References


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On Kan extensions of cohomology theories and Serre classes of groups

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1. Introduction. This paper constitutes a continuation of the investigation initiated in [4], [5]. In those papers we introduced a process, involving the Kan extension of a functor, for extending a cohomology theory from a category $J_0$ of based topological spaces to a larger category $J_1$. This process generalizes a characterization of Čech cohomology which has been noted by Eilenberg and Steenrod and studied by Dold. However, the process partakes far more of the spirit of Kan's work on extending functors than of the original description of Čech cohomology, so that the examples of cohomology theories expressible as Kan extensions take one very far from Čech cohomology, while retaining a certain generalized continuity property.

We should mention here that Lee and Raymond [11] have studied generalized Čech theories in a somewhat different sense, more strongly motivated by the classical description of Čech theory. There is some small overlap with the present authors' work, and a comparison of the two approaches will form the subject of a later paper (1).

A principal concern in [4], [5] is that of deciding under what conditions the Kan extension (1), $h_1$, to $J_1$ of a cohomology theory $h$ on $J_0$ (or maximal extension in the terminology of [2]) is itself a cohomology theory. We always require that the categories $J$ considered suitable for supporting a cohomology theory be admissible; that is, they should be non-empty full subcategories of the category of based spaces and based maps, they should admit mapping cones, and should contain entire homotopy types. We can state the axioms for $h$ or, more precisely, $(h, \sigma)$, where $\sigma$ is the suspension transformation $h^* \mapsto h^{*+1}$, to be a cohomology theory in any admissible category; but the Kan extension of a cohomology theory from an admissible category $J_0$ to an admissible category $J_1$ may well fail to be a cohomology theory.

After some preliminary algebraic argument in Section 2, we formulate a criterion for the Kan extension


(2) We use here the notation $h$, in preference to the $\phi$ notation of [4], [5].

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of a cohomology theory to satisfy the axioms in Sections 2 and 3. This criterion is much more general than that of [5] and presumably comes reasonably close to constituting a set of necessary and sufficient conditions for any cohomology theory on $\mathcal{J}_1$ to extend to a cohomology theory on $\mathcal{J}_0$. It divides itself into two parts; there is a condition for the extended theory to satisfy the exactness axiom, which is expressed by means of a local pull-back property (Theorem 2.14), and a condition for the extended theory to satisfy the suspension axiom, which is expressed by means of a local right-adjointness property (Theorem 3.5). This latter notion is closely related to that of a locally adjointable functor due to Kapul [10]. Indeed both Kapul's definition and ours involve an existence and a universal statement for a factorization of a morphism $f$; the existence statements are identical, but our universal statement is less restrictive than Kapul's. In addition, our concept is related to a pair of categories $(\mathcal{J}_1, \mathcal{J}_0)$ and so only requires the factorization if $f: \Sigma X \to Y$, $X \in \mathcal{J}_0$, $Y \in \mathcal{J}_1$. The main advantage of the new criterion over that of [5] is that the sufficient conditions of [5] are localized. For example the weak local pull-back property asserts that a diagram of homotopy classes of maps,

$$
\begin{array}{c}
\begin{array}{c}
X \\
\downarrow f_1 \\
\downarrow f_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Y_0 \\
\downarrow u_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X \in \mathcal{J}_1, \quad Y_1 \in \mathcal{J}_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Y \\
\downarrow u_1
\end{array}
\end{array}
\end{array}
$$

may be embedded in a diagram

$$
\begin{array}{c}
\begin{array}{c}
X \\
\downarrow f_1 \\
\downarrow f_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Z \\
\downarrow u_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Y_1 \\
\downarrow u_1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Y \in \mathcal{J}_1
\end{array}
\end{array}
$$

that is, the morphisms $g, v_2, v_1$ may be chosen after $f_0$ and $f_1$ are given, unlike the situation prevailing for a weak (global) pull-back of $u_0$ and $u_1$. We indicate by two examples at the end of Section 3 how the new criterion enables us to guarantee that certain Kan extensions, in situations not covered by [5], are indeed cohomology theories.

