Algebraic equivalence of ordinal numbers

by

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In this paper we shall investigate various problems of the arithmetic of ordinal numbers by considering certain substructures of the class On of all ordinals, these structures being defined by closure under some or all of the ordinal operations. In particular, we prove a theorem which solves all ordinal Diophantine equations and inequalities in just one (infinite) variable.

If $S \subseteq On$ we say

S is *closed under exp* iff $\alpha, \beta \in S$ implies $\alpha^\beta \in S$.

S is *closed under subtraction* iff $\alpha, \beta \in S$, $\alpha < \beta$ implies $(-\alpha+\beta) \in S$.

S is *closed under division* iff $\alpha, \alpha\gamma+\delta \in S$, $\delta < \alpha$ implies $\gamma \in S$.

S is *logarithmically closed* iff $\alpha, \alpha^\gamma\delta+\pi \in S$, $\delta < \alpha$, $\pi < \alpha^\gamma$ implies $\gamma \in S$.

Now if $S \subseteq On$, and ω $(=\{0, 1, 2, ...\}) \subseteq S$, we call S a

Semigroup if S is closed under $+$,

Group if S is closed under $+$, $-$,

Ring if S is closed under $+$, $\cdot$,

Field if S is closed under $+$, $\cdot$, $-$, $\div$,

Arithmetic if S is closed under $+$, $\cdot$, exp,

Divarithmetic if S is closed under $+$, $\cdot$, exp, $-$, $\div$,

Logarithmetic if S is closed under $+$, $\cdot$, exp, $-$, $\div$, log.

If $T \subseteq On$ we use $S[T]$, $(G[T]$, etc.,) for the least Semigroup, (Group, etc.,) containing T; if $T = \{\alpha\}$, $\alpha \in On$, we write $S\{\alpha\}$ for $S[T]$. A map between subsets of On is called an S-isomorphism (G-isomorphism, etc.,) if it preserves order and the operations defining a Semigroup (Group, etc.). For $\alpha, \beta \in On$, and $K \in \{S, G, R, F, A, D, L\}$, if there is a K-isomorphism between $K\{\alpha\}$ and $K\{\beta\}$ which takes α to β, we say α and β are *K-equivalent* and write $\alpha \underset{K}{=} \beta$.

We shall determine the K-equivalence classes of ordinals for each K; for instance for A-equivalence (the most interesting case) the infinite classes fall into seven forms, each depending on one or two finite parameters. Using this, we can solve any ordinal Diophantine equation in one infinite variable — the set of solutions is a union of equivalence classes, and we determine which classes are in the union by substituting the smallest or most convenient member of each class into the equation.

Conventions and notation. For any two ordinals α and β, there is a unique β-expansion of α (if $\beta > 1$), $\alpha = \beta^{\gamma_0}\delta_0 + \beta^{\gamma_1}\delta_1 + \ldots + \beta^{\gamma_n}\delta_n$, where $\gamma_0 > \gamma_1 > \ldots > \gamma_n \geqslant 0$ and $0 < \delta_i < \beta$ for each δ_i. In particular, the Cantor normal form of α is the ω-expansion of α, which we shall write as $\alpha = \omega^{\alpha_0}a_0 + \ldots + \omega^{\alpha_A}a_A + a$, where $a_A > 0$ and $a \geqslant 0$. Thus a (b, etc.) is understood to be the finite part of α (β, etc.); we write $\overline{\alpha}$ for the infinite part of α. Also, we write α^* for $\omega^{-1+\alpha_0}a_0 + \ldots + \omega^{-1+\alpha_A}a_A$; so $\overline{\alpha} = \omega \cdot \alpha^*$, and $\alpha = \overline{\alpha} + a$.

A γ-number, (δ-number, ε-number) is any infinite ordinal π such that for all $\alpha < \pi$, $\alpha \geqslant 0$ ($\alpha \geqslant 1$, $\alpha \geqslant 2$) we have $\alpha + \pi = \pi$ ($\alpha \cdot \pi = \pi$, $\alpha^\pi = \pi$). It is known that the ε-numbers are the solutions of $2^\varepsilon = \varepsilon$. We enumerate the ε-numbers, ordered by magnitude, as $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$; thus $\varepsilon_0 = \omega$.

If $S \subseteq On$, we say $\pi \in S$ is a γ-number of S or S-γ-number (δ-number of S or S-δ-number, ε-number of S or S-ε-number) if $\alpha + \pi = \pi$ for all $\alpha \in S$, $\alpha < \pi$ ($\alpha \cdot \pi = \pi$ for all $\alpha \in S$, $0 < \alpha < \pi$; $\alpha^\pi = \pi$ for all $\alpha \in S$, $1 < \alpha < \pi$). Since for any ordinals π, α, $\pi > \alpha > 1$, $2^\pi = \pi \Rightarrow \alpha^\pi = \pi$, we have that if $2 \in S \subseteq On$, then the ε-numbers of S are absolute in that the S-ε-numbers are just the ε-numbers *in* S.

We define $\varphi(\alpha, \beta, \gamma, \ldots)$ to be an S-function (G-function, etc.) iff it may be expressed in terms of only those operations with which a Semigroup (Group, etc.,) was defined.

S-equivalence and G-equivalence. If α is infinite, the set of all ordinals $\alpha m + n$ ($m, n \in \omega$) is a Semigroup, and is plainly the semigroup $S\{\alpha\}$. Since

(1) $$(\alpha m + n) + (\alpha M + N) = \alpha(m + M) + N \quad (\text{if } M \neq 0)$$

there is an S-isomorphism taking $\alpha m + n$ to $\beta m + n$ (β also infinite), and so $\alpha \underset{S}{=} \beta$ for all infinite α and β. Since $S\{\alpha\}$ is already a group, we have $S\{\alpha\} = G\{\alpha\}$ and so also $\alpha \underset{G}{=} \beta$ for all infinite α and β. Each finite ordinal defines a separate S-equivalence (and G-equivalence) class.

