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A normal space X for which $X \times I$ is not normal

by

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The purpose of this paper is to construct (without using any set theoretic conditions beyond the axiom of choice) a normal Hausdorff space X whose Cartesian product with the closed unit interval I is not normal. Such a space is often called a *Dowker space*. The question of the existence of such a space is an old and natural one [3].

In 1951, C. H. Dowker [4] proved that a normal Hausdorff space is not countably paracompact if and only if its Cartesian product with I is not normal. Other interesting equivalences are given by C. H. Dowker and M. Katětov in [4] and [8], and one is a useful tool for constructing a Dowker space. M. Katětov [8] proved there is no perfectly normal Dowker space and B. J. Ball [1] proved there is no linear Dowker space.

In [10] I proved that the existence of a Souslin line implies the existence of a Dowker space. And, more recently, I observed that almost the same proof yields: if $\varkappa$ is a regular cardinal which is not the successor of a singular cardinal, then the existence of a Souslin tree of cardinality $\varkappa$ implies the existence of a Dowker space. The existence of a Souslin line and Souslin trees of these cardinalities has been proved consistent with the usual axioms of set theory ([13], [11], [7]).

I am indebted to N. Howes [6] for the idea that a *singular* cardinal might be useful in constructing a Dowker space. Howes also introduced me to the example of A. Miščenko given in [9] which I was able to prove is not normal. But successive modification of this example led me to the Dowker space X described below.

I. The definition of X and some notation will be given. We use the usual convention that an ordinal λ *is* the set of all ordinals less than λ. An ordinal γ is *cofinal* [5] with λ if there is a subset Γ of λ order isomorphic with γ such that $\alpha < \lambda$ implies there is a $\beta \in \Gamma$ such that $\alpha \leqslant \beta$. Let $\mathrm{cf}(\lambda)$ denote the smallest ordinal cofinal with λ.

Let N denote the set of all positive integers.

Let $F = \{f\colon N \to \omega_\omega \mid f(n) \leqslant \omega_n \text{ for all } n \in N\}$.

Let $X = \{f \in F \mid \exists i \in N \text{ such that } \omega_0 < \mathrm{cf}(f(n)) < \omega_i \text{ for all } n \in N\}$.

Suppose f and g belong to F. If $f(n) < g(n)$ for all $n \in N$, we say $f < g$. If $f(n) \leqslant g(n)$ for all $n \in N$, we say $f \leqslant g$. And if $i \in N$ and $f(n) < g(n)$ for all $n \geqslant i$, we say $f <_i g$. Define $U_{f,g} = \{h \in X \mid f < h \leqslant g\}$.

The set of all $U_{f,g}$ for f and g in F is a basis for a topology on X, and we prove this space is a Dowker space. It is obviously Hausdorff.

II. We prove that $X \times I$ is not normal. We prove there is a simple sequence $D_1 \supset D_2 \supset \ldots$ of sets closed in X such that $\bigcap_{n \in N} D_n = \emptyset$, but $\bigcap_{n \in N} U_n \neq \emptyset$ if each U_n is open in X and contains D_n. Thus (as proved by Dowker in [4]), $X \times 0$ cannot be separated from $\bigcup_{n \in N} (D_n \times 1/n)$ in $X \times I$. So $X \times I$ is not normal.

For all $n \in N$, let

$$D_n = \{f \in X \mid \exists i \geqslant n \text{ such that } f(i) = \omega_i\} .$$

And for $1 < n$, let

$$C_n = \{f \in X \mid f(i) = \omega_i \text{ for all } i < n \text{ and } f(i) < \omega_i \text{ for all } i \geqslant n\} .$$

Observe that $\bigcap_{n \in N} D_n = \emptyset$. Also, each D_n is closed. For suppose $f \in X - D_n$; then $\{g \in X \mid g \leqslant f\}$ is open and does not intersect D_n.

Suppose that U_n is an open set containing D_n. Claim: $\bigcap_{n \in N} U_n \neq \emptyset$. In fact, we prove that $\bigcap_{n \in N} U_n \cap C_2 \neq \emptyset$.