The second of these two examples constitutes essentially the subject matter of Section 4. We consider the *acyclic ring*, or *Serre class* [13], [14], $\mathcal{C}_P$, of torsion abelian groups $A$ whose elements have orders which are multiples of primes in a certain family $P$ (which could, in particular, consist of a single prime $p$ or of all primes). If $\mathcal{J}_0$ is the category of 1-connected finite CW-complexes whose homology groups belong to $\mathcal{C}_P$ and if $\mathcal{J}_1$ is the category of 1-connected finite dimensional CW-complexes, then the Kan extension, $h_1$, to $\mathcal{J}_1$, of any cohomology theory $h$ defined on $\mathcal{J}_0$ is again a cohomology theory and we identify it in the case that $h$ is representable with finitely-generated coefficients. It turns out that, in fact, (see (4.2)),

$$
\begin{align}
&h^i(X) \cong h^{i-1}(X; \mathcal{Z}_P), \quad X \in \mathcal{J}_1.
\end{align}
$$

Here $h(X; \mathcal{Z}_P)$ refers to the cohomology theory obtained from $h$ by putting in the coefficient group

$$
\mathcal{Z}_P = \bigoplus_{p \in P} \mathcal{Z}_p
$$

in the sense of [9]; it is proved in that reference that the groups $h(X; \mathcal{Z}_P)$ are determined by $X$ and $P$ up to natural isomorphism if $2 \not\in P$ and up to 'quasi-natural' isomorphism if $2 \in P$. It is worth remarking that if $h = H$, ordinary reduced cohomology with integer coefficients, then $h_1$ is a theory which does not even satisfy the restricted wedge axiom,

$$
h_1(\vee S^n) \neq \bigoplus h_1(S^n)
$$

over any infinite wedge of spheres.

There is also established in [9] a universal coefficient theorem for $h(X; \mathcal{Z}_P)$ which plays a crucial role in the proof of our main result, Theorem 4.1. The proof is, in fact, broken up into a sequence of lemmas in order to display the role of the various elements in the hypotheses of the theorem and to enunciate certain features of the Kan extension process which should play a role in any future attempt to identify the nature of a particular extended theory. The paper ends with two variants of Theorem 4.1, in which we alter the hypotheses but obtain essentially the same conclusion. Thus, in each case considered in Section 4, (1.1) holds. This relation serves to illustrate the violence which can be done to a cohomology theory defined on $\mathcal{J}_1$ by replacing it by the Kan extension of its restriction to $\mathcal{J}_0$. If we start with ordinary cohomology with integer coefficients, and carry out the process described, then (taking $P$ for the purpose of this remark, to be the set of all primes), the groups of an $n$-sphere are given by

$$
\begin{align}
h^{i-1}(S^n) &= \mathbb{Q},
\end{align}
$$

This result illustrates a fact to which we hope to give attention in a subsequent paper, namely, that there exists a close connection between the Kan extension process for cohomology theories and Adams' notion of completing a space with respect to a homology theory (4).

We remark that the arguments used in the proof of Theorem 1.1 involve a study of colimits for functors to sets; since this theory is so much more elementary than that involved in a study of functors to groups, we have felt it to be adequate simply to refer to the arguments at the beginning of Section 2 and ask the reader, in considering functors to sets, to ignore anything in those arguments referring specifically to groups rather than sets.

It is hoped to devote a subsequent paper to applications of the lemmas of Section 4 to other examples of the Kan extension of a cohomology theory, different from those treated in this paper.

It is a pleasure to acknowledge the benefit of very helpful conversations with Guido Mislin in connection with specific aspects of this work.

2. The construction of colimits of functors to groups. In order to motivate the abstract study of colimits undertaken in this section, we first recall explicitly the definition of the Kan extension of a cohomology theory \( h \), defined on \( J \), a category of based spaces and based maps, to a larger category \( J' \), of based spaces and based maps, as given in [5]. Let \( J' \) be the category of based spaces and based maps, as given in [5]. Let \( J \), \( J' \) be the category of based spaces and based maps, as given in [5]. Let \( X, Y \) be the homotopy categories associated with \( J, J' \), respectively, the morphisms of \( J' \) being isomorphism classes of based maps in \( J \).

Let \( X, Y \) and \( F, G \) be the objects of \( J' \) and \( J \), respectively, the morphisms of \( J \) being isomorphism classes of based maps in \( J \). Let \( X, Y \) and \( F, G \) be the objects of \( J' \) and \( J \), respectively, the morphisms of \( J \) being isomorphism classes of based maps in \( J \).