R-equivalence and F-equivalence. If α is infinite, the set of *polynomials* in α, $\alpha^n A_n + \alpha^{n-1}A_{n-1} + \ldots + A_0 = (A_n, A_{n-1}, \ldots, A_0)(\alpha)$, say, is already a ring, and is therefore $R\{\alpha\}$. If $\alpha = \overline{\alpha}$ is a limit number, then

(2) $$(A_n, \ldots, A_0)(\overline{\alpha}) \cdot (B_m, \ldots, B_0)(\overline{\alpha}) = (B_m, \ldots, B_1, A_n B_0, A_{n-1}, \ldots, A_0)(\overline{\alpha})$$

except that if $B_0 = 0$ the terms $A_{n-1}, \ldots, A_0$ are replaced by zeros.

It follows that for infinite α and β the map taking $(A_n, \ldots, A_0)(\overline{\alpha})$ to $(A_n, \ldots, A_0)(\overline{\beta})$ is an R-isomorphism, so that if α and β have the same finite part, they are R-equivalent. Conversely, the equation $2\alpha = \alpha + a$ (a the finite part of α) shows that the R-equivalence class of α determines its finite part, and so infinite ordinals are R-equivalent iff their finite parts are equal.

Now if $\alpha \neq 0$, it follows from $\alpha^2 = (\alpha+1)(\overline{\alpha} + a - 1) + a$ that any field containing α contains $\alpha - 1$, by division. In particular, $F\{\alpha\} = F\{\overline{\alpha}\}$. But (2) shows that $R\{\overline{\alpha}\}$ is already a field, and so infinite ordinals are F-equivalent iff they are R-equivalent. It is again true that each finite α defines a separate equivalence class.

While on the subject of fields we give a further result of interest, that any field F is generated (as a ring) by the set of F-δ-numbers, i.e., any $\alpha \in F$ can be written as $\delta_{11}\delta_{12} \ldots \delta_{1n} + \delta_{21} \ldots \delta_{2m} + \ldots + \delta_{p1} \ldots \delta_{pq}$, where the δ_{ij} are all F-δ-numbers. For, let α be the smallest element of F not so expressible. Then α is not itself an F-δ-number, so $\gamma\alpha > \alpha$ for some $\gamma \in F$, $\gamma < \alpha$. In $\alpha = \gamma\delta + \pi$, $\pi < \gamma$, we have $\delta < \alpha$, for otherwise $\gamma\alpha = \alpha$, and so γ, δ, and π are all expressible as sums of products of F-δ-numbers. Using the fact that any F-δ-number is a limit number (since fields are closed under predecessors) we may write $\gamma\delta + \pi$ in this form.

There is a converse, that if $S \subseteq On$ is such that every $\alpha \in S$ is an S-δ-number, then $R[S]$ is a field, and S is the set of $R[S]$-δ-numbers.

The corresponding results for groups are valid but rather trivial; any group G is generated (as a semigroup) by the set of G-γ-numbers; if $S \subseteq On$ is such that every $\alpha \in S$ is an S-γ-number then $S[S]$ is a group, and S is the set of $S[S]$-γ-numbers.

***L*-equivalence.** Let $\alpha \in On$, and let $\alpha = 2^\beta + 2^\gamma + \ldots + 2^\pi$ be the binary expansion of α. By replacing each term 2^δ by $2^{\overline{\delta}}2^d$ and collecting terms, we get a form $\alpha = 2^{\overline{\beta}}b' + 2^{\overline{\gamma}}c' + \ldots + 2^{\overline{\pi}}p'$ ($b', c', \ldots, p' \in \omega$) say, in which the indices are limit numbers (or zero). By repeating this process on the indices, we can express α as an arithmetic function of ε-numbers; we define the *Binary ε-number expansion*, or *Bee*, of α as follows:

(i) if α is an ε-number, then α is the *Bee* of α,

(ii) if not, then the *Bee* of α is the formal expression

$$2^{Bee\text{ of }\overline{\beta}} \cdot b' + \ldots + 2^{Bee\text{ of }\overline{\pi}} \cdot p'.$$

This inductive process must always terminate, since by the well-order of On we can never have an infinite decreasing sequence of ordinals. The ε-numbers that appear in the *Bee* of α are called the *Beesic* numbers of α.

Now given the *Bees* of α and β we show how to determine the *Bees* of $\alpha + \beta$, $\alpha \cdot \beta$, α^β using only the ordering among the *Beesics* of α and β.

Notation

$$\alpha = 2^{\alpha(0)}a(0)+2^{\alpha(1)}a(1)+\ldots,$$

$$\alpha(ij\ldots k) = 2^{\alpha(ij\ldots k0)}a(ij\ldots k0)+2^{\alpha(ij\ldots k1)}a(ij\ldots k1)+\ldots$$

where $\alpha(ij\ldots lm)$ is a limit number for all $(ij\ldots lm)$.

We first note that $\alpha<\beta$ iff for some i we have $\alpha(j)=\beta(j)$ and $a(j)=b(j)$ for all $j<i$, and either $\alpha(i)<\beta(i)$ or $\alpha(i)=\beta(i)$ and $a(i)<b(i)$.

The *Bee* of $\alpha+\beta$ is found from

$$\alpha+\beta = 2^{\alpha(0)}a(0)+\ldots+2^{\alpha(i)}a(i)+2^{\beta(0)}b(0)+\ldots$$

(where i is maximal subject to $\alpha(i)\geqslant\beta(0)$ and the corresponding terms are collected if $\alpha(i)=\beta(0)$).

The *Bee* of $\alpha\cdot\beta$ is found by addition from those of $\alpha\bar{\beta}$ and α (since $\alpha\beta=\alpha\bar{\beta}+\alpha b$), using 1) the *Bee* of $\bar{\beta}$ is that of β with term of zero index dropped, and 2) the *Bee* of $\alpha\bar{\beta}$ is $2^{\alpha(0)+\bar{\beta}(0)}b(0)+2^{\alpha(0)+\bar{\beta}(1)}b(1)+\ldots$ Finally, the *Bee* of α^{β} is obtained by multiplication from those of $\alpha^{\bar{\beta}}$ and α (since $\alpha^{\beta}=\alpha^{\bar{\beta}}\alpha^{b}$), the *Bee* of $\alpha^{\bar{\beta}}$ being $2^{\alpha(0)\bar{\beta}}$.

From these statements we may infer the following:

(i) Each logarithmetic is generated by a unique set of ε-numbers. In particular, $L\{\alpha\}=L[\{$*Beesic* numbers of $\alpha\}]$.

(ii) Two logarithmetics are L-isomorphic iff their generating sets of ε-numbers have the same order-type.

