set. So, we may assume $0 < \lambda \leqslant 1$. Let

$$A = \{c\colon\ |\{p\colon\ p \in L_\theta(x, h); f(p) \geqslant c\}| \geqslant \lambda h\} .$$

Then if $c_2 \in A$ and $c_1 < c_2$, then $c_1 \in A$. Also $\sup A \in A$. Let

$$\varphi(x, h, \theta, \lambda) = \sup A .$$

Then φ is a function of x for fixed values of h, θ and λ. It can easily be verified that

$$E = \{x\colon\ \varphi(x, h, \theta, \lambda) < 0\} .$$

So, the lemma will be proved if we show that the function φ is an upper semicontinuous function of x for fixed values of h, θ and λ.

Let x_0 be any arbitrary value of x and let $\varphi(x_0, h, \theta, \lambda) = c_0$. Then for any arbitrary $\varepsilon > 0$

$$|\{p\colon\ p \in L_\theta(x_0, h);\ f(p) \geqslant c_0 + \varepsilon\}| < \lambda h .$$

Let $|\{p\colon p \in L_\theta(x_0, h);\ f(p) \geqslant c_0+\varepsilon\}| = \lambda h - \sigma$. Then

$$|\{p\colon p \in L_\theta(x_0, h);\ f(p) \geqslant c_0+\varepsilon;\ |p-x_0| > \sigma/2\}| \leqslant \lambda h - \sigma .$$

So,

$$|\{p\colon p \in L_\theta(x_0, h);\ f(p) < c_0+\varepsilon;\ |p-x_0| > \sigma/2\}| \geqslant h - \lambda h + \sigma/2 .$$

Since f is continuous in H, it is uniformly continuous in any bounded closed region in H and hence there is a $\delta > 0$ such that $|x-x_0| < \delta$ implies

$$|\{p\colon p \in L_\theta(x, h);\ f(p) < c_0+2\varepsilon;\ |p-x| > \sigma/2\}| \geqslant h - \lambda h + \sigma/2 ,$$

i.e.,

$$|\{p\colon p \in L_\theta(x, h);\ f(p) \geqslant c_0+2\varepsilon;\ |p-x| > \sigma/2\}| \leqslant \lambda h - \sigma$$

and hence

$$|\{p\colon p \in L_\theta(x, h);\ f(p) \geqslant c_0+2\varepsilon\}| \leqslant \lambda h - \sigma/2 < \lambda h.$$

So, if $|x-x_0| < \delta$ then $\varphi(x, h, \theta, \lambda) < c_0+2\varepsilon$, showing that

$$\limsup_{x\to x_0} \varphi(x, h, \theta, \lambda) \leqslant c_0 + 2\varepsilon .$$

Since ε is arbitrary, $\limsup_{x\to x_0}\varphi(x, h, \theta, \lambda) \leqslant c_0$. This shows that φ is upper semicontinuous at x_0. Since x_0 is arbitrary, φ is an upper semicontinuous function of x for fixed h, θ and λ. This completes the proof.

LEMMA 3. *Let $f\colon H\to R$ be continuous and let θ be a direction. Let $M(x, \theta, \gamma) = \limsup_{h\to 0} m(x, h, \theta, \gamma)/h$, where $m(x, h, \theta, \gamma) = |\{p\colon p \in L_\theta(x, h);$ $f(p) \geqslant \gamma\}|$. Then $M(x, \theta, \gamma)$ is a measurable function of x for fixed values of θ and γ.*

Proof. Let $A(x, h, \theta, \gamma) = \sup\{m(x, l, \theta, \gamma)/l;\ \ 0 < l < h\}$. Since $m(x, l, \theta, \gamma)/l$ is a continuous function of l in $(0, h)$, the upper bound is the same if only rational values of l are considered. Then $A(x, h, \theta, \gamma)$, being the upper bound of an enumerable set of measurable functions, is measurable.

