each $i^{-1}(V) \cap A$ is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$), so that by theorem 12, $i^{-1}(V)$ is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$). Since X is in $\mathfrak{A}_G$, $i|\ i^{-1}(V)\colon i^{-1}(V) \to V$ is a topological quotient map. Thus by theorem 12, V is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$).

Corollary (Franklin [6]). *Each of the properties "Hausdorff compactly generated" and "sequential" is open-hereditary.*

Proof. Clearly by theorem 13, $\mathfrak{H}(\mathfrak{K} \cap \mathfrak{H}) = \mathfrak{T}(\mathfrak{K}) \cap \mathfrak{H}$. Also by the corollary to proposition 9 every open subset of a compact Hausdorff space is compactly generated. Similarly, every open subset of every convergent sequence is a convergent sequence (possibly finite) and so is sequential.

References

[1] A. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Dokl. Akad. Nauk, SSSR, 150 (9) (1963).

[2] C. E. Aull, *Sequences in topological spaces*, Notices Amer. Math. Soc. 12 (1965), p. 222.

[3] R. Brown, *Ten topologies for $X \times Y$*, Quant. J. Math., Oxford, 14 (2) (1963), pp. 303–319.

[4] R. M. Dudley, *On sequential convergence*, Trans. Amer. Math. Soc. 112 (1964), pp. 483–504.

[5] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), pp. 107–116.

[6] — *Natural covers*, Compositio Math. 21 (1969), pp. 253–261.

[7] — *The categories of k-spaces and sequential spaces*, class notes, Mathematics Department, Carnegie-Mellon University.

[8] P. Freyd, *Abelian Categories*, New York 1964.

[9] P. Gabriel, *Réalisation géomètrique des ensembles semi-simpliciaux* (rédigé d'après un exposé de C. Godbillon), Séminaire Homotopique, Strasbourg 1963/64.

[10] A. M. Gleason, *Universal locally connected refinements*, Illinois J. Math. 7 (1963), pp. 521–531.

[11] H. Herrlich and G. E. Strecker, *Coreflective subcategories*, Trans. Amer. Math. Soc. 157 (1971), pp. 205–226.

[12] J. F. Kennison, *Reflective functors in general topology and elsewhere*, Trans. Amer. Math. Soc. 118 (1965), pp. 303–315.

[13] B. Mitchell, *Theory of Categories*, New York 1965.

[14] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967), pp. 133–153.

[15] G. S. Young, *The introduction of local connectivity by change of topology*, Amer. J. Math. 68 (1946), pp. 479–494.

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Svenonius sentences and Lindström's theory on preservation theorems

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Introduction

Let us call the infinitary sentence $q \bigwedge \Theta$ a *Svenonius sentence* or *S-sentence* if q is a prefix of length $\leqslant \omega$ and Θ is a countable set of finitary first order formulas appropriate for a fixed countable similarity type ξ. It is immediate from the semantics of infinitary formulas (see [5], [6], [7] and § 0 below) that any class K of structures of type ξ, definable by an S-sentence, is a PC_Δ class. Svenonius [12] showed a partial converse of this fact.

Let us call K a PC_δ^- class if K is the class of the reducts to ξ of the countable models of a countable set of ordinary sentences. Svenonius' theorem says that the PC_δ^- classes are exactly the classes of countable models of S-sentences (1). Svenonius [12] also showed that Craig's interpolation theorem [1] is an easy consequence of this theorem. Considered from this point of view, (the proof of) Svenonius' theorem yields perhaps the most elementary model-theoretical proof of the interpolation theorem, or more particularly, it demonstrates that the ideas of Henkin [4], if properly applied, are sufficient for proving the interpolation theorem.

Knowing the close connection between certain preservation theorems and interpolation theorems, it is natural to ask whether there exist analogs involving S-sentences of known preservation theorems such that the original theorems are consequences of the new ones. This paper gives a positive answer to this question.

Call an S-sentence $q \bigwedge \Theta$ positive if every element of Θ is positive in the usual sense. Our Corollary 2.4 (a) says that K is a PC_δ^- class closed under homomorphisms iff K is the class of countable models of a positive S-sentence. We also show that Lyndon's well-known preservation theorem [9] is an easy consequence of this result.

(1) See Theorem 2 in [12]. In [12] a different terminology is used and S-sentences are mentioned only in passing.

After having found the last-mentioned Corollary 2.4 (a) and a series of analogous results, the author came across Per Lindström's important work [8], which unifies the treatment of many preservation theorems concerning binary relations between structures. Lindström introduces games G involving ω-type sequences of choices by two players of finite strings of elements in two structures $\mathfrak{A}$ and $\mathfrak{B}$ and associates a relation $R(G)$ with G between structures such that $\mathfrak{A}R(G)\mathfrak{B}$ iff Player II, has a winning strategy in G played on $(\mathfrak{A}, \mathfrak{B})$ (for precise definitions see § 0 below).

The present paper combines the author's earlier findings, originating in Svenonius [12] with Lindström's theory. Say that K is *closed* ($\aleph_0$-closed) *under* R if for any (countable) $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \in K$ and $\mathfrak{A}R\mathfrak{B}$ imply $\mathfrak{B} \in K$. We associate with a Lindström game G a class $S(G)$ of S-sentences all preserved under $R(G)$ such that K is a PC_δ class $\aleph_0$-closed under $R(G)$ iff K is the class of countable structures of an element of $S(G)$ (Corollary 2.1). Our methods of proof are closely related to those of Svenonius [12]. Svenonius' theorem is, in fact, a special case of our Corollary 2.1.

Corollary 2.4 (a), concerning homomorphisms, is obtained as a special case too. It should be mentioned that a direct proof avoiding Lindström games can be given for this Corollary through a considerable simplification of the proof of the main theorem. For the proof of Lyndon's theorem thus resulting we could claim a similar status as we did above for the proof of Craig's interpolation theorem via Svenonius' theorem.

We also obtain a version of Lindström's [8] main result as Corollary 2.2.

Our results were announced in [9], [10].

§ 0. Preliminaries

Except for some slight modifications mentioned below, we use the set-theoretical and logical notation that is common in the literature of model theory. E.g., ω is the set of natural numbers, $n = \{m: m < n\}$ for $n < \omega$.

The symbols $f \restriction a$ and $f \circ g$ will be used in a more general sense than usual. We put $f \restriction a \overset{\text{df}}{=} f \cap ((a \cap \operatorname{dom} f) \times \operatorname{rn} f)$, i.e. $f \restriction a = f \restriction (a \cap \operatorname{dom} f)$. Also, $f \circ g$ is the function h such that $\operatorname{dom} h = \{x \in \operatorname{dom} g: g(x) \in \operatorname{dom} f\}$ and $h(x) = f(g(x))$ for $x \in \operatorname{dom} h$.

In the formal languages considered we identify the variables with the natural numbers. Sometimes v_n is written for n. If not stated explicitly otherwise, "formula" ("sentence") always means "finitary first-order formula (sentence) appropriate for ξ" with a fixed *countable* similarity type ξ. Also, all structures considered are of type ξ.