LEMMA 1. *Suppose $1 < n \in N$, U is open, $f \in C_{n+1}$, and $U \supset \{h \in C_{n+1} \mid f <_{n+1} h\}$. Then there exists $g \in C_n$ such that $U \supset \{h \in C_n \mid g <_n h\}$.*

Proof. Define $k \in C_n$ by letting $k(i) = f(i)$ for all $i \neq n$ and $k(n) = 0$. Define $K = \{k_\alpha \mid \alpha < \lambda\}$ to be a maximal well-ordered family of terms of $C_n - U$ such that $k <_n k_\alpha <_n k_\beta$ for all $\alpha < \beta < \lambda$. Without loss of generality we assume $\lambda \neq 0$ since $\lambda = 0$ implies k has the property of the g in our lemma. Let g' be the term of F defined by letting $g'(i) = \sup\{k_\alpha(i) \mid \alpha < \lambda\}$ for all $i \in N$.

Clearly $\lambda \leqslant \omega_n$. Suppose $\lambda = \omega_n$. Then $g' \in C_{n+1}$ and $f <_{n+1} g'$; so, by the hypothesis of the lemma, $g' \in U$ and there exists $g^* < g'$ in F such that $U_{g^*,g'} \subset U$. But there is an $\alpha_i < \lambda$ for each $i \geqslant n$ such that $k_{\alpha_i}(i) > g^*(i)$. Let $\alpha = \sup\{\alpha_i \mid i \geqslant n\}$. Then $g^* < k_\alpha \leqslant g'$; hence $k_\alpha \in U$ which is a contradiction.

Thus $\lambda < \omega_n$. Define $g \in X$ by letting $g(i) = g'(i) + \omega_1$ for all $i \geqslant n$ and $g(i) = \omega_i$ for all $i < n$. Then $g \in C_n$. And, by the maximality of K, $U \supset \{h \in C_n \mid g <_n h\}$.

LEMMA 2. *Suppose $1 < n \in N$. There is a term f of C_2 such that $U_n \supset \{h \in C_2 \mid f < h\}$.*

Proof. Since the g of Lemma 1 depends only on f and U, denote the g guaranteed by Lemma 1 by $g_{f,U}$. By induction down we define $k_i \in C_i$ for $2 \leqslant i \leqslant n+1$. Select any $k_{n+1} \in C_{n+1}$. Since $U_n \supset D_n \supset C_{n+1}$, $U_n \supset \{h \in C_{n+1} \mid k_{n+1} <_{n+1} h\}$. So we can define $k_n = g_{k_{n+1},U_n}$; and in general, define $k_i = g_{k_{i+1},U_n}$ for $2 \leqslant i \leqslant n$. Thus, $f = k_2$ has the desired properties for Lemma 2.

LEMMA 3. $\bigcap_{n \in N} U_n \neq \emptyset$.

Proof. For $1 < n \in N$, let f_n be the f guaranteed by Lemma 2. Then for $i \in N$, select an ordinal α_i such that $\mathrm{cf}(\alpha_i) = \omega_1$ and, for all $n > 1$ $f_n(i) < \alpha_i < \omega_i$. Define $\alpha_1 = \omega_1$. Now define $g \in X$ by letting $g(i) = \alpha_i$ for all $i \in N$. Clearly $g \in U_n \cap C_2$ for $n > 1$. And $g \in X = D_1 \subset U_1$.

III. We prove X is (collectionwise) normal. Suppose $\mathcal{H} = \{H_j\}_{j \in J}$ is a collection of disjoint closed subsets of X such that $L \subset J$ implies $\bigcup_{j \in L} H_j$ is closed. We show that there is a collection $\{U_j\}_{j \in J}$ of disjoint open sets such that $U_j \supset H_j$ for each $j \in J$. This shows that X is *collectionwise normal* [2]. And, by the special case where J has exactly two members, X is normal.

Let H be the union of the members of $\mathcal{H}$. And if $U \subset F$, define t_U in F by letting $t_U(n) = \sup\{f(n) \mid f \in U\}$ for each $n \in N$. Clearly $U \supset V$ implies $t_V \leqslant t_U$.