(iii) In particular each logarithmetic is L-isomorphic to the logarithmetic generated by $\{\varepsilon_\beta\colon \beta<\alpha\}$ for some α. Since this is generated as an arithmetic by its ε-numbers, we have

(iv) Each arithmetic generated by ε-numbers is already a logarithmetic, and in particular

(v) $\alpha \underset{L}{=} \beta$ iff there is an order preserving 1-1 map between the *Beesic* numbers of α and those of β which takes the *Bee* of α onto that of β.

The following disconnected results also follow from (i)–(v):

(vi) If $f(\alpha_0,\alpha_1,\ldots,\pi_0,\ldots)=g(\alpha_0,\alpha_1,\ldots,\pi_0,\ldots)$ is an ordinal Diophantine equation (or inequality) in which f and g are logarithmetic functions of the π_i (unknowns), and the α_i are parameters, then there is an A-isomorphism φ such that 1) $\varphi(\alpha_i)<\varepsilon_{\omega^2}$ for each α_i, and 2) for any solution of the equation $\pi_0,\pi_1,\ldots,\varphi$ can be extended so as to make $\varphi(\pi_i)<\varepsilon_{\omega^2}$ for each π_i.

The same is true of any finite system of ordinal Diophantine equations and inequalities; we may say that all the number theory of the ordinals has 'happened' before ε_{ω^2}.

(vii) We may add a new infinite limit ordinal ∞, $\infty<\omega$, to On without disturbing the arithmetic properties of On. To see this we need only note that On is L-isomorphic to a class of ordinals not containing ω (by the map defined by $\varepsilon_\alpha\to\varepsilon_{1+\alpha}$).

In particular it is interesting (and useful!) to consider the ∞-expansions of (real) ordinals. The rules for manipulating these are somewhat simpler than those for the ω-expansions; the indices of the ∞-expansion of any ordinal α are real limit ordinals, and the coefficients are finite.

Arithmetic equivalence. Considering the cases of semigroups and groups, rings and fields, it would seem natural to expect that ordinals are A-equivalent iff they are L-equivalent; however this is by no means the case. We shall find the following concept of *Abbreviated Arithmetic* (or A.A.) useful.

If F, G, H are classes of ordinals, and Ω is a particular ordinal, we say that Ω is an *F, G-base for H* if, in the Ω-expansion of any $\alpha\in H$ we have each index in G and each coefficient in F. Now define $X\subseteq On$ to be an A.A. iff (i) X is closed under predecessors, and (ii) X has a maximal element Ω, which is an X, X-base for $X+X$, $X\cdot X$, and X^X (where X^X means, of course, $\{\alpha^\beta\colon \alpha,\beta\in X\}$).

THEOREM. *If X is an* A.A., *then*

(i) $0\in X$, *and* $\alpha\in X\backslash\{\Omega\}$ *implies* $\alpha+1\in X$.

(ii) *If Ω is a limit number, it is an X, $A[X]$-base for $A[X]$.*

(iii) *If Ω is a limit number, any isomorphism between X and another* A.A. *extends to an A-isomorphism between the arithmetics they generate.*

(iv) *If Ω is a limit number and $X\backslash\{\Omega\}$ a field, then $A[X]$ is also a field.*

Proof. (i) The Ω-expansion of $\Omega+\Omega$ is $\Omega^1\cdot 2$, so $1\in X$, so $0\in X$. The Ω-expansion for $\alpha+1$ is $\alpha+1$, since $\alpha+1\leqslant\Omega$.

(ii) Let $\gamma\in A[X]$. We wish to show that the indices and coefficients of the Ω-expansion of γ are in $A[X]$, X respectively. If $\gamma\in X$, this is trivial; otherwise γ will be $\alpha+\beta$, $\alpha\cdot\beta$, or α^β, where the Ω-expansions of α and β are as stated.

1) $\gamma = \big(\Omega^{\alpha(0)}a(0)+\ldots+\Omega^{\alpha(A)}a(A)\big)+\big(\Omega^{\beta(0)}b(0)+\ldots+\Omega^{\beta(B)}b(B)\big)$ say, with $\alpha(i),\beta(j)\in A[X]$ for all $\alpha(i),\beta(j)$, and $a(i),b(j)\in X$ for all $a(i),b(j)$,

$= \Omega^{\alpha(0)}a(0)+\ldots+\Omega^{\alpha(i)}a(i)+\Omega^{\beta(0)}b(0)+\ldots+\Omega^{\beta(B)}b(B)$ where $\alpha(i)\geqslant\beta(0)$. If $\alpha(i)>\beta(0)$, this is of the required form; if not, then

$\gamma = \Omega^{\alpha(0)}a(0)+\ldots+\Omega^{\beta(0)}\big(a(i)+b(0)\big)+\ldots+\Omega^{\beta(B)}b(B)$. We have then two cases to consider, namely either $a(i)+b(0)\in X$ (when the expansion is as required), or $a(i)+b(0) = \pi+\Omega$ for some $\pi \in X$, when

$\gamma = \Omega^{\alpha(0)}a(0)+\ldots+\Omega^{\alpha(i-1)}a(i-1)+\Omega^{\beta(0)+1}+\Omega^{\beta(0)}\pi+\ldots$, of required form unless $\alpha(i-1) = \beta(0)+1$, in which case we repeat this process with $a(i-1)+1$ instead of $a(i)+b(0)$.

2) $\gamma = \left(\Omega^{\alpha(0)}a(0)+\ldots+\Omega^{\alpha(A)}a(A)\right)\cdot\left(\Omega^{\beta(0)}b(0)+\ldots+\Omega^{\beta(B)}b(B)\right)$
$= \Omega^{\alpha(0)+\beta(0)}b(0)+\ldots+\Omega^{\alpha(0)+\beta(B)}b(B)$ $[+\Omega^{\alpha(1)}a(1)+\ldots$ if β is a successor] — in either case of the required form.

3) $\gamma = \left(\Omega^{\alpha(0)}a(0)+\ldots+\Omega^{\alpha(A)}a(A)\right)^{(\Omega^{\beta(0)}b(0)+\ldots+\Omega^{\beta(B)}b(B))}$
$= \Omega^{\alpha(0)\cdot\beta}$ if $\alpha(0) \neq 0$, $\beta(B) \neq 0$;

Or $= \Omega^{\alpha(0)\cdot(\Omega^{\beta(0)}b(0)+\ldots+\overline{b(B)})}\cdot a^{\text{finite part of } b(B)}$ if $\alpha(0) \neq 0$, $\beta(B) = 0$ — of required form since $\overline{b(B)} \in X$, and using 2).