If σ is a function with $\operatorname{dom}\sigma \subset \omega$, $\operatorname{rn}\sigma \subset \omega$, and φ is a formula, then $\varphi(\sigma)$ denotes the result of substituting $\sigma(x)$ for x ($x \in \operatorname{dom}\sigma$) in φ at every free occurrence of x in φ after renaming bound variables to make $\sigma(x)$ free for x in φ. $\varphi(y/x)$ stands for $\varphi(\{\langle x, y\rangle\})$. We write $\varphi \sim \psi$ to denote that ψ differs from φ in the names of bound variables only.

We include the empty disjunction $\bigvee 0$ and the empty conjunction $\bigwedge 0$ among the (finitary) formulas. The former is identically false, the latter is identically true, and both are considered positive.

In this paper we deal with (possibly) infinitary sentences Φ of the form

$$\langle Q_0 x_0 \dots Q_n x_n \dots \rangle_{n<\lambda} \bigwedge \Theta$$

where $\lambda \leqslant \omega$, Θ is a countable set of (ordinary) formulas, Q_n is $\forall$ or $\exists$, and the x_n are distinct variables. Such a sentence is called a *Svenonius-sentence* or *S-sentence*. If $\lambda < \omega$ and Θ is finite, Φ is a finitary sentence. Here we reproduce the truth-definition for S-sentences by specializing that given in [5], [6], [7] for a more general type of infinitary formulas. For the sake of simplicity we assume $x_n = n = v_n$ in the above $q = \langle Q_n x_n: n < \lambda\rangle$ and we identify q with $\langle Q_n: n < \lambda\rangle$. In this case q is called a *simple prefix*.

Denote by $\forall(q)$ and $\exists(q)$ the sets $\{n < \lambda: q(n) = \forall\}$ and $\{n < \lambda: q(n) = \exists\} = \lambda - \forall(q)$, respectively. A *q-strategy on A* is a function $f: {}^{\forall(q)}A \to {}^{\exists(q)}A$ such that for $a_\forall, a'_\forall \in {}^{\forall(q)}A$ and $n < \lambda$, $a_\forall \restriction n = a'_\forall \restriction n$ implies $f(a_\forall) \restriction n = f(a'_\forall) \restriction n$ or, in other words, $f(a) \restriction n$ depends only on $a \restriction n$. f is a *winning strategy for* "$\mathfrak{A} \models q \bigwedge \Theta$" if for any $a_\forall \in {}^{\forall(q)}A$ and $\vartheta \in \Theta$ we have $\mathfrak{A} \models \vartheta[a_\forall \cup f(a_\forall)]$. We write $\mathfrak{A} \models \Phi$ or $\mathfrak{A} \in \operatorname{Mod}(\Phi)$ if there exists a winning strategy for "$\mathfrak{A} \models \Phi$" and in this case we say that Φ is *true in* $\mathfrak{A}$ and that $\mathfrak{A}$ is a *model of* Φ. Note that in the case where Φ is a finitary sentence (i.e., $\lambda < \omega$ and Θ is finite), Φ is true in $\mathfrak{A}$ in the new sense just in case Φ is true in $\mathfrak{A}$ in the ordinary sense.

A *finite approximation* of $q \bigwedge \Theta$ is a finitary sentence $q' \bigwedge \Theta'$ such that q' is a finite initial segment of q and Θ' is a finite subset of Θ. This notion is a special case of a notion of Keisler [6]. The next proposition is obvious.

0.1. *A model of an S-sentence is a model of any finite approximation of it.* (Proof. Cut the winning strategy to the appropriate size.)

The next proposition can be proved easily by (i) translating the S-sentence $q \bigwedge \Theta$ into a theory with (Skolem-) function symbols corresponding to the functions f_k: ${}^{\forall(q)\cap k}A \to A$ such that $f_k(a) = (f(a_\forall))(k)$ for some (any) $a_\forall$ extending a and f is a q-strategy on A, and then (ii) using the Löwenheim–Skolem theorem and the compactness theorem. It is contained implicitly in [12] (see Theorem 5 and the proof of "Craig's lemma", pp. 388–389).

0.2. *Assume that every countable model of the S-sentence Φ is a model of the (finitary) sentence ψ. Then $\varphi \models \psi$ for some finite approximation φ of Φ.*

The method of Skolem functions also yields

0.3. $\mathrm{Mod}(\Phi)$ is a PC_δ class, i.e. it is the class of reducts of some elementary (EC_Δ) class of structures of some *countable* type $\xi' \supset \xi$.

For $\alpha \in {}^{\omega}A$, we say that $(\mathfrak{A}, \alpha)$ is a *model* of the set Σ of formulas if $\mathfrak{A} \models \varphi[\alpha]$ for all $\varphi \in \Sigma$. The formula φ is said to be a logical consequence of Σ *with the variables held constant* if any model $(\mathfrak{A}, \alpha)$ of Σ is a model of φ.

The next lemma is a version of a part of Henkin's proof [4] of the completeness theorem.

The part in question consists in showing that a consistent set of formulas, if extended by an appropriate collection of formulas of the form $\exists z\varphi \to \varphi(y/z)$, remains consistent.

0.4. Assume we are given the simple prefix $q = \langle Q_n\colon n < \omega\rangle$, and for every $k < \omega$, the natural number y_k and the formula $\exists z_k \varphi_k$ such that

(i) $k_1 < k_2$ implies $y_{k_1} < y_{k_2}$,

(ii) $y_k \in \exists(q)$,

(iii) every free variable of $\exists z_k \varphi_k$ is $< y_k$.

Put $H = \{\exists z_k \varphi_k \to \varphi_k(y_k/z_k)\colon k < \omega\}$. Then any structure is a model of $q \wedge H$. Consequently, if $\mathfrak{A}$ is a model of the set T of sentences and each element of Θ is a consequence of $T \cup H$ with the variables held constant, then $\mathfrak{A}$ is a model of $q \wedge \Theta$.

In the rest of this section we reproduce Lindström's [8] definition of "regular relations".

Let $p = \langle P_n u_n\colon n < \omega\rangle$ be a *generalized prefix* where each P_n is $\forall$ or $\exists$ and each u_n is a non-empty finite set of variables. Let Γ be an arbitrary set of formulas. We refer to the pair $G = (p, \Gamma)$ as a *Lindström game.* Following [8], we associate with G a relation $R(G)$ between structures as follows. To give first an illustration, assume e.g. that $P_0 = \exists$, $P_1 = \forall$, ... and $u_i = \{x_i^0, \ldots, x_i^{k_i}\}$. Then, by definition, $\mathfrak{A} R(G) \mathfrak{B}$ iff the following infinitary statement "with a prefix of type ω" holds:

$$(\forall a_0^0 \in A, \ldots, \forall a_0^{k_0} \in A)(\exists b_0^0 \in B, \ldots, \exists b_0^{k_0} \in B)(\forall b_1^0 \in B, \ldots, \forall b_1^{k_1} \in B)\times$$
$$\times(\exists a_1^0 \in A, \ldots, \exists a_1^{k_1} \in A) \ldots$$
$$\bigwedge_{\gamma \in \Gamma} \{\mathfrak{A} \models \gamma[\ldots a_i^j/x_i^j \ldots] \Rightarrow \mathfrak{B} \models \gamma[\ldots b_i^j/x_i^j \ldots]\}.$$

Though this definition is essentially precise, for practical purposes we give a more formal one in the case where p is a simple prefix (i.e. $u_n = \{v_n\}$; now p is identified with $\langle P_n\colon n < \omega\rangle$).

