Dimension raising maps for which polyhedra are mapped to polyhedra*

by

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If $I = [0, 1]$ and $S = I \times I$ then there exists a map f from I onto S such that the image of every polyhedron is a topological polyhedron. This suggests dimension raising mappings from one complex to another may map tame sets onto tame sets.

THEOREM 1. *There exists a map f from I onto S such that the image of every subinterval is a 2-cell.*

Proof. Consider E. H. Moore's Crinkly curve C^3, [1]. It is a map f of I onto S given as the uniform limit of a sequence of continuous maps

$$f_n(t) = \big(\varphi_n(t), \psi_n(t)\big), \qquad t \in I$$

whose images are curves $C_n^3 \subset S$. In the definition of C_n^3, I is partitioned into 3^{2n} equal subintervals and S into the same number of congruent squares. The partitions for C_{n+1}^3 refine those of C_n^3. The construction of C_n^3 is such that if $f_n(t_a)$ is an endpoint (Moore's nodes) of a line segment of C_n^3 in a square of the corresponding subdivision of S then $f_m(t_a) = f_n(t_a)$ for all $m \geqslant n$. Further if I' is an interval of a partition of I corresponding to the square S' of a partition of S, then $f(I') = S'$.

Let $[a, b]$ be a subinterval of I. Now by the Hahn–Mazurkiewicz Theorem, as $f([a, b])$ is a Peano space, it is compact connected and locally connected. Thus by Theorem 13.1, p. 160, Newman [2], $\mathrm{int}(f[a, b])$ is uniformly locally connected. If we assume that $\mathrm{int}(f[a, b])$ is simply connected then $\mathrm{Bd}(f[a, b])$ is a simple closed curve by Theorem 16.2, p. 167, Newman [2]. Then by the Schoenflies Theorem $f([a, b])$ would be a 2-cell.

To see that $\mathrm{int}(f[a, b])$ is simply connected assume there exists a simple closed curve $C \subset \mathrm{int}(f[a, b])$ which is not homotopic to a point in $\mathrm{int}(f[a, b])$. Let p be a point in the bounded complementary domain of C in E^2 which does not belong to $\mathrm{int}(f[a, b])$. Further suppose $p \notin f[a, b]$

* I am indebted to R. D. Anderson for suggesting E. H. Moore's Crinkly curves.

for if not then $p \in \mathrm{Bd} f[a, b]$. So every neighborhood of p meets $E^2 \setminus f[a, b]$. Thus there is a $p' \in E^2 \setminus f[a, b]$ in the bounded complementary domain of C. So there is an open set U_p about p such that $U_p \cap f[a, b] = \emptyset$. Therefore there exists an n and map f_n with corresponding partition of I into subintervals such that for some subinterval of this partition, say I_n^a, $f(I_n^a) \subset U_p$. But then from the definition of C and $f_n([a, b])$, $I_n^a \subset [a, b]$ which is not possible. Therefore $\mathrm{int}(f[a, b])$ is simply connected.

THEOREM 2. *Let $f\colon I \to S$ be the map for Moore's Crinkly curve C^3 and let $\{R_i\}$, $i = 1, 2, \ldots, p$, be a finite disjoint sequence of closed intervals in I. Then*

$$\bigcup_{i=1}^{p} f(R_i) = f\Big(\bigcup_{i=1}^{p} R_i\Big)$$

is tame.

Proof. By induction and repeated use of theorems such as 11.7, Wilder [4], p. 31, and 4.42, Whyburn [3], p. 40, together with radial extension of homeomorphisms on boundary of disks, it suffices to show if $f(R_i)$ and $f(R_j)$, $i \neq j$, are 2-cells which meet in their boundaries (by the definition of f they do not meet in their interiors) then $\mathrm{Bd} f(R_i) \cap \cap \mathrm{Bd} f(R_j)$ consists of at most two components. For suppose not, $\mathrm{Bd} f(R_i) \setminus (\mathrm{Bd} f(R_i) \cap \mathrm{Bd} f(R_j))$ consists of open intervals, choose two of these intervals in $\mathrm{Bd} f(R_j)$ not accessible in $E^2 \setminus (f(R_i) \cup f(R_j))$ from unbounded complimentary domain of $\mathrm{Bd} f(R_i) \cup \mathrm{Bd} f(R_j)$. Choose two corresponding closed intervals in $\mathrm{Bd} f(R_i)$ which together with the open intervals form two simple closed curves not accessible in the same sense. These simple closed curves bound open disks not accessible as before and which do not meet $f(R_i) \cup f(R_j)$. Now there exists an n, f_n and partition of I into 3^{2n} intervals such that both disks contain a square in the corresponding partition of S. But as before it is easily seen that this is not possible.

References

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Involutions on solenoidal spaces

by

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1. Introduction. A *weak solenoidal sequence* (*solenoidal sequence*) of closed manifolds is an inverse limit sequence (X, f) such that each factor space X_n is a closed manifold and each bonding map $f_m^n\colon X_n \to X_m$ is a covering map (regular covering map). The limit space X_∞ is called a *weak solenoidal space* (*solenoidal space*).

In section 5, we present a general technique for constructing weak solenoidal spaces from solenoidal spaces. Suppose that (X, f) is a solenoidal sequence such that each factor space X_n admits a free involution that commutes with the bonding maps. These involutions induce an involution on the solenoidal space X_∞; moreover, if Y_∞ is the orbit space of this free involution on X_∞, then Y_∞ is a weak solenoidal space.

The importance of this technique is not only that we can construct new examples of weak solenoidal spaces, but we can obtain a keen insight into the internal structure of the spaces. Moreover, if we can construct a weak solenoidal space in a geometric manner and then show that we can obtain the same space as the orbit space of a known free involution on a solenoidal space, then we have tools to investigate both the global and local properties of the spaces.

We carry out this program in section 6, where we present a weak solenoidal space $M_\infty = \lim(M, f)$ which has the following properties: (1) each factor space M_n is homeomorphic to the Klein bottle; (2) each bonding map f_n^{n+1} is regular (although compositions of bonding maps are not regular); (3) the fundamental groups of any two path components of M_∞ are isomorphic; (4) M_∞ is not homogeneous; (5) there are exactly two different homeomorphism classes of path components, with only one path component in the first class; and (6) M_∞ is double-covered by the product of S^1 and the dyadic solenoid.

In section 3, we give a convenient characterization of the path component of a weak solenoidal space; this characterization is a valuable tool in the succeeding sections. In the process we obtain some interesting results (in the general theory of inverse limit spaces) concerning the