Or $= a(A)^{(\Omega^{\beta(0)}b(0)+\ldots+\Omega^{\beta(B)}b(B))}$ if $\alpha(0) = 0$, which is a product of terms $a(A)^{\Omega^{\beta(i)}b(i)} = \left(a(A)^{\Omega}\right)^{\Omega^{-1+\beta(i)}b(i)} = \left(\Omega^{\pi(0)}p(0)+\ldots+\Omega^{\pi(P)}p(P)\right)^{\Omega^{-1+\beta(i)}b(i)}$ where $\pi(0) > 0$, so by previous cases this may be written in the required form.

(iii) We observe that the rules for finding Ω-expansions of elements of $A[X]$ use only properties invarient under an isomorphism.

(iv) We show by similar calculations that if the Ω-expansions of α and β have the given property then so do those of $-\beta+\alpha$ (if $\beta < \alpha$) and γ, δ, where $\alpha = \beta\gamma+\delta$ $(\delta < \beta)$. We omit the details.

The point of this theorem is that the structure of an arithmetic is often determined by a suitable (sometimes quite drastic) 'abbreviation' of it. We now consider how we can simplest specify the structure of this abbreviation.

THEOREM. *Consider an A.A. X such that Ω is an X-δ-number and $X\setminus\{\Omega\}$ is a field. Let the X-δ-numbers be $\delta_0, \delta_1, \ldots, \delta_\gamma = \Omega$, in order of magnitude. Then*

(i) *For any $\alpha, \beta < \gamma$, $\delta_\alpha^{\delta_\beta}$ and 2^{δ_β} may be written in the form $\Omega^\pi\cdot\delta_{\pi(1)}\cdot\delta_{\pi(2)}\ldots\delta_{\pi(n)}$ for some $\pi \in X$, $\pi(1), \ldots, \pi(n) < \gamma$.*

(ii) *These expressions for 2^{δ_β}, $\delta_\alpha^{\delta_\beta}$, for all $\alpha, \beta < \gamma$, completely determine the structure of X.*

Proof. (i) $\delta_\alpha^{\delta_\beta} \in X^X$, so $\delta_\alpha^{\delta_\beta} = \Omega^{\alpha(1)}\alpha(2)+\alpha(3)$, with $\alpha(1)$, $\alpha(2)$, $\alpha(3) \in X$. But $\delta_\alpha^{\delta_\beta}$ is easily seen to be an X-γ-number (if $\pi < \delta_\alpha^{\delta_\beta}$, $\pi+\delta_\alpha^{\delta_\beta} > \delta_\alpha^{\delta_\beta}$, consider the δ_α-expansion of π), and so cannot be a non-trivial sum. Hence if $\alpha(2) \neq 0$, then $\alpha(3) = 0$ and $\alpha(2)$ (a sum of products of X-δ-numbers) is a product of X-δ-numbers. If $\alpha(2) = 0$, $\delta_\alpha^{\delta_\beta} = \alpha(3)$ is a product of X-δ-numbers. 2^{δ_β} is considered similarly.

(ii) The order-type of the X-δ-numbers defines the structure of $X\setminus\{\Omega\}$ as a field. Since $X\setminus\{\Omega\}$ is a field, $\alpha+\beta$, $\alpha\cdot\beta \in X$ for $\alpha, \beta \in X\setminus\{\Omega\}$, and hence their Ω-expansions are trivial. We need to find suitable expansions for α^β, α^Ω, for arbitrary $\alpha, \beta \in X$

$$\alpha^\beta = (\delta_{\alpha(11)}\cdot\delta_{\alpha(12)}\ldots\delta_{\alpha(1n)})^{\overline{\beta}}\cdot\alpha^b = (\delta_{\alpha(11)})^{\overline{\beta}}\cdot\alpha^b$$
$$= \delta^{\delta_{\beta(11)}\delta_{\beta(12)}\ldots}\cdot\delta^{\delta_{\beta(21)}\delta_{\beta(22)}\ldots}\ldots\delta^{\delta_{\beta(m1)}\delta_{\beta(m2)}\ldots}\cdot\alpha^b.$$

Since each $\delta^{\delta_{\beta(i1)}}$ is of the form $\Omega^\pi\cdot\delta_{\pi(1)}\ldots\delta_{\pi(n)}$, we may (if necessary if $\pi = 0$, repeating the process) determine $\gamma(1)$, $\gamma(2)$ in terms of X-δ-numbers so that $\alpha^\beta = \Omega^{\gamma(1)}\gamma(2)$.

$$\alpha^\Omega = (\delta_{\alpha(11)})^\Omega = \Omega^\pi\cdot\delta_{\pi(1)}\ldots\delta_{\pi(n)} \quad \text{for known } \pi, \delta_{\pi(i)}.$$

Hence given two A.A.'s satisfying the conditions of the theorem, and an order preserving 1-1 map between their δ-numbers homomorphic with respect to the relations $\delta_\alpha^{\delta_\beta} = \Omega^\pi\cdot\delta_{\pi(1)}\cdot\delta_{\pi(2)}\ldots\delta_{\pi(n)}$, π being written in terms of the δ-numbers, and the similar relations for 2^{δ_α}, we may extend it to an isomorphism between the A.A.'s, and hence to an A-isomorphism between the arithmetics they generate.

We shall also need the following result;

LEMMA. *Any ordinal may be expressed uniquely in the form $\varepsilon^\alpha\beta+\pi$, where ε is an ε-number, $\beta < \varepsilon$, $\pi < \varepsilon^\alpha$, and $\alpha < \varepsilon'$, where ε' is the least ε-number greater than ε.*

Proof. The ε-function is continuous, i.e., $\lim_{\alpha<\beta}(\varepsilon_\alpha) = \varepsilon_\beta$ for β a limit number, so for any ordinal δ there is a greatest ε-number less than δ. Call this ε-number ε; then in the ε-expansion of δ, $\delta = \varepsilon^\alpha\beta+\ldots$ say, we will have $\alpha < \varepsilon'$ as required.