We call a mapping $h\colon {}^{\exists(p)}A \times {}^{\forall(p)}B \to {}^{\forall(p)}A \times {}^{\exists(p)}B$ a *p-strategy on* $A \times B$ if, with $(\alpha', \beta') \overset{\mathrm{df}}{=} h(\alpha, \beta)$, "$\alpha' \restriction n$ and $\beta' \restriction n$ depend only on $(\alpha \restriction n, \beta \restriction n)$", i.e., more precisely, $(\alpha_i', \beta_i') = h(\alpha_i, \beta_i)$ $(i = 1, 2)$ and $\alpha_1 \restriction n = \alpha_2 \restriction n$, $\beta_1 \restriction n = \beta_2 \restriction n$ imply $\alpha_1' \restriction n = \alpha_2' \restriction n$ and $\beta_1' \restriction n = \beta_2' \restriction n$. p is called a *winning strategy for* "$\mathfrak{A} R(G) \mathfrak{B}$" if, in addition, for any $(\alpha_\exists, \beta_\forall) \in {}^{\exists(p)}A \times {}^{\forall(p)}B$ and $(\alpha_\forall, \beta_\exists) = h(\alpha_\exists, \beta_\forall)$, $\mathfrak{A} \models \gamma(\alpha_\forall \cup \alpha_\exists)$ implies $\mathfrak{B} \models \gamma(\beta_\forall \cup \beta_\exists)$ for all $\gamma \in \Gamma$. $\mathfrak{A} R(G) \mathfrak{B}$ holds by definition iff there is a winning strategy for "$\mathfrak{A} R(G) \mathfrak{B}$".

Lindström [8] has shown that many ordinary algebraic relations R coincide with $R(G)$ for suitable G when R and $R(G)$ are both restricted to countable structures. An example of an R where we need more than one (in fact, two) variables in each u_n quantified in p at a time is the relation "$\mathfrak{A} \times \mathfrak{A}$ is isomorphic to $\mathfrak{B} \times \mathfrak{B}$".

§ 1. The main result

(a) Definition of the class $S(G)$. We call ϱ a *regular tree* if (i) ϱ is a (reflexive) partial ordering with field $\lambda \leqslant \omega$, (ii) ϱ is a tree, i.e. $n[\varrho] \overset{\mathrm{df}}{=} \{m\colon m\varrho n\}$ is totally ordered by ϱ, (iii) $m\varrho n$ implies $m \leqslant n$.

Let ϱ be a regular tree. Clearly, $n[\varrho] \subset n+1$. Let the *level* $l_\varrho(n)$ (or simply $l(n)$) *of* n be $\overline{\overline{n[\varrho]}}-1$. Obviously, $m\varrho n$ implies $l(m) \leqslant l(n)$, and also conversely provided that both m and n are in some set $k[\varrho]$. We say that n is a *ϱ-successor of* m if $m\varrho n$ and $l(n) = l(m)+1$. We introduce one more symbol. Let σ_ϱ^n (or simply σ^n) be the function $\sigma^n = (l \restriction n[\varrho])^{-1}$. σ^n is the enumeration of the elements of $n[\varrho]$ in the order of their levels.

Let the Lindström game $G = (p, \Gamma)$ be given with a simple prefix p. Let ϱ be a regular tree with field λ. Let $q = q[p, \varrho]$ be the simple prefix $q = \langle p(l_\varrho(n))\colon n < \lambda\rangle$. Note that $n \in \exists(q)\big(n \in \forall(q)\big)$ iff $l(n) \in \exists(p)\big(l(n) \in \forall(p)\big)$. We call a formula a G, ϱ-formula if it is of the form $\gamma(\sigma_\varrho^n)$ such that $\gamma \in \Gamma$ and all the free variables of γ are in $\mathrm{dom}(\sigma_\varrho^n)$.

For a given Lindström game $G = (p, \Gamma)$, we define $S(G)$ to be the class of all S-sentences Φ such that for some regular tree ϱ, Φ is $q[p, \varrho] \wedge \Theta$ and each element of Θ is a finite (possibly empty) disjunction of G, ϱ-formulas.

Remark 1. Consider the case $\varrho = \leqslant$. Now the σ^n are identity mappings and $q[p, q] = p$. Thus the above Φ becomes $p \wedge \Theta$, where the elements of Θ are finite disjunctions of elements of Γ. It is easy to see that this Φ is preserved under $R(G)$.

Next we give the definition of $S(G)$ for $G = (p, \Gamma)$ where $p = \langle P_n u_n\colon n < \omega\rangle$ is a generalized prefix. Let us write Qs for $\langle Qv_0, \ldots, Qv_k\rangle$ if $s = \langle v_0, \ldots, v_k)$ and $\bigcap_{n<\lambda} Q_n s_n$ for the concatenation $Q_0 s_0 \frown Q_1 s_1 \ldots$ Now, let ϱ be again a regular tree with field λ and $\varepsilon = \langle \varepsilon_n \colon n \in \omega \rangle$ a system of mappings $\varepsilon_n \colon u_{l(n)} \to \omega$ such that $rn(\varepsilon_1) \cap rn(\varepsilon_2) = 0$ for $n_1 \neq n_2$. For each n, let s_n be a 1-1 sequence such that $rn(s_n) = rn(f_n)$. Put $q \overset{\text{df}}{=} \bigcap_{n<\lambda} P_{l(n)} s_n$ and let Θ be any set of finite disjunctions of formulas of the form $\gamma(\bigcup_{m \varrho n} \varepsilon_m)$ such that $n < \lambda$, $\gamma \in \Gamma$ and the free variables of γ are in $\operatorname{dom}(\bigcup_{m \varrho n} \varepsilon_m)$. We define $S(G)$ to be the class of all S-sentences $q \wedge \Theta$ thus obtained.

Remark 2. To facilitate reading, in the proofs we restrict ourselves to Lindström games (p, Γ) with a simple p. If we wanted to write out the proofs for the general case, we would use generalized prefixes in writing S-sentences. When doing so, the above s_n are superfluous and q becomes $\langle P_{l(n)} rn(\varepsilon_n) \colon n < \omega \rangle$. Thus concatenation is avoided too.

(b) Properties of $S(G)$.

Proposition 1.1. *Every element of $S(G)$ is preserved under $R(G)$, i.e. $\Phi \in S(G)$, $\mathfrak{A} R(G) \mathfrak{B}$ and $\mathfrak{A} \models \Phi$ imply $\mathfrak{B} \models \Phi$.*

Proof. Let $G = (p, \Gamma)$. We assume that p is simple.