We wish to study conditions on \( I \) which guarantee that the standard construction of the direct limit when \( I \) is a directed set still apply to our case; plainly, however, \( I \) is not a directed set and, indeed, it may well fail to be even quasi-filtered (see [3]). Thus we require a substantial generalization of the usual theory although this generalization may be known in the folklore. In addition we are, of course, hoping that the colimit of \( F_I \), viewed as a functor on \( J' \), will inherit from \( h \) the exactness property required of a cohomology theory.

We suppose now that \( I \) is a connected category such that

(a) given \( X \rightarrow Y \) in \( I \), we may construct a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

in \( I \); and

(b) given \( X \rightarrow Y \) in \( I \), we may construct a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

in \( I \).

We will then say that \( I \) is adapted (for colimits) notice that conditions (a) and (b) together yield the definition (in a suitable 'universe') of a quasi-filtered category. We remark that condition (a) implies that any two objects of \( I \) may be connected by a path \( \cdots \rightarrow \cdots \). We also remark that if \( I \) satisfies (a) and

(b') \( I \) admits finite coproducts,

then \( I \) satisfies (b) and hence is adapted.

Now let \( F_i : I \rightarrow \mathbf{B} \) be a functor. We will write \( F_i \) for \( F_i(i) \), \( i \in |I| \), and \( \psi : F_i \rightarrow F_j \) for \( \psi : F_i \rightarrow F_j \) where \( \psi : i \rightarrow f \) in \( I \). We introduce a relation in \( \bigcup F_i \) by declaring

\[
\begin{align*}
x_i & \sim x_j, \quad x_i \in F_i, \quad x_j \in F_j \\
i & \sim \mu \quad \text{if there is } i \rightarrow \mu \rightarrow j \text{ in } I \text{ with } \varphi(x_i) = \varphi(x_j)
\end{align*}
\]

Theorem 2.3. If \( I \) is a connected category satisfying condition (a), then (2.2) is an equivalence relation.

Proof. Only transitivity is in question, so we suppose \( x_i \sim x_j, x_j \sim x_k \), thus,

\[
\begin{align*}
i & \xrightarrow{\sim} \mu \xrightarrow{\sim} j, \quad \varphi(x_i) = \varphi(x_j) \\
\mu & \xrightarrow{\sim} \lambda \xrightarrow{\sim} k, \quad \varphi(x_j) = \varphi(x_k)
\end{align*}
\]

By condition (a) we have a commutative square in \( I \)

\[
\begin{array}{ccc}
i & \xrightarrow{\sim} \mu & \xrightarrow{\sim} k \\
\downarrow & & \downarrow \\
\theta & \xrightarrow{\sim} \psi & \xrightarrow{\sim} \varphi
\end{array}
\]
Then \( i \overset{\sigma}{\rightarrow} n \leftarrow m \rightarrow j \) in \( I \) and

\[ \varphi(x_1) = \varphi(x_2) = \sigma(y_1) = \sigma(y_2) , \]

so \( x_1 \sim x_2 \).

We write \([a_1]\) for the equivalence class of \( x_1 \). We define a product operation among the equivalence classes by the rule

\[ [x_1][x_2] = [\varphi(x_1) \cdot \varphi(x_2)] , \]

where \( i \overset{\lambda}{\rightarrow} k \leftarrow j \) in \( I \).

**Theorem 2.5.** If \( I \) is adapted then (2.4) is well-defined.

**Proof.** We first show that the function \( (x_1, x_2) \rightarrow [\varphi(x_1) \cdot \varphi(x_2)] \) is well-defined. Thus suppose

\[ i \overset{x_1}{\rightarrow} k \overset{\lambda}{\leftarrow} j \text{ in } I , \quad \lambda = 1, 2 . \]

We construct, by (a), the commutative square

\[ \begin{array}{ccc}
  i & \rightarrow & k' \\
  \downarrow & & \downarrow \\
  j' & \rightarrow & j \\
\end{array} \]

and then use (b) to construct the commutative diagram

\[ \begin{array}{ccc}
  i & \rightarrow & k' \\
  \downarrow & & \downarrow \\
  j' & \rightarrow & j \\
  \downarrow & & \downarrow \\
  k & \rightarrow & k \\
\end{array} \]