Now as h decreases, $A(x, h, \theta, \gamma)$ decreases and hence $\lim_{h\to 0} A(x, h, \theta, \gamma)$ is a measurable function of x for fixed θ and γ. Since $M(x, \theta, \gamma) = \lim_{h\to 0} A(x, h, \theta, \gamma)$, the result follows.

LEMMA 4. *Let $f\colon H\to R$ be continuous and let θ be a direction. Then $\limsup_{p\to x,\theta} \operatorname{ap} f(p)$ is a measurable function of x.*

Proof. Let γ be any real number. We shall show that the set

$$E = \{x\colon \limsup_{p\to x,\theta} \operatorname{ap} f(p) > \gamma\}$$

is measurable. Consider a decreasing sequence $\{\gamma_n\}$ such that $\gamma_n \to \gamma$ as $n\to\infty$. Then by Lemma 3, the function $M(x, \theta, \gamma_n)$ is measurable for each n and hence the set $\{x\colon M(x, \theta, \gamma_n) > 0\}$ is measurable for each n. Now it is easy to verify that

$$E = \bigcup_{n=1}^{\infty} \{x\colon M(x, \theta, \gamma_n) > 0\} ,$$

which shows that the set E is measurable. This completes the proof.

LEMMA 5. *Let $f\colon H\to R$ be continuous and let θ be a direction. Let $m(x, h, \theta, \gamma) = |\{p\colon p \in L_\theta(x, h);\ f(p) \geqslant \gamma\}|$. Then the sets*

$$E(\theta, \gamma, \lambda, n) = \{x\colon m(x, h, \theta, \gamma)/h \leqslant \lambda \text{ for all } h,\ 0 < h < 1/n\}$$

and

$$F(\theta, \gamma, \mu, n) = \{x\colon m(x, h, \theta, \gamma)/h \geqslant \mu \text{ for all } h,\ 0 < h < 1/n\}$$

are measurable for fixed values of θ, γ, λ, μ and n, where n is any positive integer.

Proof. For a fixed value of h, $0 < h < 1/n$, we write

$$E_h(\theta, \gamma, \lambda) = \{x\colon m(x, h, \theta, \gamma)/h \leqslant \lambda\} .$$

Then

$$E(\theta, \gamma, \lambda, n) = \bigcap_{0<h<1/n} E_h(\theta, \gamma, \lambda) .$$

Now since $m(x, h, \theta, \gamma)/h$ is a continuous function of h in $(0, 1/n)$, the intersection $\bigcap_{0<h<1/n} E_h(\theta, \gamma, \lambda)$ is the same if only rational values of h in $(0, 1/n)$ are considered. That is,

$$E(\theta, \gamma, \lambda, n) = \bigcap_{0<h_r<1/n} E_{h_r}(\theta, \gamma, \lambda) ,$$

where h_r are the rational values of h in $(0, 1/n)$.

By Lemma 2 the set $E_{h_r}(\theta, \gamma, \lambda)$ is measurable for each h_r and hence the set $E(\theta, \gamma, \lambda, n)$ is measurable. Similarly, it can be shown that the set $F(\theta, \gamma, \mu, n)$ is measurable.

THEOREM 1. *If $f\colon H\to R$ is continuous and θ_1, θ_2 are directions, then the set*

$$E = \{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) > \limsup_{p\to x,\theta_2} f(p)\}$$

is of measure zero and of the first category.

Proof. For a fixed rational number γ and a fixed positive integer n let $E_{\gamma n}$ denote the set of all points x such that

(1) $\qquad f(p) \leqslant \gamma$ for all $p \in L_{\theta_2}(x, 1/n)$

and

(2) $$\limsup_{p\to x,\theta_1} \operatorname{ap} f(p) > \gamma .$$

Clearly,

(3) $$E \subset \bigcup E_{\gamma n}$$

where the union is taken over the set of all rational numbers and the set of all positive integers. Since