(I) Construction of a q-strategy on B. Let ϱ be a regular tree with field λ, $q = q[p, \varrho]$. Let $h \colon {}^{\exists(p)}A \times {}^{\forall(p)}B \to {}^{\forall(p)}A \times {}^{\exists(p)}B$ be a p-strategy on $A \times B$ and $f \colon {}^{\forall(q)}A \to {}^{\exists(q)}A$ a q-strategy on A. Extend the definition of h by defining $h(\alpha, \beta)$, for any $l < \omega$ and $(\alpha, \beta) \in {}^{\exists(p) \cap l}A \times {}^{\forall(p) \cap l}B$, to be the uniquely determined pair $(\alpha', \beta') \in {}^{\forall(p) \cap l}A \times {}^{\exists(p) \cap l}B$ such that for some (any) extension $\alpha_1 \in {}^{\exists(p)}A$, $\beta_1 \in {}^{\forall(p)}B$ of α and β, respectively, and for $(\alpha_1', \beta_1') = h(\alpha_1, \beta_1)$, α_1' and β_1' are extensions of α' and β', respectively. Note that if $h(\alpha_i, \beta_i) = (\alpha_i', \beta_i')$ for $i = 1, 2$ and $\alpha_1 \subset \alpha_2$, $\beta_1 \subset \beta_2$, then $\alpha_1' \subset \alpha_2'$, $\beta_1' \subset \beta_2'$, i.e. h is monotone.

Now choose $\beta_\forall \in {}^{\forall(q)}B$ arbitrarily. We claim that there exist unique $\alpha_\forall \in {}^{\forall(q)}A$ and $\beta_\exists \in {}^{\exists(q)}B$ such that for each $n < \lambda$ we have

(1) $$h\big((f\alpha_\forall) \circ \sigma^n, \beta_\forall \circ \sigma^n\big) = (\alpha_\forall \circ \sigma^n, \beta_\exists \circ \sigma^n)\,.$$

Observe that in (1) the pair of arguments on the left-hand side is in ${}^{\exists(p) \cap l}A \times {}^{\forall(p) \cap l}B$ and the pair on the right-hand side is in ${}^{\forall(p) \cap l}A \times {}^{\exists(p) \cap l}B$ for $l = l(n)+1$.

To establish the claim, we define the values $\alpha_\forall(m)$ for $m \in \forall(q)$ by induction on $m < \lambda$, $m \in \forall(q)$. Assume $\alpha_\forall(m')$ is defined for $m' < m$, $m' \in \forall(q)$, i.e. $\alpha_\forall \restriction m$ is defined. Since f is a q-strategy and $m \notin \exists(q) = \operatorname{dom}(f\alpha_\forall)$, $(f\alpha_\forall) \restriction (m+1) = (f\alpha_\forall) \restriction m$ is defined too (and is equal to $(f\alpha_\forall') \restriction m$ for any $\alpha_\forall' \in {}^{\forall(p)}A$ such that $\alpha_\forall' \restriction m = \alpha_\forall \restriction m$). Using $rn(\sigma^m) \subset m+1$, we see that $(f\alpha_\forall) \circ \sigma^m$ is defined. Put

(2) $$(\alpha_1, \beta_1) \overset{\text{df}}{=} h\big((f\alpha_\forall) \circ \sigma^m, \beta_\forall \circ \sigma^m\big)$$

and finally

(3) $$\alpha_\forall(m) \overset{\text{df}}{=} \alpha_1\big(l(m)\big) \qquad \big(m \in \forall(q)\big)\,.$$

Having $\alpha_\forall$, we define $\beta_\exists \in {}^{\exists(q)}B$ by

(4) $$\beta_\exists(m) \overset{\text{df}}{=} \beta_1\big(l(m)\big) \qquad \big(m \in \exists(q)\big)$$

for any $m \in \exists(q)$ where β_1 is defined by (2).

To show (1), choose $n < \lambda$ and let

(5) $$(\alpha_2, \beta_2) \overset{\text{df}}{=} h\big((f\alpha_\forall) \circ \sigma^n, \beta_\forall \circ \sigma_n\big)\,.$$

Let $k \leqslant l(n)$ be arbitrary, put $m = \sigma^n(k)$; hence we have $k = l(m)$. Taking the restrictions to $k+1$ on both sides in (5) and using the fact that h is monotone, we obtain

(6) $$\big(\alpha_2 \restriction (k+1), \beta_2 \restriction (k+1)\big) = h\big((f\alpha_\forall) \circ \sigma^m, \beta_\forall \circ \sigma^m\big)\,.$$

Comparing (2), (3) and (6), we infer for $k \in \forall(p)$

$$\alpha_2(k) = \big(\alpha_2 \restriction (k+1)\big)(k) = \alpha_1\big(l(m)\big) = \alpha_\forall(m) = (\alpha_\forall \circ \sigma^n)(k)$$

and similarly for $k \in \exists(p)$

$$\beta_2(k) = (\beta_\exists \circ \sigma^n)(k)\,.$$

Hence

$$\alpha_2 = \alpha_\forall \circ \sigma^n \quad \text{and} \quad \beta_2 = \beta_\exists \circ \sigma^n\,.$$

The last two equalities together with (5) establish (1) as desired.

Conversely, it is clear that if (1) holds, then (3) and (4) hold with definition (2). This shows the uniqueness of $\alpha_\forall$ and $\beta_\exists$, and even more, viz. that $\alpha_\forall(m)$ $\big($for $m \in \forall(q)\big)$ and $\beta_\exists(m)$ $\big($for $m \in \exists(q)\big)$ depend only on $\beta_\forall \circ \sigma^m$, i.e. they depend only on $\beta_\forall \restriction (m+1)$. Otherwise expressed, $\alpha_\forall \restriction n$, $\beta_\exists \restriction n$ depend only on $\beta_\forall \restriction n$ for $n < \omega$. Hence, if we put $g(\beta_\forall) = \beta_\exists$ such that we have (1) for $n < \lambda$, g is a q-strategy.

(II) Showing that g is a winning strategy for "$\mathfrak{B} \models \Phi$". Now let $\Phi \in S(G)$ and let ϱ be a regular tree such that $\Phi = q[p, \varrho] \wedge \Theta$ and the elements of Θ are finite disjunctions of G, ϱ-formulas.

Assume $\mathfrak{A} R(G) \mathfrak{B}$ and $\mathfrak{A} \models \Phi$. Let h be a winning strategy for "$\mathfrak{A} R(G) \mathfrak{B}$" and f a winning strategy for "$\mathfrak{A} \models \Phi$". Let g be the q-strategy on B defined in (I), i.e. for every $\beta_\forall \in {}^{\forall(q)}B$ there is an $\alpha_\forall \in {}^{\forall(q)}A$ such that

(7) $$h\big((f\alpha_\forall) \circ \sigma^n, \beta_\forall \circ \sigma_n\big) = \big(\alpha_\forall \circ \sigma^n, (g\beta_\forall) \circ \sigma^n\big)$$

for $n < \lambda$. Let $\beta_\forall \in {}^{\forall(q)}B$ and $\vartheta \in \Theta$. Choose $\alpha_\forall$ such that (7) holds. Since f is a winning strategy for "$\mathfrak{A} \models \Phi$", we have $\mathfrak{A} \models \gamma[\alpha_\forall \cup f\alpha_\forall]$; hence $\mathfrak{A} \models \gamma(\sigma^n)[\alpha_\forall \cup f\alpha_\forall]$ for some G, ϱ-formula $\gamma(\sigma^n)$ which is a disjunct of ϑ. Here $\gamma \in \Gamma$ and the free variables of γ are in $\operatorname{dom}(\sigma^n)$. The last fact allows us to infer $\mathfrak{A} \models \gamma[a_{\forall} \circ \sigma^n \cup (fa_{\forall}) \circ \sigma^n]$. This and (7) yield $\mathfrak{B} \models \gamma[\beta_{\forall} \circ \sigma^n \cup (g\beta_{\forall}) \circ \sigma^n]$ since h is a winning strategy for "$\mathfrak{A}R(G)\mathfrak{B}$". Consequently $\mathfrak{B} \models \vartheta[\beta_{\forall} \cup g\beta_{\forall}]$, which shows that g is a winning strategy for "$\mathfrak{B} \models \Phi$", completing the proof of 1.1.