If we set \( x_2 = x_2' \), \( \lambda = 1, 2 \), we have the commutative diagram

\[ \begin{array}{ccc}
  i & \rightarrow & k' \\
  \downarrow & & \downarrow \\
  j' & \rightarrow & j \\
  \downarrow & & \downarrow \\
  k & \rightarrow & k \\
\end{array} \]

Thus

\[ [\varphi(x_1) \cdot \varphi(x_2)] = [\varphi(x_1) \cdot \varphi(x_2)] = [\varphi(x_1) \cdot \varphi(x_2)] , \]

so that the function \( (x_1, x_2) \rightarrow [\varphi(x_1) \cdot \varphi(x_2)] \) is well-defined.

Second we show that \( [\varphi(x_1) \cdot \varphi(x_2)] \) depends only on \([x_1]\) and \([x_2]\). It is plainly sufficient to show that it is independent of the choice of \( x_1 \) from \([x_1]\); and it is then obvious that we need only consider the effect of replacing \( x_1 \) by \( \mu(x_1) \) where \( \mu : i \rightarrow i \) in \( I \). There is a diagram

\[ \begin{array}{ccc}
  i & \rightarrow & m \\
  \downarrow & & \downarrow \\
  j & \rightarrow & j \\
\end{array} \]

in \( I \) and by what we have already shown, since we have \( i \overset{\mu}{\rightarrow} m \leftarrow j \) in \( I \),

\[ [\varphi(x_1) \cdot \varphi(x_2)] = [\varphi(x_1) \cdot \varphi(x_2)] . \]

However the right-hand side is the image of \( \mu(x_1), x_2 \) under the given function, so that our claim is established and the theorem proved.

Let us write \( G \) for the collection of equivalence classes under (2.2) and \( \delta : F_i \rightarrow G \) for the function \( \delta(x_1) = [x_1] \). Plainly \( \delta \) is homomorphic with respect to (2.4) and if \( \varphi : i \rightarrow j \) in \( I \) then

\[ \delta \varphi = \delta \varphi . \]

**Theorem 2.9.** If \( I \) is adapted, then \( G \) is a group under (2.4) and

\[ \lim F = (G, \delta_0) . \]

**Proof.** If \( \epsilon_i \) is the identity of \( F_i \), then plainly \( \epsilon_i \sim \epsilon_i \) for any \( i, j \in I \) and \( \epsilon = [\epsilon_1] \) is the identity of \( G \). Equally plainly \([\epsilon^{-1}_1]\) is the inverse of \([\epsilon_1]\) and associativity follows immediately from the observation that, given \( i, j, k \in I \), we can find

\[ \begin{array}{ccc}
  i & \rightarrow & j \\
  \downarrow & & \downarrow \\
  k & \rightarrow & k \\
\end{array} \]

in \( I \). For \([x_1][x_2][x_3] = [x_1][x_2][x_3] = [\varphi(x_1) \varphi(x_2) \varphi(x_3)] , \]

Now suppose given a group \( H \) and homomorphisms \( \phi : F_i \rightarrow H \) such that if \( \varphi : i \rightarrow j \) in \( I \) then \( \phi \varphi = \phi \varphi \). We define a homomorphism \( \varphi : G \rightarrow H \) by \( \varphi(x_1) = \varphi(x_2) \). If \( i \overset{\lambda}{\rightarrow} k \overset{\lambda}{\leftarrow} j \) in \( I \) and if \( \varphi(x_1) = \varphi(x_2) \), then \( \varphi(x_1) = \varphi(x_2) = \varphi(x_1) = \varphi(x_2) \), so that \( \varphi \) is well-defined. It is easy to see that \( \varphi \) is a homomorphism and that

\[ \varphi \epsilon_1 = \epsilon \varphi . \]

Moreover, \( \varphi \) is obviously uniquely determined by (2.10) so that the theorem is completely proved.

We now consider how to apply Theorem 2.9 to our example (2.1), so that \( I = I(X) = \mathcal{F}\text{-}	ext{op} \), \( F = \mathcal{K} \).