Let us now return to the problem of A-equivalence; we shall divide the ordinals into subclasses depending on properties of their forms $\varepsilon^\alpha\beta+\pi$ (as given in the lemma above), and demonstrate that no two ordinals of different subclasses can be A-equivalent. Then we shall show that two ordinals of the same class are A-equivalent. First we consider the case of limit ordinals.

We may divide these ordinals into two non-A-equivalent classes according as $\alpha 2^\alpha > 2^\alpha$ or $\alpha 2^\alpha = 2^\alpha$; if $\alpha 2^\alpha > 2^\alpha$ call α of *finite type*; otherwise of *infinite type*.

Case 1): α of finite type. Let $\alpha = \varepsilon^\gamma\delta+\pi$, with ε an ε-number, $0 < \delta < \varepsilon$, $\pi < \varepsilon^\gamma$, $0 < \gamma < \varepsilon'$ where ε' is the least ε-number greater than ε. Then

$$\alpha 2^\alpha > 2^\alpha \Leftrightarrow \varepsilon^\gamma\cdot 2^{\varepsilon^\gamma\delta} > 2^{\varepsilon^\gamma\delta} \Leftrightarrow \varepsilon\cdot\gamma+\varepsilon^\gamma\delta > \varepsilon^\gamma\delta$$
$$\Leftrightarrow \gamma = 1, \quad \delta < \omega.$$

So $\alpha = \varepsilon \cdot m + \pi$, $\pi < \varepsilon$. But also

$$\alpha^{(n-1)2} < 2^{\alpha 2} < \alpha^{n2-1} \Leftrightarrow \alpha = \varepsilon(n-1)+\pi, \qquad \omega \leqslant \pi < \varepsilon,$$

$$\alpha^{n2-1} < 2^{\alpha 2} \leqslant \alpha^{n2} \Leftrightarrow \alpha = \varepsilon \cdot n.$$

Hence we assert that the following classes of ordinals are not A-equivalent:

$\{\varepsilon \cdot n$: n fixed, finite; ε any ε-number$\}$—call such ordinals of *type* $\varepsilon \cdot n$

$\{\varepsilon n + \pi$: n fixed, finite; ε any ε-number, π any limit ordinal $\omega \leqslant \pi < \varepsilon\}$—call such ordinals of *type* $\varepsilon^* \cdot n$.

Case 2). α of infinite type. Let $\alpha = \varepsilon^\gamma \delta + \pi$ as above, and call α of

type ζ if $\gamma = 1$, $\omega \leqslant \delta$, $\delta \cdot \pi = \pi$.

type ζ^* if $\gamma = 1$, $\omega \leqslant \delta$, $\delta \cdot \pi > \pi$.

type η if $2 \leqslant \gamma < \omega$.

type θ if $\gamma \geqslant \omega$, $\varepsilon \cdot \gamma \cdot \pi = \pi$.

type θ^* if $\gamma \geqslant \omega$, $\varepsilon \cdot \gamma \cdot \pi > \pi$.

The table below shows whether or not an arbitrary element of each type satisfies certain equations; these may be verified by substitution.

	Type of α				
	ζ	ζ^*	η	θ^*	θ
$\alpha^\alpha = 2^{\alpha^2}$	Satisfied	N.S.	N.S.	N.S.	N.S.
$\alpha^\alpha = 2^\alpha$	Not Satisfied	N.S.	N.S.	N.S.	S.
$2^\alpha \cdot \alpha^\alpha = \alpha^\alpha$		S.	S.	N.S.	
$\alpha^\alpha \cdot 2^{\alpha^2} = 2^{\alpha^3}$		N.S.	S.	S.	

Thus we have that no two limit ordinals of different types are A-equivalent. We shall now show that two limit ordinals of the same type are A-equivalent.

Type $\varepsilon \cdot n$. The L-isomorphism between $L\{\varepsilon_\alpha\}$ and $L(\varepsilon_\beta\}$ restricts to an A-isomorphism between $A\{\varepsilon_\alpha n\}$ and $A\{\varepsilon_\beta n\}$.

Type ε^*. This is the most difficult case, because $A\{\alpha\}$ (α of type ε^*) is not a field and so we are not able to make use of our structure theorem. Let $\alpha = \varepsilon + \pi$, ε an ε-number and $\pi < \varepsilon$. Let $R_\alpha = R[\{\alpha, 2^{\alpha n}(n < \omega), 2^{2^{\alpha n}}(n < \omega), 2^{2^{\alpha n.\alpha}}(n < \omega)\}]$; the order of these elements is

$$2 < \alpha < 2^\alpha < 2^{2^\alpha} < 2^{\alpha^2} < 2^{2^{\alpha_\alpha}} < 2^{2^{\alpha 2}} < \ldots < 2^{\alpha^n} < 2^{2^{\alpha(n-1).\alpha}} < 2^{2^{\alpha n}} < 2^{\alpha^{n+1}} < \ldots$$

Since for any two of these elements, γ_1 and γ_2 say, with $\gamma_1 < \gamma_2$, we have $\gamma_1 + \gamma_2 = \gamma_2$ and $\gamma_1 \cdot \gamma_2 = \gamma_2$ or γ_2^2 (which one determined independently of α), we may write any element of R_α uniquely as a descending sum of descending products of the defining numbers; since the sum and product of any two such is determined independently of α, R_α and R_β will be R-isomorphic, for any α and β of type ε^*.

Now let $G_\alpha = R_\alpha \cup \{\Omega\}$, where $\Omega = 2^{2^{2^\alpha}}$; to show G_α is an A.A. we need to demonstrate suitable Ω-expansions for γ^β, for $\gamma \in R_\alpha$ and $\beta \in G_\alpha$. We may suppose $\beta = \bar{\beta}$ is infinite, and hence that γ is one of the defining numbers (using the relation in On, $[\text{poly in } \gamma_1, \gamma_2, \ldots]^{\bar{\beta}} = [\mathrm{Max}(\gamma_1, \gamma_2, \ldots)]^{\bar{\beta}}$). We give the following table for γ^β in certain cases, noting first that $\Omega^{2^{\alpha^2}} = 2^{(2^{2^\alpha})(2^{\alpha^2})} = 2^{2^{\varepsilon 2^\pi} \cdot \varepsilon^{\varepsilon+\pi}} = 2^{\varepsilon^{2^\pi + \varepsilon + \pi}} = 2^{\varepsilon^{\varepsilon+\pi}} = 2^{2^{\alpha^2}}$, and similarly $\Omega^{2^{2^{\alpha_\alpha}}} = 2^{2^{2^{\alpha_\alpha}}}$ and $\Omega^\Omega = 2^\Omega$.