(4) $$E_{\gamma n} = \{x\colon f(p) \leqslant \gamma \text{ for all } p \in L_{\theta_2}(x, 1/n)\} \cap \{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) > \gamma\},$$

the set $E_{\gamma n}$ is measurable for each γ and n by Lemma 1 and Lemma 4. We shall show that $E_{\gamma n}$ is of measure zero for each γ and n. If possible, suppose $|E_{\gamma n}| \neq 0$ for some γ and n. Let $x_0 \in E_{\gamma n}$ be a point of density of $E_{\gamma n}$. From (2) we conclude that x_0 is a point of positive upper density of the set

(5) $$\{p\colon p \in L_{\theta_1}(x_0);\ f(p) > \gamma\}.$$

Let

(6) $$\bar{d}(\{p\colon p \in L_{\theta_1}(x_0);\ f(p) > \gamma\};\ x_0;\ \theta_1) = \sigma > 0.$$

Suppose $\theta_1 > \theta_2$. For each $x \in E_{\gamma n}$, $x < x_0$, let $L_{\theta_2}(x)$ intersect $L_{\theta_1}(x_0)$ at the point $q(x)$. Since x_0 is a point of density of the set $E_{\gamma n}$, x_0 is also a (one-sided) point of density of the set

(7) $$\{q(x)\colon x \in E_{\gamma n};\ x < x_0\}.$$

Hence and from (6) we conclude that there is an interval $J \subset L_{\theta_1}(x_0)$ with x_0 as one of its end-points such that

(8) $$|J| < \frac{1}{n}\cdot\frac{\sin\theta_2}{\sin\theta_1}$$

(9) $$|J \cap \{p\colon p \in L_{\theta_1}(x_0);\ f(p) > \gamma\}| > |J|\cdot\tfrac{3}{4}\sigma$$

and

(10) $$|J \cap \{q(x)\colon x \in E_{\gamma n};\ x < x_0\}| > |J|\cdot\left(1 - \frac{\sigma}{2}\right).$$

From (9) and (10) we conclude that the sets (5) and (7) have common points in J. Let $p_0 \in J$ be one of such common point. Then

(11) $$f(p_0) > \gamma.$$

Also since $p_0 \in \{q(x);\ x \in E_{\gamma n};\ x < x_0\}$, we have $x' \in E_{\gamma n}$, $x' < x_0$, and $p_0 = q(x')$. Hence by (8) $p_0 \in L_{\theta_2}(x', 1/n)$. So, from (1)

(12) $$f(p_0) \leqslant \gamma.$$

Since (11) and (12) are contradictory, we conclude that $|E_{\gamma n}| = 0$ for each γ and n. The relation (3) shows that the set E is of measure zero.

The case $\theta_1 < \theta_2$ can be similarly treated by considering points x of $E_{\gamma n}$ such that $x > x_0$ and the interval $J \subset L_{\theta_1}(x_0)$ such that $|J| < \frac{1}{n}\cdot\frac{\sin\theta_2}{\sin\theta_1}$. Thus in any case $|E| = 0$.

To prove that the set E is of the first category it suffices to show that the sets $E_{\gamma n}$ are nowhere dense. Suppose, on the contrary, that there is an interval $I = [a, b]$ in which one of the sets $E_{\gamma n}$, say $E_{\gamma_0 n_0}$, is dense. So, from (4) the set

$$\{x\colon f(p) \leqslant \gamma_0 \quad \text{for all } p \in L_{\theta_2}(x, 1/n_0)\}$$

is dense in I. Since by Lemma 1 the above set is closed, we conclude that for every $x \in I$

$$f(p) \leqslant \gamma_0 \quad \text{for all } p \in L_{\theta_2}(x, 1/n_0).$$

Let $J = [c, d]$ be any interval such that $a < c < d < b$. Then

$$\limsup_{p\to x,\theta_1} f(p) \leqslant \gamma_0 \quad \text{for all } x \in J.$$

Thus

(13) $$J \cap \{x\colon \limsup_{p\to x,\theta_1} f(p) > \gamma_0\} = 0.$$

Now by (4) the set $\{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) > \gamma_0\}$ is dense in I, and hence there is a ξ such that

(14) $$\xi \in J \cap \{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) > \gamma_0\}.$$

Since (13) and (14) are contradictory, we conclude that the sets $E_{\gamma n}$ are nowhere dense and this completes the proof.