A. Our aim will be to define for each countable ordinal α, by induction, a cover $\mathfrak{J}_\alpha$ of H by disjoint open sets having the following property: If $\beta < \alpha < \omega_1$ and $V \in \mathfrak{J}_\alpha$, then there exists a $U \in \mathfrak{J}_\beta$ such that

(1) $V \subset U$,

(2) if V intersects at least two members of $\mathcal{H}$, then $t_V \neq t_U$, and

(3) if U intersects at most one member of $\mathcal{H}$, then $U = V$.

B. First let us prove that the existence of $\mathfrak{J}_\alpha$ as described in A is sufficient to find a set of disjoint open sets $\{U_j\}_{j \in J}$ such that $U_j \supset H_j$.

Suppose $f \in H$. Since $\mathfrak{J}_\alpha$ covers H with *disjoint* open sets, $\alpha < \omega_1$ implies there is a unique $U_\alpha \in \mathfrak{J}_\alpha$ such that $f \in U_\alpha$. Since the terms of $\mathfrak{J}_\beta$ are disjoint by (1) of A, $\beta < \alpha < \omega_1$ implies $U_\alpha \subset U_\beta$; thus $t_{U_\alpha} \leqslant t_{U_\beta}$. And if U_α intersects more than one term of $\mathcal{H}$, by (2) of A, there is at least one $n \in N$ such that $t_{U_\alpha}(n) < t_{U_\beta}(n)$. Hence, since for any one n, one can move backward in ω_n only finitely many steps, there is an $\alpha_f < \omega_1$ such that U_{α_f} intersects at most one term of $\mathcal{H}$. So, by (3) of A, $\alpha_f < \beta < \omega_1$ implies $U_\beta = U_{\alpha_f}$.

For all $j \in J$, define $U_j = \bigcup_{f \in H_j} U_{\alpha_f}$. Suppose f and g belong to different terms of $\mathcal{H}$. Now there is $\alpha < \omega_1$ greater than α_f or α_g; thus the term of $\mathfrak{J}_\alpha$ to which f belongs is U_{α_f} and the term to which g belongs is U_{α_g}. Since the terms of $\mathfrak{J}_\alpha$ are disjoint and $U_{\alpha f}$ intersects only one term of $\mathcal{K}$, $U_{\alpha_f} \cap \cap U_{\alpha_g} = \emptyset$. Hence the terms of $\{U_j\}_{j\in J}$ are disjoint.

C. We now prove the existence of $\mathfrak{J}_\alpha$ as described in A.

Define $\mathfrak{J}_0 = \{X\}$.

Suppose $\mathfrak{J}_\beta$ has been defined for all $0 \leqslant \beta < \alpha < \omega_1$, and let us define $\mathfrak{J}_\alpha$.

C.1. Suppose α is a limit ordinal. If $f \in H$ and $\beta < \alpha$, define $U_f(\beta)$ to be the term of $\mathfrak{J}_\beta$ to which f belongs, and define $U_f = \bigcap_{\beta<\alpha} U_f(\beta)$. Define $\mathfrak{J}_\alpha = \{U_f |\ f \in H\}$. By Lemma 4 below, each U_f is open since α is countable. And the terms of $\mathfrak{J}_\alpha$ are disjoint. If $\beta < \alpha$, $U_f(\beta)$ is the term of $\mathfrak{J}_\beta$ containing U_f. If U_f intersects two terms of $\mathcal{K}$, then so do $U_f(\beta)$ and $U_f(\beta+1)$. So $t_{U_f(\beta+1)} \neq t_{U_f(\beta)}$ and $t_{U_f} \leqslant t_{U_f(\beta+1)} \leqslant t_{U_f(\beta)}$. Hence, $t_{U_f} \neq t_{U_f(\beta)}$ and (2) of A is satisfied. If $U_f(\beta)$ intersects only one term of $\mathcal{K}$, then $U_f(\beta) = U_f(\gamma)$ for all $\beta < \gamma < \alpha$; hence $U_f = U_f(\beta)$ and (3) of A is satisfied.