Let us write $C_R(K)$ for the class of all $\mathfrak{B}$ such that for some $\mathfrak{A} \in K$ we have $\mathfrak{A} R \mathfrak{B}$. Let $\overline{\mathrm{Mod}}(T)$, $\bar{C}_R(K)$ denote the class of countable members of $\mathrm{Mod}(T)$, $C_R(K)$, respectively.

Our main result is

THEOREM 1.2. *Let G be a Lindström game. For any set T of sentences there is a certain $\Phi \in S(G)$ such that we have*

$$C_{R(G)}\big(\mathrm{Mod}(T)\big) \subset \mathrm{Mod}(\Phi)\,,$$

$$\bar{C}_{R(G)}\big(\mathrm{Mod}(T)\big) = \bar{C}_{R(G)}\big(\overline{\mathrm{Mod}}(T)\big) = \overline{\mathrm{Mod}}(\Phi)\,.$$

Proof. We again consider only the case where p is simple, $G = (p, \Gamma)$.

(I) Construction of Φ. To simplify the terminology, we say in connection with a regular tree ϱ that the elements of level 0 are ϱ-successors of -1 and we extend the definition of the function l_ϱ by putting $l_\varrho(-1) = -1$.

We first construct a regular tree ϱ and, in case $\exists(p) \neq 0$, the sequences $\langle \exists z_k \varphi_k \colon k < \omega\rangle$, $\langle y_k \colon k < \omega\rangle$ of formulas $\exists z_k \varphi_k$ and variables y_k, respectively, such that:

(i) For any $n \in \omega \cup \{-1\}$, there are infinitely many ϱ-successors of n,

and for $\exists(p) \neq 0$,

(ii) if x occurs free in $\exists z_k \varphi_k$, then $x < y_k$ $(x, k < \omega)$,

(iii) $k_1 < k_2$ implies $y_{k_1} < y_{k_2}$ $(k_1, k_2 < \omega)$,

(iv) $l_\varrho(y_k) \in \exists(p)$ $(k < \omega)$,

(v) for any $n \in \omega \cup \{-1\}$ with $l(n)+1 \in \exists(p)$ and for any formula $\exists z\varphi$ there is a $k < \omega$ such that y_k is a ϱ-successor of n and $\exists z_k \varphi_k \sim \exists z \varphi$.[2]

Note that (i) means that (ω, ϱ) is isomorphic to the set S of non-empty finite sequences of natural numbers partially ordered by the inclusion $\subset$. Also, the regularity of ϱ means that for some (any) isomorphism τ of (ω, ϱ) onto $(S, \subset)$, $\tau(n_1) \subset \tau(n_2)$ implies $n_1 \leqslant n_2$. This indicates why we proceed as follows.

If $\exists(p) = 0$, the construction of ϱ with (i) is trivial. Suppose $\exists(p) \neq 0$.

Let us call a subset S' of S *convex* if $s_1 \in S'$, $s_2 \in S$, $s_2 \subset s_1$ imply $s_2 \in S'$. A 1-1 enumeration τ of a convex subset of S with $\lambda = \mathrm{dom}(\tau) \leqslant \omega$ is called *regular* if, for $n_1, n_2 < \lambda$, $\tau(n_1) \subset \tau(n_2)$ implies $n_1 \leqslant n_2$. Note the

[2] The reader may skip the following elementary argument up to the definition of Θ.

following elementary fact: If S_1, S_2 are finite convex subsets of S, $S_1 \subset S_2$ and τ_1 is a regular enumeration of S_1, then there is a regular enumeration τ_2 of S_2 such that $\tau_1 \subset \tau_2$.

Let $\langle s_k \colon k < \omega\rangle$ be an enumeration of S and $\langle (t_k, \exists w_k \psi_k) \colon k < \omega\rangle$ an enumeration of all pairs $(t, \exists w\psi)$ such that $t \in S \cup \{0\}$ and $lh(t)$ (= the length of t) $\in \exists(p)$. We define an increasing sequence $\langle \tau_k \colon k < \omega\rangle$ of regular enumerations of finite convex subsets of S, as well as $\langle y_k \colon k < \omega\rangle$ by induction on k as follows. Assume that $\tau_{k'}$ is defined for $k' < k$. Let X be the smallest convex subset of S such that X contains

$$rn(\tau_{k-1}) \cup \{s_m \colon m \text{ is free in } \exists w_k \psi_k\} \cup \{t_k\} \cup \{s_k\}$$

(if $k = 0$, omit $rn(\tau_{k-1})$ and if $t_k = 0$, omit $\{t_k\}$). Let $t'_k = t_k \cup \{(lh(t_k), i)\}$ for a certain (the smallest) i such that $t'_k \notin X$. By the above remark, let τ' be a regular enumeration of X such that if $k > 0$, then $\tau_{k-1} \subset \tau'$, put $y_k \overset{\mathrm{df}}{=} \mathrm{dom}(\tau') = lh(\tau')$ and finally $\tau_k \overset{\mathrm{df}}{=} \tau \cup \{(y_k, t'_k)\}$. Clearly, τ_k is again a regular enumeration of a convex subset of S.

Put $\tau = \bigcup_{k<\omega} \tau_k$. Since $s_k \in rn(\tau_k)$, τ is a regular enumeration of S. Hence the partial ordering ϱ of ω defined by $m\varrho n \Leftrightarrow \tau(m) \subset \tau(n)$ is a regular tree and we trivially have (i). Now let the 1-1 onto function $\mu \colon \omega \to \omega$ be defined by $\tau\big(\mu(m)\big) = s_m$ and put $\exists z_k \varphi_k \overset{\mathrm{df}}{=} (\exists w_k \psi_k)(\mu)$. We leave the verification of (ii)–(v) to the reader.

Note that the items ϱ, $\exists z_k \varphi_k$, y_k constructed depend only on p and the underlying similarity type.

Put $H = \{\exists z_k \varphi_k \to \varphi_k(y_k/z_k)\}$ if $\exists(p) \neq 0$ and $H = 0$ if $\exists(p) = 0$. Let Θ be the set of all finite disjunction ϑ of G, ϱ-formulas such that ϑ is a logical consequence of $T \cup H$ *with the variables held constant*. Put $\Phi = q[p, \varrho] \wedge \Theta$. Clearly, $\Phi \in S(G)$.