THEOREM 2. *If $f\colon H \to R$ is continuous and θ_1, θ_2 are directions, then the set*

$$E = \{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) < \liminf_{p\to x,\theta_2} \operatorname{ap} f(p)\}.$$

is of measure zero.

Proof. For a fixed pair of rational numbers γ and s, $\gamma < s$, and a fixed positive integer n, let $E_{\gamma s n}$ denote the set of all points x such that for all h, $0 < h < 1/n$, the following relations hold:

(1) $$|\{p\colon p \in L_{\theta_1}(x, h);\ f(p) \geqslant \gamma\}| \leqslant h/3$$

and

(2) $$|\{p\colon p \in L_{\theta_2}(x, h);\ f(p) \geqslant s\}| \geqslant 2h/3.$$

Then it is easy to verify that

(3) $$E \subset \bigcup E_{\gamma s n}$$

where the union is taken over the set of all pairs of rational numbers γ and s, $\gamma < s$ and over the set of all positive integers n. Then $E_{\gamma sn}$ is measurable for each γ, s and n. Indeed, letting

$$E(\theta_1, \gamma, 1/3, n) = \{x\colon m(x, h, \theta_1, \gamma)/h \leqslant 1/3 \text{ for all } h,\ 0 < h < 1/n\}$$

and

$$F(\theta_2, s, 2/3, n) = \{x\colon m(x, h, \theta_2, s)/h \geqslant 2/3 \text{ for all } h,\ 0 < h < 1/n\}$$

where

$$m(x, h, \theta, t) = |\{p\colon p \in L_\theta(x, h);\ f(p) \geqslant t\}|,$$

we see that

$$E_{\gamma sn} = E(\theta_1, \gamma, 1/3, n) \cap F(\theta_2, s, 2/3, n),$$

and since $E(\theta_1, \gamma, 1/3, n)$ and $F(\theta_2, s, 2/3, n)$ are measurable by Lemma 5, the set $E_{\gamma sn}$ is measurable. We shall prove that $E_{\gamma sn}$ is of measure zero for each γ, s and n. If possible, let $|E_{\gamma sn}| \neq 0$. Let $x_0 \in E_{\gamma sn}$ be a point of density of $E_{\gamma sn}$. So, there is an interval I containing x_0 in its interior such that

(4) $$|E_{\gamma sn} \cap I| > \tfrac{15}{16}|I|.$$

Let T be the triangle in H whose base is I and whose other two sides, designated as L_1 and L_2, are in the directions θ_1 and θ_2 respectively. We consider the interval I to be so small in length such that both of L_1 and L_2 are of length less than $1/n$. For each $x \in I$, let $h_1(x)$ denote the length of the line segment in the direction θ_1 joining x with L_2 and let $h_2(x)$ denote the length of the line segment in the direction θ_2 joining x with L_1.

Now

$$\{p\colon p \in T;\ f(p) > \gamma\} \subset \{p\colon p \in L_{\theta_1}(x, h_1(x));\ f(p) > \gamma;\ x \in I \cap E_{\gamma sn}\} \cup$$
$$\cup \{p\colon p \in L_{\theta_1}(x, h_1(x));\ x \in I - E_{\gamma sn}\}.$$