LEMMA 4. *If $\{U_n\}_{n\in N}$ is a collection of open sets, then $\bigcap_{n\in N} U_n$ is open.*

Proof. Suppose $f \in \bigcap_{n\in N} U_n$; for all $n \in N$ there exists $g_n \in F$ such that $g_n < f$ and $U_{g_n,f} \subset U_n$. Let $g \in F$ be defined by letting $g(i) = \sup\{g_n(i) |\ n \in N\}$ for each $i \in N$. Then, since $g \leqslant f$ and $\mathrm{cf}(g(i)) \leqslant \omega_0 < \mathrm{cf}(f(i))$ for each $i \in N$, $g < f$. So $U_{g,f} \subset \bigcap_{n\in N} U_n$. Thus $\bigcap_{n\in N} U_n$ is open.

C.2. Suppose $\alpha = \beta+1$. For all $U \in \mathfrak{J}_\beta$ we shall define a set $\mathfrak{J}_U$ of disjoint open subsets of U covering $U \cap H$ such that $V \in \mathfrak{J}_U$ implies (2) and (3) of A are satisfied. Clearly $\mathfrak{J}_\alpha = \bigcup_{U\in\mathfrak{J}_\beta} \mathfrak{J}_U$ thus has the desired properties.

Assume $U \in \mathfrak{J}_\beta$ and let t denote t_U.

Case 1. If U intersects at most one term of $\mathcal{K}$, then define $\mathfrak{J}_U = \{U\}$. Clearly (3), and trivially (2), of A are satisfied. In all other cases we need only satisfy (2) since (3) then becomes trivial.

Case 2. Assume U intersects at least two members of $\mathcal{K}$ and there is an $i \in N$ such that $\mathrm{cf}(t(i)) \leqslant \omega_0$. Since $U \neq \emptyset$, $t(i) \neq 0$. If $0 < \mathrm{cf}(t(i)) < \omega_0$, then $t(i) = \gamma+1$ for some ordinal γ. But $f \in U$ implies $\mathrm{cf}(f(i))$ is uncountable and $f(i) \leqslant t(i)$, so $f(i) < \gamma$; but this contradicts $t = t_U$. Thus $\mathrm{cf}(t(i)) = \omega_0$. Let $\{\lambda_n\}_{n\in N}$ be an increasing sequence of terms of $t(i)$ cofinal with $t(i)$. Define $V_1 = \{f \in U |\ f(i) \leqslant \lambda_1\}$ and, for $1 < n < \omega_0$, define $V_n = \{f \in U |\ \lambda_{n-1} < f(i) \leqslant \lambda_n\}$. Then $\mathfrak{J}_U = \{V_n |\ n \in N\}$ is a set of disjoint open subsets of U covering U. And $n \in N$ implies $t_{V_n}(i) < t(i)$. Thus $\mathfrak{J}_U$ has the desired properties.

Case 3. Assume U intersects at least two members of $\mathcal{K}$ and $\mathrm{cf}(t(n)) > \omega_0$ for all $n \in N$. This is the hard case and we need a lemma.

LEMMA 5. *There exists an $f \in F$ such that $f < t$ and $\{h \in U |\ f < h\}$ intersects at most one term of $\mathcal{K}$.*

From Lemma 5, we define $\mathfrak{J}_U$ in Case 3 as follows. If $M \subset N$, define

$$V_M = \{h \in U |\ h(n) \leqslant f(n) \text{ for all } n \in M \text{ and } h(n) > f(n) \text{ for all } n \in (N-M)\}.$$

Observe that, if $t_{V_M} = t$, then $M = 0$. But V_0 intersects at most one term of $\mathcal{K}$. So, if $\mathfrak{J}_U = \{V_M |\ M \subset N\}$, then $\mathfrak{J}_U$ has the desired properties.

In order to prove Lemma 5, we need Lemma 6. For each $n \in N$, define

$$U_n = \{h \in U |\ \mathrm{cf}(h(i)) \leqslant \omega_n \text{ for all } i \in N\}; \text{ clearly } U = \bigcup_{n\in N} U_n .$$

LEMMA 6. *Suppose $n \in N$. There exists $g \in F$ such that $g < t$ and $\{h \in U_n |\ g < h\}$ intersects at most one term of $\mathcal{K}$.*

Before proving Lemma 6, let us show that Lemma 6 implies Lemma 5. Let g_n be the g guaranteed by Lemma 6 for n. Define $f \in F$ by letting $f(i) = \sup\{g_n(i) |\ n \in N\}$ for each $i \in N$. Then $f < t$ by the assumption of Case 3. Let us show that $\{h \in U |\ f < h\}$ intersects at most one term of $\mathcal{K}$. Suppose $f < h \in U \cap H$ and $f < k \in U \cap H$. There exists i and j in N such that $h \in U_i$ and $k \in U_j$; let $n = i+j$. Then $g_n < h$ and $g_n < k$ and $h \in U_n$ and $k \in U_n$; thus h and k belong to the same term of $\mathcal{K}$.