(II) Verification of the relationships stated in the theorem. We will assume $\exists(p) \neq 0$. Since for $q = q[p, \varrho]$ we have $n \in \exists(q)$ iff $l(n) \in \exists(p)$, it follows from (I) (iv) that $y_k \in \exists(q)$. Thus by (I) (ii) and (iii) the conditions of 0.4 are met and we may conclude that any model of T is a model of Φ. Hence by 1.1 we have

$$C_{R(G)}\big(\mathrm{Mod}(T)\big) \subset \mathrm{Mod}\,\Phi\,. \tag{8}$$

Now suppose that $\mathfrak{B}$ is a countable model of Φ. Our goal is to construct a countable model $\mathfrak{A}$ of T such that $\mathfrak{A} R(G) \mathfrak{B}$.

Let g be a winning strategy for "$\mathfrak{B} \models \Phi$". Since by (i) the set S_n of ϱ-successors of any $n \in \omega \cup \{-1\}$ is infinite and $\mathfrak{B}$ is countable, there is a $\beta_{\forall} \in {}^{\forall(q)}B$ such that

$$rn(\beta_{\forall} \upharpoonright S_n) = B \qquad \big(S_n \subset \forall(q)\big) \tag{9}$$

for every $n \in \omega \cup \{-1\}$ with $l(n)+1 \in \forall(q)$, i.e. $S_n \subset \forall(q)$. Put $\beta = \beta_\forall \cup \cup g(\beta_\forall)$. Then we have

$$\mathfrak{B} \models \vartheta[\beta]$$

for every $\vartheta \in \Theta$.

Consider the set Σ_0 of formulas $\neg\varphi$ such that φ is a G, ϱ-formula and $\mathfrak{B} \models \neg\varphi[\beta]$. Put $\Sigma = T \cup H \cup \Sigma_0$. We claim that Σ is consistent, i.e. that it has a model $(\mathfrak{A}', \alpha)$. Indeed, if not, then by the compactness theorem there is n, $0 \leqslant n < \omega$ and $\neg\varphi_i \in \Sigma_0$ for each $i < n$ such that $\neg \bigwedge_{i<n} \neg\varphi_i$ is a logical consequence of $T \cup H$ with the variables held constant; hence so is $\vartheta \overset{\mathrm{df}}{=} \bigvee_{i<n} \varphi_i$. Thus $\vartheta \in \Theta$, and consequently $\mathfrak{B} \models \vartheta[\beta]$, i.e. $\mathfrak{B} \models \varphi_i[\beta]$ for a certain $i < n$, which is a contradiction to $\neg\varphi_i \in \Sigma_0$.

Let $(\mathfrak{A}', \alpha)$ be a model of Σ, i.e. $\mathfrak{A}' \models \varphi[\alpha]$ for $\varphi \in \Sigma$. Take $n_0 \in \exists(q) \neq 0$. Note that $\mathfrak{A}' \models \big(\exists z_k \varphi_k \to \varphi_k(y_k/z_k)\big)[\alpha]$ for $k < \omega$. Hence, by (I) (v) applied for $n = n_0$, it follows that for any formula of the form $\exists z\varphi$ there is a $y = y_k \in S_{n_0}$ such that $\mathfrak{A}' \models \big(\exists z\varphi \to \varphi(y/z)\big)[\alpha]$. Therefore, by the Elementary Substructure Criterion of Tarski and Vaught [13], we may conclude that the substructure $\mathfrak{A}$ of $\mathfrak{A}'$ with domain $A = rn(\alpha)$ exists [3] and is an elementary substructure of $\mathfrak{A}'$. A fortiori, $(\mathfrak{A}, \alpha)$ is a model of Σ. Obviously, A is countable.

We claim that

$$rn(\alpha \upharpoonright S_n) = A \qquad \big(S_n \subset \exists(q)\big) \tag{10}$$

for any n such that $S_n \subset \exists(q)$. Let n be such that $S_n \subset \exists(q)$ and $a \in A$, i.e. $a = \alpha(m)$ for a certain $m < \omega$. Consider the logically valid formula $\exists v_{m'}[v_m \approx v_{m'}]$ with some $m' \neq m$. Apply (I) (v) to find k such that $\exists z_k\varphi_k \sim \exists v_{m'}[v_m \approx v_{m'}]$ and $y = y_k \in S_n$. Hence we have $\mathfrak{A}' \models (v_m \approx y_k)[\alpha]$, i.e. $a = \alpha(m) = \alpha(y_k) \in rn(\alpha \upharpoonright S_n)$, as was to be shown.

Note that the fact that $(\mathfrak{A}, \alpha)$ is a model of $\Sigma_0 \subset \Sigma$ can be expressed equivalently by saying that

(vi) $\mathfrak{A} \models \varphi[\alpha]$ implies $B \models \varphi[\beta]$ for any G, ϱ-formula φ.

Now we show that $\mathfrak{A} R(G) \mathfrak{B}$. To define a winning strategy $h\colon {}^{\exists(p)}A \times {}^{\forall(p)}B \to {}^{\forall(p)}A \times {}^{\exists(p)}B$ for "$\mathfrak{A} R(G) \mathfrak{B}$", take $\alpha'_\exists \in {}^{\exists(p)}A$, $\beta'_\forall \in {}^{\forall(p)}B$. By induction on k we define $n_k < \omega$ as follows: n_k is the smallest natural number such that $l(n_k) = k$, $n_{k-1} \varrho n_k$ if $k < 0$, and

$$\alpha'_\exists(k) = \alpha(n_k) \quad \text{for} \quad k \in \exists(p)\,, \tag{11}$$

$$\beta'_\forall(k) = \beta(n_k) \quad \text{for} \quad k \in \forall(p)\,. \tag{12}$$

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(3) The underlying similarity type may contain operation symbols.

By (9) and (10), the n_k exist. Next we define $\alpha'_\forall \in {}^{\forall(p)}A$, $\beta'_\exists \in {}^{\exists(p)}B$ by

$$\alpha'_\forall(k) = \alpha(n_k) \quad \text{for} \quad k \in \forall(p)\,, \tag{13}$$

$$\beta'_\exists(k) = \beta(n_k) \quad \text{for} \quad k \in \exists(p)\,. \tag{14}$$

Put $h(\alpha'_\exists, \beta'_\forall) \overset{\mathrm{df}}{=} (\alpha'_\forall, \beta'_\exists)$.

It is obvious from the definition that $\alpha'_\forall \upharpoonright n$, $\beta'_\exists \upharpoonright n$ depend only on $(\alpha'_\exists \upharpoonright n, \beta'_\forall \upharpoonright n)$; hence h is a p-strategy on $A \times B$. Now choose $\gamma \in \Gamma$ arbitrarily, let m be a natural number not smaller than any free variable of γ, and put $n = n_m$. σ^n maps $\{0, \ldots, m\}$ onto $\{n_0, \ldots, n_m\}$ such that $\sigma^n(k) = n_k$. Assume $\mathfrak{A} \models \gamma[\alpha'_\forall \cup \alpha'_\exists]$. This means that $\mathfrak{A} \models \gamma(\sigma^n)[\alpha]$ by (11) and (13). Hence by (vi) $\mathfrak{B} \models \gamma(\sigma^n)[\beta]$, i.e. $\mathfrak{B} \models \gamma[\beta'_\forall \cup \beta'_\exists]$ by (12) and (14). We have shown that h is a winning strategy for "$\mathfrak{A} R(G) \mathfrak{B}$".

