A solenoidal and monothetic minimally almost periodic group

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A topological group G is called *monothetic* if it contains a dense infinite cyclic subgroup.

A topological group G is called *solenoidal* if there is a continuous injective homomorphism of the additive group of the real numbers with its usual interval topology into G, with dense image. For more details the reader is referred to [1] and [3], and the references cited there.

Clearly, there is a 1-1 correspondence between complete monothetic groups with given dense infinite cyclic subgroup on one hand and group topologies on $\boldsymbol{Z}$ on the other hand; likewise for complete solenoidal groups with given dense copy of $\boldsymbol{R}$, and group topologies on $\boldsymbol{R}$.

A monothetic and solenoidal group G is commutative hence its characters *separate points*, that is: for each $x \in G$, there is a homomorphism $\chi\colon G \to \boldsymbol{R}/\boldsymbol{Z}$ such that $\chi(x) \neq 0 \bmod 1$.

If G is solenoidal or monothetic and G is locally compact, then the continuous characters separate points, too. We extend the idea of minimal almost periodicity, as defined in [2] as follows.

DEFINITION. A topological group G is said to be *minimally almost periodic* if and only if the only continuous homomorphism of G into any compact group is the trivial homomorphism.

Apparently an abelian topological group is minimally almost periodic if and only if the only continuous character is the zero-character.

The theorem in this paper proves that there exists a metric solenoidal and monothetic group without continuous characters, equivalently, that there exists a topology on the real numbers, weaker than the usual topology by which it is a metrizable topological group without continuous characters.

Such a group cannot, of course, be locally compact.

S. Rolewicz in [3] has exhibited an example of a not locally compact metric monothetic group.

In fact, be proved that a certain subgroup G of the infinite-dimensional torus $\boldsymbol{T}^\infty = (\boldsymbol{R}/\boldsymbol{Z})^\infty$ is such, if endowed with a suitable metric.

Let $|\cdot|$ be the norm on $\boldsymbol{R}/\boldsymbol{Z}$ defined by $|x+\boldsymbol{Z}| = \min\{|x+z|:\ z \in \boldsymbol{Z}\}$. Let G be the subgroup of $\boldsymbol{T}^\infty$ consisting of all sequences $\{x_n\}$ such that $\lim|x_n| = 0$ and define a norm on G by $\|\{x_n\}\| = \max\{|x_n|:\ n \in \boldsymbol{N}\}$. G becomes then a complete not locally compact group which contains an element $\{\lambda_n+\boldsymbol{Z}\}$ that generates a dense subgroup of G. G is solenoidal. To see this, define Λ: $\boldsymbol{R}\to G$ by $\Lambda(r) = \{\lambda_n r+\boldsymbol{Z}\}$. It is easy to check that Λ is continuous. We will prove

THEOREM. *There exists a discrete infinite cyclic subgroup A of G, such that $A \cap \Lambda(\boldsymbol{R}) = \{0\}$ and such that G/A is minimally almost periodic and monothetic and solenoidal.*

To prove this we will need the following

LEMMA. *The natural injection $G\to\boldsymbol{T}^\infty$ is the injection of G into its almost periodic compactification, or, equivalently, each continuous character on G is extendable to $\boldsymbol{T}^\infty$.*

Firstly, we prove the lemma.

Let χ be a character on G, that is, a continuous homomorphism $G\to\boldsymbol{R}/\boldsymbol{Z}$. Restricted to the subgroup $\boldsymbol{T}_p$ of G consisting of all sequences $\{x_n\}$ such that $x_n = 0$ for $n \neq p$, it is of the form $\boldsymbol{R}/\boldsymbol{Z}\to\boldsymbol{R}/\boldsymbol{Z}$: $x\to N_p x$, with $N_p \in \boldsymbol{Z}$.

Suppose there exists an infinite number of p such that $N_p \neq 0$. Because χ is continuous, there exists an $\varepsilon < 1/4$, such that $\|\{x_n\}\| < \varepsilon$ implies $\chi(\{x_n\}) \in (-1/3,\ +1/3)+\boldsymbol{Z}\subset \boldsymbol{R}/\boldsymbol{Z}$. Let $p_1, p_2, \ldots, p_k \in \boldsymbol{N}$ be such that $N_{p_i} \neq 0$ for $1 \leqslant i \leqslant k$ and $k = [6/\varepsilon]+1$. For each i, $1 \leqslant i \leqslant k$ choose $z_i \in (-1/3,\ +1/3)+\boldsymbol{Z}$ such that $N_{p_i}z_i = \varepsilon/2+\boldsymbol{Z}$. Then $\sum_{i=1}^{k} N_{p_i}z_i = \varepsilon k/2+\boldsymbol{Z}$.