		β				
		α	2^α	2^{2^α}	2^{α^2}	$2^{2^{\alpha_\alpha}}$
	2	2^α	2^{2^α}	Ω	$\Omega^{2^{\alpha^2}}$	$\Omega^{2^{2^{\alpha_\alpha}}}$
	α	2^{α^2}	$2^{2^{\alpha 2}}$	Ω	$\Omega^{2^{\alpha^2}}$	$\Omega^{2^{2^{\alpha_\alpha}}}$
γ	2^{α^n}	$2^{\alpha^{n+1}}$	$2^{2^{\alpha(n+1)}}$	Ω	$\Omega^{2^{\alpha^2}}$	$\Omega^{2^{2^{\alpha_\alpha}}}$
	$2^{2^{\alpha n}}$	$2^{2^{\alpha n_\alpha}}$	$2^{2^{\alpha(n+1)}}$	Ω	$\Omega^{2^{\alpha^2}}$	$\Omega^{2^{2^{\alpha_\alpha}}}$
	$2^{2^{\alpha n_\alpha}}$	$2^{\alpha^{n+2}}$	$2^{2^{\alpha(n+2)}}$	Ω	$\Omega^{2^{\alpha 2}}$	$\Omega^{2^{2^{\alpha_\alpha}}}$

Now consider $\beta = \beta_1 \cdot \beta_2 \cdot \ldots$, a product (descending) of the defining numbers. Then $\gamma^\beta = \gamma_1^\beta$ (γ_1 one of the defining numbers) $= \gamma_1^{\beta_1\beta_2\ldots}$. If $\beta_1 \leqslant 2^{2^{\alpha_\alpha}}$ we may use the above table to determine $\gamma_1^{\beta_1}$; if $\gamma_1^{\beta_1} < \Omega$, repeat the process; if $\gamma_1^{\beta_1} \geqslant \Omega$, then $\gamma^\beta = \gamma_1^\beta$ is a power of Ω with index in R_α, which is an Ω-expansion of the required form. If $\beta_1 > 2^{2^{\alpha_\alpha}}$ then $2^{\alpha^2} \cdot \beta_1 = \beta_1$, so $\gamma_1^\beta = \gamma_1^{(2^{\alpha^2})\beta} = \Omega^{(2^{\alpha^2})\beta} = \Omega^\beta$, also as required.

Finally let β be a sum of products of the defining numbers; then γ^β is a product of elements of R_α and powers of Ω with indices in R_α, which, since Ω is an R_α-δ-number, may be written $\Omega^{\gamma_1} \cdot \gamma_2$ with γ_1, γ_2 in R_α. Hence G_α is an A.A.

Furthermore, since all the processes involved above are independent of the actual value of α, the R-isomorphism between R_α and R_β, α and β of type ε^*, is isomorphic with respect to their structure as A.A.'s: by a theorem above it may be extended to an A-isomorphism between $A\{\alpha\}$ $(=A[G_\alpha])$ and $A\{\beta\}$ $(=A[G_\beta])$.

Type $\varepsilon^* n$ ($n > 1$). Any ordinal of type $\varepsilon^* n$ may be written as $\alpha = \gamma \cdot n$, where γ is of type ε^*. Let $\beta = \gamma' \cdot n$ be another such; then the A-isomorphism between $A\{\gamma\}$ and $A\{\gamma'\}$ restricts to an A-isomorphism between $A\{\alpha\}$ and $A\{\beta\}$.

Type ζ. For α of type ζ let $G_\alpha = F\{\alpha\} \cup \{\Omega\}$, where $\Omega = 2^\alpha$; Ω is a maximal element of G_α, and the only δ-numbers of G_α are α and Ω. We have the following table for $\delta_1^{\delta_2}$, independently of the value of α.

		δ_2	
		α	Ω
δ_1	2	2^α	Ω^{2^α}
	α	Ω^α	Ω^{2^α}

So G_α is an A.A., and the map $\alpha \to \beta$, $2^\alpha \to 2^\beta$, where β is also of type ζ, extends to an isomorphism between G_α and G_β, and then again to an A-isomorphism between $A\{\alpha\}$ and $A\{\beta\}$.

Type ζ^*. Let α be of type ζ^*, so $\alpha = \varepsilon\delta + \pi$ where $\delta \geqslant \omega$ and $\delta \cdot \pi > \pi$. Let $G_\alpha = F[\{\alpha, 2^\alpha, \delta_1, \delta_2, \delta_3, \ldots\}] \cup \{\Omega\}$, where $\Omega = \alpha^\alpha$, δ_1 is defined by $2^{\alpha^2} = \Omega \cdot \delta_1$, and δ_{i+1} is defined by $\delta_i^\alpha = \Omega \cdot \delta_{i+1}$. Then $\delta_1 = \varepsilon^{-\pi+\delta\pi} > 2^\alpha$, and it may be seen that $2^\alpha < \delta_1 < \delta_2 < \ldots$, and that the δ_i are all G_α-δ-numbers with the following exponential table:

		Index			
		α	2^α	δ_j	Ω
Base	2	2^α	Ω^{2^α}	Ω^{δ_j}	Ω^Ω
	α	Ω	Ω^{2^α}	Ω^{δ_j}	Ω^Ω
	2^α	$\Omega \cdot \delta_1$	Ω^{2^α}	Ω^{δ_j}	Ω^Ω
	δ_i	$\Omega \cdot \delta_{i+1}$	Ω^{2^α}	Ω^{δ_j}	Ω^Ω

G_α is therefore an A.A., and if β is also of type ζ^* the map $\alpha \to \beta$, $2^\alpha \to 2^\beta$, etc., extends to an isomorphism between G_α and G_β, and then to an A-isomorphism between $A[G_\alpha]$ and $A[G_\beta]$. In this we have $\alpha \to \beta$, so it will restrict to an A-isomorphism between $A\{\alpha\}$ and $A\{\beta\}$.