**Definition 2.11.** \( \mathcal{F} \) has weak local pull-backs relative to \( \mathcal{F} \) if and only if, given the diagram

\[ \begin{array}{ccc}
  X & \rightarrow & Y \\\n  \downarrow \epsilon & & \downarrow \epsilon \\\n  Y_1 & \rightarrow & Y_2 \\
\end{array} \]
Now define \( h_0(X) = \lim h_X \). This clearly defines a contravariant functor \( h_0 : J_0 \to \text{Ab} \), called the Kan extension of \( h \). We prove (compare Theorems 3.9, 3.10 of [5]):

**Theorem 2.11.** If \( J_0 \) is \( J_1 \)-adapted, \( h_0 \) satisfies the exactness axiom.

**Proof.** Let \( X' \to X \to X' \) be a mapping cone sequence in \( J_1 \); we must show

\[
h_0(X') = \lim h_X = h_0(X'')
\]

exact. Since \( I(X) \) is adapted for all \( X \in \mathcal{J}_1 \), we may represent an element \( \alpha \) of \( h_0(X') \) by means of a pair \( (a, f) \) where \( f : X \to Y \in J_0 \) and \( a \in h(Y) \). Moreover \( h_0(g')[(a, f)] = [a, f'] \). Thus if \( h_0(g')[(a, f)] = 0 \), \( [a, f'] = 0 \) so that, in view of (2.1), there exists \( u' : f' \to f' \) in \( J_0(X) \) such that \( h(u') \alpha = 0 \).

Consider the diagram

\[
\begin{array}{ccc}
X' & \to & X' \\
\downarrow f' & & \downarrow f \\
Y' & \to & Y'
\end{array}
\]

where the bottom row is also a mapping cone sequence (and so in \( J_0 \)). Since the top row is a mapping cone sequence there exists \( f'' : X'' \to X'' \) such that the right-hand commutative (in 2.15) is homotopy-commutative; and since \( h(u') \alpha = 0 \), there exists \( a' \in h(Y'') \) with \( h(u'')a' = a. \) Then \( u' : f'' \to f' \) in \( J_0(X) \), so that

\[
h_0(g'')[(a', f'')] = [a', f''(g'')] = [a', u''f] = [h(u'')a', f] = [a, f]
\]

and the theorem is proved. We draw particular attention to the representation of an element of \( h_0(X) \) as

\[
[a, f], f : X \to Y \in J_0, \quad a \in h(X)
\]

Since the homotopy axiom is automatically satisfied by \( h_0 \), it remains only to consider the suspension axiom. This will be handled in Section 3, always under the hypothesis that \( J_0 \) is \( J_1 \)-adapted.

**3. Invariance under suspension.** Let \( J_0 \) be \( J_1 \)-adapted and let \( (a, f) \) be a cohomology theory on \( J_0 \). We extend \( h_0 : J_0 \to \text{Ab} \) as in Section 2. Then the natural equivalence \( h_0^a : h_0^{a} \to h_0^{a} \Sigma : J_0 \to \text{Ab} \) extends to a natural transformation \( a_0 : h_0^a \to h_0^{a} \Sigma : J_0 \to \text{Ab} \) and we seek conditions in this section under which \( a_0 \) is an equivalence. We remark first of all that \( a_0 \) is given by

\[
a_0[a, f] = [a, f]
\]

(\( \Sigma \) as in [5], we write \( [a, f] \) for the equivalence class containing \( (a, f) \)).
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We now introduce a condition on the triple \((J_1, J_2, \Sigma)\) which will guarantee the result we seek to achieve. Recall that \(J\) is the category obtained from \(J\) by replacing maps by homotopy classes.

**Definition 3.2.** We say that \(\Sigma: J_1 \rightarrow J_1\) is locally right-\(J_1\)-adjunctable if \((i)\) given \(f: \Sigma X \rightarrow Y\) in \(J_1\), \(X \in [\Sigma X]\), there exist \(Z \in [\Sigma Z]\); \(g: X \rightarrow Z\) in \(J_1\), \(w: \Sigma Z \rightarrow \Sigma Y\) in \(J_2\) such that \(f = u \circ \Sigma g\); and \((ii)\) given

\[
Z_1, Z_2, Y \in [\Sigma Y], \quad u_1 \circ \Sigma g_1 = u_2 \circ \Sigma g_2,
\]

there exist \(\tau_1: Z_1 \rightarrow Z_2\), \(\tau_2: Z_1 \rightarrow Z_2\), \(w: \Sigma Z \rightarrow \Sigma Y\) in \(J_2\) with \(\tau_1 g_1 = \tau_2 g_2\), \(w \circ \Sigma \tau_1 = u_1\), \(w \circ \Sigma \tau_2 = u_2\).