Hence

(5) $$|\{p\colon p \in T;\ f(p) > \gamma\}| \leqslant \sin\theta_1 \int_{I \cap E_{\gamma sn}} \big|\{p\colon p \in L_{\theta_1}(x, h_1(x));\ f(p) > \gamma\}\big|\, dx$$
$$+ \sin\theta_1 \int_{I - E_{\gamma sn}} \big|\{p\colon p \in L_{\theta_1}(x, h_1(x))\}\big|\, dx$$
$$\leqslant \tfrac{1}{3}\sin\theta_1 \int_{I \cap E_{\gamma sn}} h_1(x)\, dx + \sin\theta_1 \int_{I - E_{\gamma sn}} h_1(x)\, dx \quad \text{from (1)}$$
$$= \sin\theta_1 \int_I h_1(x)\, dx - \tfrac{2}{3}\sin\theta_1 \int_{I \cap E_{\gamma sn}} h_1(x)\, dx$$
$$= |T| - \tfrac{2}{3}\sin\theta_1 \int_{I \cap E_{\gamma sn}} h_1(x)\, dx.$$

Also since

$$\{p\colon p \in T;\ f(p) \geqslant s\} \supset \{p\colon p \in L_{\theta_2}(x, h_2(x));\ f(p) \geqslant s;\ x \in I \cap E_{\gamma sn}\},$$

we have

(6) $$|\{p\colon p \in T;\ f(p) \geqslant s\}| \geqslant \sin\theta_2 \int_{I \cap E_{\gamma sn}} \big|\{p\colon p \in L_{\theta_2}(x, h_2(x));\ f(p) \geqslant s\}\big|\, dx$$
$$\geqslant \tfrac{2}{3}\sin\theta_2 \int_{I \cap E_{\gamma sn}} h_2(x)\, dx \quad \text{from (2).}$$

From (5) and (6),

$$|T| \geqslant \tfrac{2}{3} \int_{I \cap E_{\gamma sn}} [\sin\theta_1 h_1(x) + \sin\theta_2 h_2(x)]\, dx$$
$$\geqslant \tfrac{2}{3} \int_I [\sin\theta_1 h_1(x) + \sin\theta_2 h_2(x)]\, dx - \tfrac{2}{3} \int_{I - E_{\gamma sn}} [\sin\theta_1 h_1(x) + \sin\theta_2 h_2(x)]\, dx$$
$$\geqslant \tfrac{2}{3} \cdot 2|T| - \tfrac{2}{3} \int_{I - E_{\gamma sn}} [\sin\theta_1 h_1(x) + \sin\theta_2 h_2(x)]\, dx,$$

i.e.,

$$|T| \leqslant 2 \int_{I - E_{\gamma sn}} [\sin\theta_1 h_1(x) + \sin\theta_2 h_2(x)]\, dx$$
$$\leqslant 2[l_1 \sin\theta_1 + l_2 \sin\theta_2] \cdot |I - E_{\gamma sn}|,$$

where l_1 and l_2 are the lengths of L_1 and L_2 respectively,

$$= 2 \cdot 2k \cdot |I - E_{\gamma sn}|,$$

where k is the altitude of the triangle T,

$$< 4k \cdot \tfrac{1}{16}|I| \quad \text{from (4)}$$
$$= \tfrac{1}{4}k \cdot |I|$$
$$= \tfrac{1}{2}|T|,$$

which is a contradiction. Hence we conclude that $|E_{\gamma sn}| = 0$. So, by (3), $|E| = 0$. This completes the proof.

Remark 1. In [2] it is proved that if $f\colon H \to R$ is continuous and if θ is a direction, then, except a set of values of x of the first category, the total cluster set of f at x is the same as the cluster set of f at x in the direction θ. From this one can deduce that if $f\colon H \to R$ is continuous and if θ_1 and θ_2 are any two directions, then the set

$$\{x\colon \limsup_{p \to x, \theta_1} f(p) > \limsup_{p \to x, \theta_2} f(p)\}$$

is of the first category. Since

$$\limsup_{p\to x,\theta_1} \operatorname{ap} f(p) \leqslant \limsup_{p\to x,\theta_1} f(p) \quad \text{for all } x,$$

we conclude that the set

$$\{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) > \limsup_{p\to x,\theta_2} f(p)\}$$

is of the first category. Hence the last part of Theorem 1 also follows from [2].