Proof of Lemma 6. Assume that Lemma 6 is false; that is, for all $f \in F$ such that $f < t$, there are terms h and k of U_n such that $f < h$ and $f < k$ and h and k belong to *different* terms of $\mathcal{K}$.

We now need more notation. Remember, n is fixed.

For $i \leqslant n$, define $M_i = \{j \in N |\ \mathrm{cf}(t(j)) = \omega_i\}$. Define $M = \{j \in N |\ \mathrm{cf}(t(j)) > \omega_n\}$. By Case 3, $N = \bigcup_{i\leqslant n} M_i \cup M$.

Let $R = \{r\colon (1, 2, \ldots, n) \to \omega_n |\ r(i) < \omega_i \text{ for all } i \in (1, \ldots, n)\}$. The cardinality of R is clearly ω_n. So we can define a set $\{r_\lambda\}_{\lambda<\omega_n}$ of terms of R such that $\lambda < \omega_n$ and $r \in R$ implies there is a γ such that $\lambda < \gamma < \omega_n$ and $r_\gamma = r$.

For all $i \leqslant n$ and $j \in M_i$, choose a subset $\{s_{j,\sigma}\}_{\sigma<\omega_i}$ of $t(j)$ cofinal with $t(j)$ such that $\sigma < \gamma < \omega_i$ implies $s_{j,\sigma} < s_{j,\gamma}$.

To return to the proof, we wish to select by induction for each ordinal $\lambda < \omega_n$ an $f_\lambda \in F$ and h_λ and k_λ belonging to U_n as follows. Define f_0 by letting $f_0(j) = s_{j,r_0(i)}$ for $j \in M_i$ and $i \leqslant n$ and letting $f_0(j) = 0$ for $j \in M$. Then choose $h_0 \in U_n$ and $k_0 \in U_n$ belonging to different terms of $\mathcal{K}$ such that $f_0 < h_0$ and $f_0 < k_0$; such a choice is possible since $f_0 < t$ and we assumed that Lemma 6 is false. Suppose h_λ and k_λ in U_n have been chosen for all $\gamma < \lambda < \omega_n$. Define f_λ by letting $f_\lambda(j) = s_{j,r_\lambda(i)}$ for $j \in M_i$ and $i \leqslant n$, and $f_\lambda(j) = \sup\{h(j) |\ h \in \bigcup_{\gamma<\lambda} \{h_\gamma, k_\gamma\}\}$ for $j \in M$. If h is h_γ or k_γ for some $\gamma < \lambda$, then $h \in U_n$ by our induction hypothesis; so $h(j) < t(j)$ for $j \in M$. Since $\lambda < \omega_n$, the definition of M gives $f_\lambda(j) < t(j)$ for all $j \in M$. And $f_\lambda(t) < t(j)$ for $j \in N-M$. Hence $f_\lambda < t$. Thus, by our assumption that Lemma 6 is false, we can choose $h_\lambda \in U_n$ and $k_\lambda \in U_n$ such that $f_\lambda < h_\lambda$ and $f_\lambda < k_\lambda$ and h_λ and k_λ belong to *different* terms of $\mathcal{H}$. The facts we need to remember are:

(a) $\lambda < \omega_n$ and $i \leqslant n$ and $j \in M_i$ implies $s_{j,r_\lambda(i)} < h_\lambda(j) \leqslant t(j)$ and $s_{j,r_\lambda(i)} < k_\lambda(j) \leqslant t(j)$.

(b) $\gamma < \lambda < \omega_n$ and $j \in M$ implies $h_\gamma(j) < k_\lambda(j) < h_{\lambda+1}(j) < t(j)$ and $h_\gamma(j) < k_\lambda(j) < h_{\lambda+1}(j) < t(j)$.