This definition may be domesticated by the following observation.

**Proposition 3.3.** Suppose \(\Sigma: J_1 \rightarrow J_1\) has a right \(J_1\)-adjoint \(\Omega: J_2 \rightarrow J_1\), that is, there is a natural equivalence

\[\Sigma_j(X, Y) \cong J_2(X, \Omega Y), \quad X \in [\Sigma X], \quad Y \in [\Sigma Y].\]

Then \(\Sigma\) is locally right-\(J_1\)-adjunctable.

Proof. Let \(e: \Sigma \Omega \rightarrow 1\) be the counit of the adjunction, and let \(f: X \rightarrow \Omega Y\) be the adjoint of \(f: \Sigma X \rightarrow Y\), so that

\[(3.4)\]

\[f = e \circ \Sigma f'.\]

We fulfill condition \((i)\) of Definition 3.2 by taking \(Z = \Omega Y\), \(g = f', u = e\).

As to condition \((ii)\), we first remark that if \(u_1 \circ \Sigma g_1 = u_2 \circ \Sigma g_2\), then \(u_1 g_1 = u_2 g_2\). Thus we fulfill condition \((ii)\) by taking \(Z = \Omega Y\), \(\tau_1 = u_1\), \(\tau_2 = u_2\), \(u = e\).

On the other hand we will produce examples below where \(\Sigma\) is locally right-\(J_1\)-adjunctable but does not have a right \(J_1\)-adjoint (let alone a right \(J_1\)-adjoint extendable to a right \(J_1\)-adjoint).

We now prove the main result of this section.

**Theorem 3.3.** If \(J_2\) is \(J_1\)-adapted and \(\Sigma\) is locally right-\(J_1\)-adjunctable, then \(a_1: h^2 \rightarrow h^2: J_1 \rightarrow Ab\) is a natural equivalence.

Proof. We must show that \(a_1\) given by \((3.1)\) is onto and one-one. \(a_1\) is onto. We construct

\[
\xymatrix{
\Sigma X \ar[r]^f & Y \\
\Sigma Z \ar[u]^w \ar[ru]^{g''}\\
}
\]

as condition \((i)\) of Definition 3.2 permits. Given \([a, f] \in h_1(\Sigma X)\), let \(\beta \in h_2(Z)\) be such that \(\alpha \beta = h(w) a\). Then

\[\alpha \beta = [\alpha \beta, Z g] = \alpha_1(\beta, g),\]

so \(\alpha_1\) is onto.

\(\alpha_1\) is one-one. Let \([a, f] \in h_1(X)\) and suppose \(\alpha_1(a, f) = 0\). This implies a commutative diagram

\[
\xymatrix{
\Sigma X \ar[r]^{f} \ar[d]^{\Sigma w} & \Sigma Y \\
\Sigma Z \ar[u]^{w} & \Sigma Z \\
}
\]

with \(h(w) a = 0\). By condition \((i)\) we may augment this diagram to

\[
\xymatrix{
\Sigma X \ar[r]^{f} \ar[d]^{\Sigma w} & \Sigma Y \\
\Sigma Z \ar[u]^{w} & \Sigma Z \\
}
\]

as \(\alpha_1(a, f) = 0\).