Type η. Let $\alpha = \varepsilon^g \cdot \delta + \pi$, with $1 < g < \omega$, $\pi < \varepsilon^g$, $\delta < \varepsilon$. Let G_α be $F[\{\alpha, 2^{\alpha^n}(n < \omega), \alpha^{\alpha^n}(n < \omega)\}] \cup \{\Omega\}$ where $\Omega = 2^{2^\alpha}$; the G_α-δ-numbers are α, $2^{\alpha^n}(n < \omega)$, $\alpha^{\alpha^n}(n < \omega)$, and Ω, and their exponential table is as follows:

		Index				
		α	2^α	$2^{\alpha^n}(n > 1)$	$\alpha^{\alpha^n}(n \geqslant 1)$	Ω
Base	2	2^α	Ω	$\Omega^{2^{\alpha^n}}$	$\Omega^{\alpha^{\alpha^n}}$	Ω^Ω
	α	α^α	Ω	$\Omega^{2^{\alpha^n}}$	$\Omega^{\alpha^{\alpha^n}}$	Ω^Ω
	$2^{\alpha^n}(n \geqslant 1)$	$2^{\alpha^{n+1}}$	Ω	$\Omega^{2^{\alpha^n}}$	$\Omega^{\alpha^{\alpha^n}}$	Ω^Ω
	$\alpha^{\alpha^n}(n \geqslant 1)$	$\alpha^{\alpha^{n+1}}$	Ω	$\Omega^{2^{\alpha^n}}$	$\Omega^{\alpha^{\alpha^n}}$	Ω^Ω

Hence G_α is an A.A. and as this table is independent of the value of α we may, as before, extend the obvious map $\alpha \to \beta$, etc., (where β is also of type η), to an A-isomorphism between $A\{\alpha\}$ and $A\{\beta\}$.

Type θ. Let α be of type θ, and let $G_\alpha = F\{\alpha\} \cup \{\Omega\}$ where $\Omega = 2^\alpha$; then Ω is a maximal element of G_α, and the G_α-δ-numbers are α and Ω with the following exponential table:

		Index	
		α	Ω
Base	2	Ω	Ω^Ω
	α	Ω	Ω^Ω

So as before G_α is an A.A. and, for β also of type θ, we may find an A-isomorphism between $A\{\alpha\}$ and $A\{\beta\}$.

Type θ^*. This case is similar to that of ζ^* — in both cases there are functions of α that are close but not equal, and again we must construct an arithmetic larger than $A\{\alpha\}$. Let $\alpha = \varepsilon^\gamma\delta + \pi$, with $\delta < \varepsilon$, $\pi < \varepsilon^\gamma$, $\omega \leqslant \gamma < \varepsilon'$ (where ε' is the least ε-number greater than ε) and $\varepsilon \cdot \gamma \cdot \pi > \pi$. Let $G_\alpha = F[\{\alpha, \delta_1, \delta_2, \ldots\}] \cup \{\Omega\}$ where $\Omega = 2^\alpha$, δ_1 is defined by $\alpha^\alpha = \Omega \cdot \delta_1$, and δ_{i+1} is defined by $\delta_i^\alpha = \Omega \cdot \delta_{i+1}$. Then $\delta_1 = 2^{-\pi+\varepsilon\gamma\pi} > \alpha$, so $\alpha < \delta_1 < \delta_2 < \delta_3 < \ldots < \Omega$. It may be shown that α, Ω, and the δ_i are the G_α-δ-numbers with the following exponential table:

		Index		
		α	δ_j	Ω
Base	2	2^α	Ω^{δ_j}	Ω^Ω
	α	$\Omega \cdot \delta_1$	Ω^{δ_j}	Ω^Ω
	δ_i	$\Omega \cdot \delta_{i+1}$	Ω^{δ_j}	Ω^Ω

So we may obtain, for β also of this type, an A-isomorphism between $A[G_\alpha]$ and $A[G_\beta]$, and this will restrict to one between $A\{\alpha\}$ and $A\{\beta\}$.

We may now consider successor ordinals. Clearly $\alpha \underset{A}{=} \beta \Rightarrow a = b$ (where $\alpha = \bar{\alpha} + a$, $\beta = \bar{\beta} + b$), for we have $2\alpha = \alpha + a$ and $2\beta = \beta + b$. Conversely, $\bar{\alpha} \underset{A}{=} \bar{\beta}$ and $a = b \Rightarrow \alpha \underset{A}{=} \beta$, for the isomorphism between $A\{\bar{\alpha}\}$ and $A\{\bar{\beta}\}$ will map $\alpha \to \beta$ and will thus restrict to an isomorphism between $A\{\alpha\}$ and $A\{\beta\}$. It remains to show that there are no two α and β with $\bar{\alpha}$ and $\bar{\beta}$ of different types yet $\alpha \underset{A}{=} \beta$; we do this by finding, for each pair of possible types for $\bar{\alpha}$ and $\bar{\beta}$, an equation satisfied by one of $\bar{\alpha} + a$ and $\bar{\beta} + a$ but not the other.

As before we say that α is of finite or infinite type according as $a2^\alpha > 2^\alpha$ or $a2^\alpha = 2^\alpha$ respectively. Then α is of finite type iff $\bar{\alpha}$ is of finite type. For α of infinite type we have the following table of equations satisfied or not satisfied by $\alpha = \overline{\alpha} + a$:

	Type of $\overline{\alpha}$				
	ζ	ζ^*	η	θ	θ^*
$2^\alpha \alpha^\alpha = \alpha^\alpha$	S.	S.	S.	N.S.	N.S.
$\alpha^\alpha 2^{\alpha^2} = 2^{\alpha^2}$	N.S.	N.S.	S.	S.	S.
$\alpha^\alpha 2^{\alpha\alpha} = 2^{\alpha^2}$	S.	N.S.			
$2^\alpha \alpha^\alpha \geqslant \alpha^\alpha$				S.	N.S.

If $\overline{\alpha}$ is of type $\varkappa$, call $\alpha = \overline{\alpha} + n$ of type $\varkappa + n$.