(c) h_λ and k_λ belong to different terms of $\mathcal{H}$.

Define $g \in F$ by letting $g(j) = t(j)$ for $j \in N-M$ and $g(j) = \sup\{h_\lambda(j) \mid \lambda < \omega_n\}$ for $j \in M$. If $j \in N-M$, then $\operatorname{cf}(g(j)) > \omega_0$ by our assumption of Case 3, and $\operatorname{cf}(g(j)) \leqslant \omega_n$ by our definition of M. For $j \in M$, $\operatorname{cf}(g(j)) = \omega_n$ by (b). Hence $g \in X$, and, by (a) and (b), $g \leqslant t$.

By our definition of $\mathcal{H}$, since $g \in X$, there exists $f \in F$ such that $f < g$ and $U_{f,g}$ intersects at most one term of $\mathcal{H}$.

Since $f < g$ and $g \leqslant t$, $f < t$.

For $i \leqslant n$ and $j \in M_i$, since $\{s_{j,\sigma}\}_{\sigma < \omega_i}$ is cofinal with $t(j)$ and $f(j) < t(j)$, we can choose $\sigma_j < \omega_i$ such that $f(j) < s_{j,\sigma_j}$. Then let $\mu_i = \sup\{\sigma_j \mid j \in M_i\}$. Since M_i is countable, $\mu_i < \omega_i$. Define $r \in R$ by letting $r(i) = \mu_i$ for all $i \leqslant n$. Then $f(j) < s_{j,r(i)}$ for all $j \in M_i$.

For $j \in M$ there is a $\sigma_j < \omega_n$ such that $h_{\sigma_j}(j) > f(j)$. Let $\sigma = \sup\{\sigma_j \mid j \in M\}$. Then $h_\gamma(j) > f(j)$ and $k_\gamma(j) > f(j)$ for all $j \in M$ and $\gamma > \sigma$ by (b).

Choose γ with $\sigma < \gamma < \omega_n$ and $r_\gamma = r$. If $j \in M$, then $f(j) < h_\gamma(j) < g(j)$. And, if $j \in M_i$ for some $i \leqslant n$, then $f(j) < s_{j,r(i)} = s_{j,r_\gamma(i)} < h_\gamma(j) \leqslant t(j) = g(j)$. Thus $f < h_\gamma \leqslant g$; and similarly $f < k_\gamma \leqslant g$. But this contradicts the fact that $U_{f,g}$ intersects at most one term of $\mathcal{H}$.

IV. Let us comment. The construction of a Dowker space makes one wonder about other, perhaps nicer Dowker spaces. There is nothing unique about the space X described here, but all other Dowker spaces I know how to construct are roughly as bad as X. Consider the following questions:

1. Does there exist a 1st-countable Dowker space?
2. Does there exist a separable Dowker space?
3. Does there exist a cardinality $\aleph_1$ Dowker space?

It is consistent with the usual axioms of set theory that the answer to each of the three questions is *yes*.(1) But I do not know how to construct such spaces using only the axiom of choice. I conjecture that (3) having a *no* answer is equivalent to Souslin's conjecture [12]. I feel (2) is a question about the cardinality of c rather than about Dowker spaces. There is probably a non-consistency example for (1).

Another question one might ask is:

4. Does there exist a *realcompact* Dowker space?

The Dowker space constructed in (5) using a Souslin line is realcompact. But the space X (as well as the Dowker spaces constructed using Souslin trees of cardinality greater than $\aleph_1$) is not realcompact. Thus a *yes* answer to (4) is again only a consistency result.

To see that X is not realcompact let us look at the space $X' = \{f \in F \mid \operatorname{cf}(f(n)) > \omega_0 \text{ for all } n\}$ for this space is interesting in itself. Mr. P. Nyikos pointed out to me that X' *is* the realcompactification of X. Clearly $X \neq X'$ but X is dense in X'. By Lemma 5 every continuous real valued function on X can be extended to X'. And X' is realcompact (and paracompact) for, given an open cover $\mathcal{U}$ of X', there is an open refinement of $\mathcal{U}$ covering X' with *disjoint* open sets. Let us prove this.