Now consider

\[
\xymatrix{
\Sigma X \ar[r]^{f} \ar[d]^{\Sigma w} & \Sigma Y \\
\Sigma Z \ar[u]^{w} & \Sigma Z \\
}
\]

By condition \((ii)\) of Definition 3.2, we may find

\[
\tau: Y \rightarrow T, \quad u: Z \rightarrow T, \quad s: \Sigma T \rightarrow \Sigma Y \quad \text{in} \quad J_0
\]

with \(sf = wg\), \(s \circ \Sigma t = 1\), \(s \circ \Sigma c = w\). Set

\[
\Sigma \Sigma Y
\]

(3.8)

\[k = sf = wg\]
and consider

\[ X \xrightarrow{f} Y \]

\[ Z \xrightarrow{g} Z' \]

\[ f \circ g \]

Define \( \beta \in \kappa(T) \) by \( h(\beta) x \alpha x = \alpha \). Then \( h(\Delta x) \beta = \alpha \), since \( g \circ \Delta x = 1 \), so that

\[ h(\beta) \alpha = \alpha \]

and \([a, f] = [\beta, k] \). Also \( h(\Delta x) \beta = h(\Delta x) \alpha = 0 \), since \( g \circ \Delta x = 0 \), so that

\[ h(\beta) \alpha = 0 \]

Thus \([a, f] = [\beta, k] = [h(\Delta x) \beta, g] = 0 \), and the theorem is proved.

**Corollary 3.3.** If \( J_a \) is \( J_c \)-adapted and \( \Sigma \) is locally right-\( J_c \)-adjointable, then the Kan extension of a cohomology theory \((h, c)\) is a cohomology theory \((h, c)\).

Since the hypotheses of Corollary 3.3 are substantially weaker than those of Theorems 3.9 and 3.10 of [5] all the examples given in that paper (see especially section 4 of [5]) are applications of Corollary 3.9. We now give two further examples (among many) to indicate the wider scope of Corollary 3.9.

**Example 3.10.** Let \( J_a \) be the category of finite CW-complexes and let \( J_c \) be the category of compact spaces. Then \( J_a \) is a category with weak local pull-backs relative to \( J_c \). For, given

\[ X \xrightarrow{f} Y \]

we can construct a first approximation

\[ \sum X \xrightarrow{\sum f} \sum Y \xrightarrow{\sum g} \sum Z \]

where \( Z' \) is the weak pull-back (see Proposition 3.2 of [5]) of \( w_0 \) and \( u_1 \). \( Z' \) is not in \( J_a \) in general but, by Milnor's Theorem [12] it may be assumed to be a CW-complex and hence \( g'X \subseteq Z' \) lies in some finite subcomplex \( Z \),

\[ g'X \subseteq Z \subseteq Z' \]

If \( g \colon X \to Z \) is just \( g' \) with restricted range, and \( v_i \colon Z \to Y_i \), \( i = 1, 2 \), is just \( v'_i \) with restricted domain, then we obtain the diagram verifying the condition given in Definition 2.11.

Finally we show that \( \Sigma \colon \sum J_a \to \sum J_c \) is locally right-\( J_c \)-adjointable. To obtain the factorization

\[ \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \]

of \( f \colon X \to Y \), we take as our first approximation

\[ \Sigma X \xrightarrow{\Sigma f} \Sigma Y \]

\( Z' \subseteq \cup Y_i \), the loop-space on \( Y_i \), \( f' \) adjoint to \( f \).

Again by Milnor's Theorem we may assume \( Z' \) to be a CW-complex and then we find \( Z \subseteq Z' \), a finite subcomplex containing \( f' \). Then we define \( g \colon X \to Z \) by restricting the range of \( f' \) to \( u_1 \) and \( \Sigma Z \to \Sigma Y \) by restricting its domain of \( c \). As to part (ii) of Definition 3.2 we take \( Z' = \cup Y_i \), \( v'_i = v_i \), adjoint to \( u_i \), \( i = 1, 2 \), \( u' = u \colon \Sigma Z \to \Sigma Y \) as our first approximation, yielding

3.11)

\[ X \]

\[ Z' \]

\[ Y \]

\[ \Sigma Y \]

\[ \Sigma Z \]

Recall that our arrows are homotopy classes, so that the left hand diagram of (3.11) represents a homotopy-commutative diagram \( \triangle f \times \triangle g \). The homotopy is a map \( F \colon X \times I \to Z \) so \( F(X \times I) \) lies in some finite subcomplex \( Z \) of \( Z' \), chosen so that \( v'_i, v'_i \) have their images in \( Z \). Then we verify part (ii) of Definition 3.2 by taking \( v_i \), \( v_i \) to be \( v'_i \), \( v'_i \) with range restricted to \( Z \), and \( u_1 \) \( \Sigma Z \to \Sigma Y \) to be the restriction of \( u' \).