For α of finite type, we have, if $\alpha \neq \overline{\alpha}$,

$$\alpha^{m2-1}2^\alpha < 2^{\alpha 2} < \alpha^{m2}2^\alpha \Leftrightarrow \overline{\alpha}^{m2-1} < 2^{\overline{\alpha 2}} < \overline{\alpha}^{m2} \Leftrightarrow \overline{\alpha} \text{ of type } \varepsilon \cdot m$$

$$\alpha^{m2}2^\alpha < 2^{\alpha 2} < \alpha^{m2+1}2^\alpha \Leftrightarrow \overline{\alpha}^{m2} < 2^{\overline{\alpha 2}} < \overline{\alpha}^{m2+1} \Leftrightarrow \overline{\alpha} \text{ of type } \varepsilon^* \cdot m .$$

The smallest infinite ordinal of each arithmetic type is given in the figure below; ε is taken to be any ε-number, ε' the least ε-number greater than ε, ε_1 to be the least ε-number greater than ω, r and n finite with $r \geqslant 0$, $n > 0$, and π a limit number.

Type	**Form**	**Least element**
$\varepsilon \cdot n + r$	$\varepsilon \cdot n + r$	$\omega \cdot n + r$
$\varepsilon^* \cdot n + r$	$\varepsilon \cdot n + \pi + r$ $(\omega \leqslant \pi < \varepsilon)$	$\varepsilon_1 n + \omega + r$
$\zeta + r$	$\varepsilon \cdot \delta + \pi + r$ $(\omega \leqslant \delta < \varepsilon,\ \pi < \varepsilon,\ \delta \cdot \pi = \pi)$	$\varepsilon_1 \omega + r$
$\zeta^* + r$	$\varepsilon \cdot \delta + \pi + r$ $(\omega \leqslant \delta < \varepsilon,\ \pi < \varepsilon,\ \delta \cdot \pi > \pi)$	$\varepsilon_1 \omega + \omega + r$
$\eta + r$	$\varepsilon^n \delta + \pi + r$ $(n > 1,\ 1 \leqslant \delta < \varepsilon,\ \pi < \varepsilon^n)$	$\omega^2 + r$
$\theta + r$	$\varepsilon^\gamma \delta + \pi + r$ $(\omega \leqslant \gamma < \varepsilon',\ 1 \leqslant \delta < \varepsilon,\ \pi < \varepsilon^\gamma,\ \varepsilon\gamma\pi = \pi)$	$\omega^\omega + r$
$\theta^* + r$	$\varepsilon^\gamma \delta + \pi + r$ $(\omega \leqslant \gamma < \varepsilon',\ 1 \leqslant \delta < \varepsilon,\ \pi < \varepsilon^\gamma,\ \varepsilon\gamma\pi > \pi)$	$\omega^\omega + \omega + r$

It is of interest to observe the succession of arithmetic types of the limit ordinals α lying between successive ε-numbers, $\varepsilon < \varepsilon'$ say; we write $[S]$ to denote an infinite succession of sequences of the form S. Then the sequence of types is

$$\varepsilon, \varepsilon 2, \varepsilon 3, \ldots, [\eta], [\theta, [\theta^*]]$$

in the case $\varepsilon = \omega$, and

$$\varepsilon, [\varepsilon^*], \varepsilon 2, [\varepsilon^* 2], \varepsilon 3, [\varepsilon^* 3], \ldots, [\zeta, [\zeta^*]], [\eta], [\theta, [\theta^*]]$$

in every other case.

We may now give complete solutions for any system of arithmetic equations and inequalities in one infinite variable which contain no infinite constants. For example, the inequality $2^\alpha \cdot \alpha^\alpha \cdot (\alpha^5 + 2^\alpha) > \alpha^\alpha \cdot 2^\alpha$; by direct substitution of the least element of each type we have that the solutions are those, and only those, ordinals of types $\varepsilon \cdot n + r$ $(n \leqslant 5)$, $\varepsilon^* n + r$ $(n \leqslant 4)$, $\theta + r$, and $\theta^* + r$ (for all finite r).

It is interesting to note that every soluble equation or inequality of this type has a solution less than $\varepsilon_1 \omega + \omega \cdot 2$, and that this is a best possible result.

Divarithmetic equivalence. Finally, let us consider D-equivalence. Clearly any two ordinals not A-equivalent will be not D-equivalent. On the other hand, the proofs of A-equivalence for infinite arithmetic types hold for D-equivalence, since the A.A.'s used in the proofs were all fields, and hence in each case either $A\{\alpha\} = A[G_\alpha] = D[G_\alpha] = D\{\alpha\}$ or $D\{\alpha\} \subseteq D[G_\alpha] = A[G_\alpha]$.

For ordinals of finite arithmetic type, those of type $\varepsilon n + m$ are clearly also D-equivalent. Let α be of arithmetic type $\varepsilon^* n + r$, i.e., $\alpha = \varepsilon n + \pi + r$, with $\omega \leqslant \pi < \varepsilon$, π a limit number, ε an ε-number. Divarithmetics are closed under predecessors, so $\varepsilon n + \pi \in D\{\alpha\}$. Consider $\overline{\alpha}^{n+1} = 2^{\overline{\alpha}}\delta + \gamma$; $\delta \in D\{\alpha\}$, but δ is either ε or $\varepsilon + 1$, and in either case $\varepsilon \in D\{\alpha\}$. So $\pi + r \in D\{\alpha\}$, and $D\{\pi + r\} \subseteq D\{\alpha\}$; since $\pi + r < \varepsilon$ every element of $D\{\pi + r\}$ is less than ε. Hence two ordinals $\varepsilon n + \pi + r$ and $\varepsilon' n + \pi' + r$ are D-equivalent iff $\pi + r$ and $\pi' + r$ are D-equivalent. We obtain, by induction,

THEOREM. *Two ordinals α and β are D-equivalent iff when expressed (uniquely) in the form*

$$\alpha = \varepsilon_1 n_1 + \varepsilon_2 n_2 + \ldots + \varepsilon_k n_k + \pi + r,$$
$$\beta = \varepsilon_1' m_1 + \varepsilon_2' m_2 + \ldots + \varepsilon_p' m_p + \pi' + r',$$

with π, π' limit, $\varepsilon_1 > \ldots > \varepsilon_k$ ε-numbers, $0 < n_i < \omega$ for each n_i, π of infinite Arithmetic type or 0, $\varepsilon_1' > \ldots > \varepsilon_p'$ ε-numbers, $0 < m_j < \omega$ for each m_j, π' of infinite arithmetic type or 0, then we have $k = p$, $n_i = m_i$ for all $i \leqslant k$, $r = r'$, and $\pi = \pi' = 0$ or π and π' of the same infinite arithmetic type.

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