Suppose $g \in G$ and $p \in X'$. Define $t(p, g)$ to be the point t of X' such that $t(n) = \omega_n$ when $p(n) \geqslant g(n)$ and $t(n) = p(n)$ when $p(n) < g(n)$. Define $U(p, g) = \{q \in X' \mid q(n) \geqslant g(n)$ when $p(n) \geqslant g(n)$ and $q(n) = p(n)$ when $p(n) < g(n)\}$. Since $\mathcal{U}$ covers $t(p, g)$, there exists a term $h(p, g) \geqslant g$ in G such that $U(t(p, g), h(p, g))$ is a subset of some term of $\mathcal{U}$.

Suppose $p \in X'$. Let $f_p, \omega_1 \to G$ be defined by:

(a) $f_p(0)$ is the term of F all of whose coordinates are 0.

(b) $f_p(\alpha+1) = h(p, f_p(\alpha))$.

(c) $f_p(\alpha)(n) = \sup\{f_p(\beta)(n) \mid \beta < \alpha\}$ when α is a limit ordinal.

If $p \in X'$ and $\alpha < \omega_1$, define $N(p, \alpha) = \{n \in N \mid p(n) < f_p(\alpha)(n)\}$. Then $\alpha < \beta$ implies $N(p, \alpha) \subset N(p, \beta)$; and if $p \notin U(p, f_p(\alpha))$, then $N(p, \alpha) \neq N(p, \beta)$. So there is a smallest ordinal α_p such that $p \in U(p, f_p(\alpha_p))$.

Define $\mathcal{V} = \{U(p, f_p(\alpha_p)) \mid p \in X'\}$. Clearly $\mathcal{V}$ is an open refinement of $\mathcal{U}$ covering X'. In order to show that the terms of $\mathcal{V}$ are disjoint, assume a point $r \in U(p, f_p(\alpha_p)) \cap U(q, f_q(\alpha_q))$.

Recall that $\beta \leqslant \alpha_p$ and $p(n) < f_p(\beta)(n)$ implies $r(n) = p(n)$, and $\beta \leqslant \alpha_p$ and $p(n) \geqslant f_p(\beta)(n)$ implies $r(n) \geqslant f_p(\beta)(n)$; and similarly for q in place of p. Without loss of generality we assume $\alpha_p \leqslant \alpha_q$.

Hence $\beta \leqslant \alpha_p \leqslant \alpha_q$ and $f_p(\beta) = f_q(\beta)$ implies $t(p, f_p(\beta)) = t(q, f_q(\beta))$ and thus $f_p(\beta+1) = f_q(\beta+1)$. So by induction $f_p(\alpha_p) = f_q(\alpha_p)$ and $t(p, f_p(\alpha_p))$

(1) The example described in [10] has cardinality $\aleph_1$ and with a simple modification can be made 1st-countable. Thus (1) and (3) exist in any model of set theory (such as $V = L$) which allows an $\aleph_1$ Souslin tree.

To prove that (2) is consistent with the usual axioms of set theory, assume $M_{\aleph_3}$, a Martin's axiom. Then one can have both (a) an $\aleph_2$ Souslin tree T and (b) an $\aleph_2$ subset S of the reals every subset of which is a relative F_σ [11]. Using T of (a) and the methods of [10], one can build an $\aleph_2$ Dowker space Y. And using S of (b) one can, in a cannonical way, add a countable discrete set C to S and topologize $C \cup S$ to be a normal space with $\overline{C} \supset S$ and S being discrete in itself. Now identify S and Y in some one-to-one fashion. The space $C \cup Y$ thus formed is a separable Dowker space.

$= t(q, f_q(a_p))$. If $q \in U(q, f_q(a_p))$ then $U(p, f_p(a_p)) = U(q, f_q(a_q))$. Otherwise there is an n such that $q(n) < f_q(a_p)(n)$ but $p(n) \geqslant f_p(a_p)(n)$. But $f_p(a_p) = f_q(a_p)$ so $r(n) < f_q(a_p)(n)$ and $r(n) \geqslant f_p(a_p)(n)$ which is a contradiction.

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