Now $6/\varepsilon\cdot\varepsilon/2 \leqslant \varepsilon k/2 < (6/\varepsilon+1)\varepsilon/2$, hence $1/3 \leqslant \varepsilon k/2 < 1/3+1/8$, hence $\varepsilon k/2+\boldsymbol{Z} \notin (-1/3,\ +1/3)+\boldsymbol{Z}$. If we define $\{x_n\} \in G$ by $x_{p_i} = z_i$ for $1 \leqslant i \leqslant k$ and $x_n = 0$ if $n \neq p_i$ for all i, we arrive at a contradiction.

Clearly every character of G is defined by its restriction to $\boldsymbol{T}_p$, $p \in \boldsymbol{N}$, because the union of all $\boldsymbol{T}_p$ generates a dense subgroup of G. This proves the lemma.

We turn now to the proof of the theorem.

Choose an infinite sequence t_n of rationally independent irrational numbers in the interval $(1/4,\ 1/2)$. Define $a = \{x_n\}$ as follows: $x_n = t_n/n+\boldsymbol{Z} \in \boldsymbol{R}/\boldsymbol{Z}$. For any $n \in \boldsymbol{N}$ holds $\|na\| \leqslant 1/4$. Hence a generates a discrete subgroup of G. We call this subgroup A. Let r be such that $\Lambda(r) \in A$, so let $k \in \boldsymbol{Z}$ be such that $\lambda_n r = kx_n$, for all $n \in \boldsymbol{N}$. To prove that $r = 0$, assume the contrary. Then $\lambda_n = k/r\cdot x_n$, for n large enough, hence $\lambda_n(\lambda_{n+1})^{-1} = |x_n|(|x_{n+1}|)^{-1}$. This contradicts the construction in [3] of the sequence λ_n, as $\lambda_n(\lambda_{n+1})^{-1}$ goes to infinity, whereas $|x_n|(|x_{n+1}|)^{-1} < 4$.

Let q denote the quotient map $G\to G/A$, and let χ be a character on G/A, then χq is a character on G that is zero on A. It has then a continuous extension to $\boldsymbol{T}^\infty$, which likewise has A in its kernel, but A is dense in $\boldsymbol{T}^\infty$, hence $\chi q = 0$ and hence $\chi = 0$, because q is onto. So G/A is minimal almost periodic and, because $q\Lambda$ is an injection and $\Lambda(\boldsymbol{R})$ is dense in G, G/A is a solenoid. Clearly G/A is also monothetic. We have proved the theorem.

The theorem may give us some information about the structure of solenoidal groups, in the following sense. Let $\mathcal{T}$ be a group topology on $\boldsymbol{Z}$, with neighborhoodbasis at 0: $\{V_i:\ i \in I\}$. Then we can define a group topology $\mathcal{T}'$ on $\boldsymbol{R}$, by taking as basis of neighborhoods at 0: $\{rV_i+ +(-d,\ +d):\ i \in I,\ d \in \boldsymbol{R},\ d > 0\}$, in which r is a fixed real number. Such a topology $\mathcal{T}'$ on $\boldsymbol{R}$ we call a characteristic topology, relative r and $\mathcal{T}$. One may ask: are all topologies on $\boldsymbol{R}$ that are weaker than the usual interval topology on $\boldsymbol{R}$ characteristic? (This question was first asked by K. H. Hofmann). The answer is no. Indeed, if $\mathcal{S}$ is a characteristic topology on $\boldsymbol{R}$, then $\boldsymbol{R}$ has a continuous character, with respect to $\mathcal{S}$. Conversely, if $\boldsymbol{R}$ has a character χ, continuous with respect to $\mathcal{S}$, then there is a local inverse to χ, in other words, $\boldsymbol{R}$ with the topology $\mathcal{S}$ is a fibre bundle over $\boldsymbol{R}/\boldsymbol{Z}$, with $\ker\chi$ and the induced topology on $\ker\chi$ as fibre (cf. [4], 7.4.), hence $\mathcal{S}$ is characteristic.

Remark. If a group topology $\mathcal{S}$ is weaker than the usual topology on $\boldsymbol{R}$, and $\boldsymbol{R}$ has no continuous characters with respect to $\mathcal{S}$, then $\{0\}$ and $\boldsymbol{R}$ are the only closed subgroups of $\boldsymbol{R}$ with respect to $\mathcal{S}$. To see this, let M be a closed subgroup of $\boldsymbol{R}$ with respect to $\mathcal{S}$. $M \neq \{0\}$, $M \neq \boldsymbol{R}$. Then M is infinite cyclic and $\boldsymbol{R}/M = \boldsymbol{R}/\boldsymbol{Z}$, algebraically, whereas the lefthand member carries a weaker topology than the usual compact topology of $\boldsymbol{R}/\boldsymbol{Z}$. So $\boldsymbol{R}/M$ is homeomorphic to $\boldsymbol{R}/\boldsymbol{Z}$ and the quotient map is a continuous character, which is a contradiction.

References

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