To sum up, we have proved that for any countable $\mathfrak{B} \in \mathrm{Mod}(\Phi)$ there is a countable $\mathfrak{A} \in \mathrm{Mod}(T)$ with $\mathfrak{A} R(G) \mathfrak{B}$, or in symbols:

$$\overline{\mathrm{Mod}}(\Phi) \subset C_{R(G)}\big(\overline{\mathrm{Mod}}(T)\big)\,. \tag{15}$$

(1) and (15) imply the equalities of the theorem, q.e.d.

§ 2. Corollaries and remarks

Let us call a class of structures a PC_δ class if it consists of the reducts to the underlying similarity type ξ of all models of a countable set of sentences appropriate for some extended similarity type, and let a class be called a $PC_{\bar\delta}$-class if it is identical to the class of countable members of a PC_δ class. Theorem 1.2 easily generalizes to PC_δ classes in place of $\mathrm{Mod}(T)$ if we note the following.

Let ξ' be a countable similarity type extending ξ. Let the Lindström game be given with a set Γ of formulas of ξ. Then G defines the relations $R(G)$ and $R'(G)$ between structures of type ξ and those of ξ', respectively. Clearly $\mathfrak{A} R'(G) \mathfrak{B}$ iff $\mathfrak{A} \upharpoonright \xi R(G) \mathfrak{B} \upharpoonright \xi$ where $\mathfrak{A} \upharpoonright \xi$ is the reduct of $\mathfrak{A}$ to ξ. Hence $\big(C_{R'(G)}(K)\big) \upharpoonright \xi = C_{R(G)}(K \upharpoonright \xi)$ for any class K of structures of ξ'. This remark and an application of 1.2 to $R'(G)$ leads to

Corollary 2.1. *K is a $PC_{\bar\delta}$ class $\aleph_0$-closed under $R(G)$ iff K is the class of all countable models of some $\Phi \in S(G)$.*

Proof. The "if" part follows from 0.3 and 1.1. Suppose next that $K = \big(\overline{\mathrm{Mod}}_{\xi'}(T)\big) \upharpoonright \xi$ and K is $\aleph_0$-closed under $R(G)$. Note that $R(G)$ is reflexive. The last two facts imply $K = \bar C_{R(G)}(K)$. By 1.2, we have $\Phi \in S(G)$ such that

$$K = \bar C_{R(G)}(K) = \bar C_{R(G)}\Big(\big(\overline{\mathrm{Mod}}_{\xi'}(T)\big) \upharpoonright \xi\Big) = \Big(\bar C_{R'(G)}\big(\overline{\mathrm{Mod}}_{\xi'}(T)\big)\Big) \upharpoonright \xi$$

$$= \overline{\mathrm{Mod}}_{\xi'}(\Phi) \upharpoonright \xi = \overline{\mathrm{Mod}}_\xi(\Phi), \text{ q.e.d.}$$

Define $L(G)$ to be the class of finitary elements of $S(G)$. Inspection of the definition of $S(G)$ shows that, if p is a simple prefix in $G=(p,\Gamma)$, every finite approximation of any element of $S(G)$ belongs to $L(G)$. (In the general case, the prefix of a finite approximation ψ of an element $q\wedge\Theta$ of $S(G)$ can be extended to another finite initial segment of q such that the resulting ψ' belongs to $L(G)$; obviously ψ is logically equivalent to ψ'.)

Now let R be a relation between structures (of ξ) such that (i) $\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A}R(G)\mathfrak{B}$ for any $\mathfrak{A}$ and $\mathfrak{B}$, and (ii) $\mathfrak{A}R\mathfrak{B}$ iff $\mathfrak{A}R(G)\mathfrak{B}$ for countable $\mathfrak{A}$ and $\mathfrak{B}$. We briefly say that G is *associated* with R. Clearly, if G is associated with R, then in 2.1 $R(G)$ can be replaced by R.

The following Corollary is a version of the main result of Lindström [8] (cf. Remark 1. below).

COROLLARY 2.2 (Lindström [8]). *Assume that G is associated with R. For any sentences φ and ψ, the following two conditions are equivalent*:

(i) *For any $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}\models\varphi$ and $\mathfrak{A}R\mathfrak{B}$ imply $\mathfrak{B}\models\psi$.*

(ii) *There is $\vartheta\in L(G)$ such that $\varphi\models\vartheta\models\psi$.*

Proof. The implication (ii)⇒(i) is a direct consequence of 1.1. Now assume (i) and let $\Phi\in S(G)$ be chosen for $T=\{\varphi\}$ according to 1.2. By (i), $C_R(\mathrm{Mod}\varphi)\subset\mathrm{Mod}\psi$, hence by 1.2. $\overline{\mathrm{Mod}}(\Phi)\subset\mathrm{Mod}(\psi)$. Using 0.2 we can find a finite approximation $\vartheta\in L(G)$ of Φ such that $\vartheta\models\psi$. Since by the reflexivity of $R(G)$, by 1.2 and by 0.1 we have

$$\mathrm{Mod}\varphi\subset C_{R(G)}(\mathrm{Mod}\varphi)\subset\mathrm{Mod}\Phi\subset\mathrm{Mod}\vartheta\,,$$

we also have $\varphi\models\vartheta$, which completes the proof.

Putting $\psi=\varphi$ in 2.2 we obtain

COROLLARY 2.3 (Lindström [8]). *If G is associated with R, a sentence is preserved under R iff it is logically equivalent to an element of $L(G)$.*

Remark 1. Lindström formulates his main result differently and derives 2.2 as a consequence. Let us call R (*strictly*) *elementary* if there exists an extended similarity type $\xi'\supset\xi$ containing the "new" unary predicate symbols A, B and if there exists a set Σ of sentences of ξ' (a sentence φ of ξ') such that $\mathfrak{A}R\mathfrak{B}$ iff for some model M of Σ we have $\mathfrak{A}\simeq(M\restriction\xi)|A^M$, $\mathfrak{B}\simeq(M\restriction\xi)|B^M$ where $M|C$ denotes the substructure of M with domain C. Lindström associates sets $\Delta(G)$, $\Delta^*(G)$ of sentences of ξ with any given G; $\Delta^*(G)$ is the closure of $\Delta(G)$ under finite disjunction and conjunction. These sets are syntactically closely related to our $L(G)$; in fact, one can show in an elementary though lengthy way that every element of $\Delta^*(G)$ is logically equivalent to an element of $L(G)$ and vice versa. This last fact, of course, also follows from 2.3 and Lindström's result, which is 2.3 with $L(G)$ replaced by $\Delta^*(G)$. Lindström's main result, restricted to a countable similarity type and slightly weakened, states that for any elementary R and for any G associated with R, if $\mathfrak{A}\models\vartheta$ implies $\mathfrak{B}\models\vartheta$ for any $\vartheta\in\Delta(G)$, then there are elementary extensions $\mathfrak{A}'$, $\mathfrak{B}'$ of A and B, respectively, such that $\mathfrak{A}'R\mathfrak{B}'$. We point out that this statement, as well as Lindström's original theorem in its full strength, can be derived in a few lines from 2.2 with $L(G)$ replaced by $\Delta^*(G)$ by means of standard model theory, i.e. 2.2 is essentially equivalent to that theorem. We also note that this cannot be said of 2.3.