Remark 2. It is also proved in [2] that if $f\colon H\to R$ is continuous and if θ_1 and θ_2 are any two directions, then the set

$$\{x\colon \limsup_{p\to x,\theta_1} \operatorname{ap} f(p) < \liminf_{p\to x,\theta_2} \operatorname{ap} f(p)\}$$

is of the first category. Thus Theorem 2 together with this result completes the analogue of the theorem of Blumberg [1] by showing that for a continuous function $f\colon H\to R$ if θ_1 and θ_2 are any two directions, then for any point $x\in L$, except a set of measure zero and of the first category, the approximate upper limit of f at x in the direction θ_1 is not exceeded by the approximate lower limit of f at x in the direction θ_2.

4. Some consequences. In this section we shall study some consequences of the results established in the last section. Let L denote the line $y = x$ in the plane and let H denote the open half-plane above the line L. Then to every pair of points x, y, $x < y$, on the real line R there corresponds a unique point (x, y) in H and to every point (x, y) in H there correspond two definite real numbers x and y such that $x < y$. Let f be and arbitrary real function of a real variable. Let us define a function F in the open half-plane H such that

$$F(x, y) = \frac{f(y)-f(x)}{y-x} \quad \text{for} \quad (x, y)\in H.$$

Then the upper right derivate and the approximate upper right derivate of f at x are the same as the upper limit and the approximate upper limit of F at (x, x) respectively in the vertical direction. The upper left derivate and the approximate upper left derivate of f at x are the same as the upper limit and the approximate upper limit of F at (x, x) respectively in the horizontal direction. The symmetric upper derivate and the approximate symmetric upper derivate of f at x are the same as the upper limit and the approximate upper limit of F at (x, x) respectively in the normal direction. The corresponding lower derivate and the approximate lower derivate of f at x are obtained from the lower limit and the approximate lower limit of F at (x, x) analogously. Further, we have to note that if $f\colon R\to R$ is continuous, then $F\colon H\to R$ is also continuous.

As usual, D^+f, D^-f, $\overline{f^{(1)}}$, AD^+f, AD^-f and $\overline{f_{\text{ap}}^{(1)}}$ will denote the right hand upper, left hand upper, symmetric upper [3], right hand approximate upper, left hand approximate upper and the symmetric approximate upper [4] derivate of f respectively; D_+f, D_-f, $\underline{f}^{(1)}$, AD_+f, AD_-f and $\underline{f}_{\text{ap}}^{(1)}$ will denote the corresponding lower derivates. Let $\mathfrak{U}$, $\mathfrak{U}_{\text{ap}}$, $\mathfrak{L}$ and $\mathfrak{L}_{\text{ap}}$ be the sets of functions defined by

$$\mathfrak{U} = \{D^+f, D^-f, \bar{f}^{(1)}\}; \quad \mathfrak{U}_{\text{ap}} = \{AD^+f, AD^-f, \bar{f}_{\text{ap}}^{(1)}\};$$

$$\mathfrak{L} = \{D_+f, D_-f, \underline{f}^{(1)}\}; \quad \mathfrak{L}_{\text{ap}} = \{AD_+f, AD_-f, \underline{f}_{\text{ap}}^{(1)}\}.$$

From Theorems 1 and 2 and Remark 2 we have the following theorems:

THEOREM 4. *If $f\colon R\to R$ is continuous, then for arbitrary $\lambda\in\mathfrak{U}$, $\mu\in\mathfrak{U}_{\text{ap}}$, the set*

$$\{x\colon \lambda(x) < \mu(x)\}$$

is of measure zero and of the first category.

Remark. An analogous result is true for functions of classes $\mathfrak{L}$ and $\mathfrak{L}_{\text{ap}}$.

THEOREM 5. *If $f\colon R\to R$ is continuous, then for arbitrary $\lambda\in\mathfrak{U}_{\text{ap}}$, $\mu\in\mathfrak{L}_{\text{ap}}$, the set*

$$\{x\colon \lambda(x) < \mu(x)\}$$

is of measure zero and of the first category.

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