Consider the relation $\mathfrak{A}R\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. Let p be the alternating prefix $\langle\exists\forall\exists\forall\ldots\rangle$ and let Γ be the set of all atomic and negated atomic formulas. As Fraisse [3] has shown (c.f. [8]), $G=(p,\Gamma)$ is associated with R. Applying 2.1, we obtain Svenonius' theorem stated in the Introduction.

Next consider the relation $\mathfrak{A}R\mathfrak{B}$ iff $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, take the above p and let Γ be the set of atomic formulas. Lindström [8] shows ([4]), $G=(p,\Gamma)$ is associated with R. Clearly, every element of $S(G)$ (of $L(G)$) is a positive S-sentence (sentence) ([5]). Hence by 2.1, 2.3 we have

COROLLARY 2.4. (a) *K is a PC_δ class closed under homomorphisms iff K consists of the countable models of some positive S-sentence.*

(b) (Lyndon [9]) *A sentence is preserved under homomorphisms iff it is logically equivalent to a positive one.*

The "if" part in (a) is trivial though it is not a consequence of our earlier results.

Remark 2. We think that 2.4 (a) is new (or at least it was new when we announced it in [9]) in spite of its straightforward character. We point out again that a direct proof avoiding Lindström games can be given for 2.4. (a), which is considerably simpler than, though in essentials quite similar to, the proof of 1.2. This fact contrasts with the original approach of Lindström [8], where Lindström games play an essential role even if restricted to special cases, such as Lyndon's theorem.

Remark 3. In [8] many elementary relations are listed with Lindström games associated with them. These examples yield preservation theorems as special cases of 2.2. We now add one more item to this list. Following Feferman [2], we call $\mathfrak{B}$ an E-extension of $\mathfrak{A}$ (where E is a binary predicate symbol) if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for $a\in A$, $b\in B$, if $bE^{\mathfrak{B}}a$ then $b\in A$. Let $\mathfrak{A}R\mathfrak{B}$ denote that $\mathfrak{B}$ is an E-extension of $\mathfrak{A}$. We exhibit $G=(p,\Gamma)$ associated with R as follows. Let p be the simple alternating prefix $\langle\exists\forall\exists\forall\ldots\rangle$. Let $\langle j_i\colon i\in\forall(p)\rangle$ be a mapping such that $j_i\in\exists(p)$,

([4]) The proof uses a "Cantor-type argument".

([5]) Note that $\vee 0$ is considered a positive sentence.

$j_i < i$, and for any $j \in \mathfrak{H}(p)$ there are infinitely many i with $j_i = j$. Let Γ be the set of formulas of the form

$$[\bigwedge_{i \in n \cap \forall(p)} [v_i \mathrm{E} v_{j_i} \vee v_i \approx v_{j_i}]] \rightarrow \sigma$$

where σ is a quantifier-free formula such that every free variable of σ is $< n$. With this G, 2.3 gives a preservation theorem for E-extensions. This example shows, however, that Lindström's (our) approach does not always give the most natural preservation theorem, in particular, it does not always give the one that could be generalized to the infinitary language $L_{\omega_1 \omega}$. (For the "natural" result that can be so generalized, c.f. Feferman [2] and also [11].) Similar situations arise with other natural relations too, c.f. [11].

This limitation is connected with the circumstance that our (Lindström's) approach yields (essentially) prenex sentences as the "special" sentences in $L(G)$ (in $\Delta^*(G)$) (at least in the case where Γ contains only quantifier-free formulas), which we cannot expect to suffice in the case of $L_{\omega_1 \omega}$.

Remark 4. Note that for any G, $R(G)$ is (i) elementary, (ii) reflexive and transitive, and even more, (iii) its transitivity "can be expressed in first order terms". By (iii) we mean that if f, g are winning strategies for "$\mathfrak{A} R(G) \mathfrak{B}$", "$\mathfrak{B} R(G) \mathfrak{C}$" and f, g are represented as sequences $\langle f_k \colon k \in \mathfrak{H}(p) \rangle$, $\langle g_k \colon k \in \mathfrak{H}(p) \rangle$ of finitary operations in the domain of a structure M containing isomorphic copies of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, then the h_k in a winning strategy $h = \langle h_k \colon k \in \mathfrak{H}(p) \rangle$ for "$\mathfrak{A} R(G) \mathfrak{B}$" can be defined over M by first-order formulas using the f_k, g_k. It would not be difficult to formulate this stronger transitivity property for any strictly elementary relation. Note that a number of familiar relations have the properties (i)–(iii) and there are Lindström games G associated with them (though they are not identical to $R(G)$). We conjecture that any strictly elementary relation with (ii) and (iii) has a Lindström game associated with it.

References

[1] W. Craig, *Three uses of the Herbrand–Gentzen theorem in relating model theory to proof theory*, J. Symb. Logic 22 (1957), pp. 269–289.

[2] S. Feferman, *Persistent and invariant formulas for outer extensions*, Compositio Math. 20 (1968), pp. 20–52.

[3] R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. École Norm. Sup., 3-éme série, 71 (1954), pp. 363–388.

[4] L. A. Henkin, *The completeness of the first-order functional calculus*, J. Symb. Logic 14 (1949), pp. 159–166.

[5] — *Some remarks on infinitely long formulas*, Infinitistic Methods, proceedings of the symposium on foundations of mathematics, Warsaw 1959, New York–Oxford–London–Paris and Warsaw 1961, pp. 167–183.

[6] H. J. Keisler, *Finite approximations of infinitely long formulas*, The Theory of Models, proceedings of the 1963 international symposium at Berkeley, Amsterdam 1965, pp. 158–169.

[7] — *Infinite quantifiers and continuous games*, Applications of Model Theory to Algebra, Analysis and Probability, ed. W. A. J. Luxemburg, New York 1969, pp. 228–264.

[8] P. Lindström, *On relations between structures*, Theoria (Lund), 1966, pp. 172–185.

[9] M. Makkai, *Preservation theorems for pseudo-elementary classes*, Notices Amer. Math. Soc. 16 (1969), p. 425.

[10] — *Regular relations and Svenonius formulas*, ibid., p. 981.

[11] — *On the model theory of denumerably long formulas with finite strings of quantifiers*, J. Symb. Logic 34 (1969), pp. 437–459.

[12] L. Svenonius, *On the denumerable models of theories with extra predicates*, The Theory of Models, proceedings of the 1963 international symposium at Berkeley, Amsterdam 1965, pp. 376–389.

[13] A. Tarski, R. L. Vaught, *Arithmetical extensions of relational systems*, Compositio Math. 13 (1957), pp. 81–102.

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