We now give an example which will figure prominently in the sequel. Let \( C \) be an acyclic ring (Serre class (!12) of abelian groups; we say that a CW-complex \( Y \) belongs to \( C \) if it is 1-connected and all its (reduced) integral homology groups or, equivalently, all its homotopy groups belong to \( C \).

(1) We always assume that our Serre classes \( C \) satisfy conditions IIA and III of [12].
Example 3.12. Let $J_0$ be the category of finite CW-complexes belonging to a Serre class $C$ and let $J_1$ be the category of 1-connected finite-dimensional CW-complexes. Then $J_0$ is admissible since the mapping cone $C_0$ of a map $f : X \to Y$ of finite 1-connected complexes is finite and 1-connected, and the homology groups of $C_0$ belong to $C$ by a simple application of the exact homology sequence. $J_0$ is plainly closed under finite products and we verify that $J_0$ has weak local pull-backs relative to $J_1$. As in Example 3.10, we start from

\[
\begin{array}{c}
X & \xrightarrow{f} & Y_1 \\
\downarrow{h} & & \downarrow{w} \\
Y_1' \xrightarrow{w_0} Y
\end{array}
\]

and construct the first approximation

\[
\begin{array}{c}
X' & \xrightarrow{f'} & Y_1' \\
\downarrow{h'} & & \downarrow{w_0'} \\
Y_1'' \xrightarrow{w_0''} Y
\end{array}
\]

where $Z'$ is the weak (homotopy-) pull-back of $u_0$ and $u_0'$. An easy argument using the homotopy sequence of a fibration shows that the homotopy groups of $Z'$ belong to $C$; and, as already argued, we may assume $Z'$ to be a CW-complex. As a second approximation we take $Z''$ to be the universal cover of $Z'$. Since $X$ is 1-connected, $g'$ lifts to $g'': X \to Z''$ and we obtain

\[
\begin{array}{c}
X & \xrightarrow{f''} & Y_1' \\
\downarrow{h''} & & \downarrow{w_0''} \\
Y_1'' \xrightarrow{w_0''} Y
\end{array}
\]

Since $X_1, Y, Y_1$ are all 1-connected finite complexes, their homotopy groups are finitely-generated. So therefore are the homotopy groups of $Z'$ and $Z''$ and hence the homology groups of $Z''$. Thus $Z'' \in C$ and the skeleta of $Z''$ may be taken to be finite complexes. Now $Z''$ admits a homology decomposition (see [6]). By varying $Z''$ within its homotopy type (but retaining the property that its skeleta are finite complexes) we may suppose that, for each $k$, there is a subcomplex $Z''(k)$ of $Z''$ such that

(i) $Z'' \subseteq Z''(k) \subseteq Z''_{k+1}$, where $Z''_k$ is the $k$-skeleton of $Z''$;

(ii) $H(Z''(k)) = 0, i > k$;

(iii) the inclusion $Z''(k) \subseteq Z''$ induces an isomorphism in homology in dimensions $\leq k$.

Conditions (ii) and (iii) guarantee that $Z''(k) \in C$ and condition (i) then guarantees that $Z''(k) \in J_0$. If $\dim X = k$, then $g''$ may be deformed into $Z''_k$ and thus we verify the condition given in Definition 2.11 by taking $Z = Z''(k)$ and defining $g : X \to Z$ by restricting the range of $g''$ (after submitting $g''$ to the necessary deformation), and $v_i : Z \to Y_1, i = 1, 2$, by restricting the domain of $v_i''$. The argument showing that $\Sigma : J_1 \to J_0$ is locally right adjointable now follows very similar lines, replacing the weak homotopy pull-back by the loop-space. Thus, without giving all the details, we obtain the required factorisation

\[
\Sigma X \xrightarrow{g} \Sigma Z \xrightarrow{v} Y
\]

of $f : \Sigma X \to Y$ by first setting $Z = \Omega Y$, then setting $Z'' = \Delta Y$, the universal cover of $\Omega Y$, and finally setting $Z = \pi_1$-suitable homology section $Z''(k)$. Again, part (ii) of Definition 3.2 is verified by first setting $Z = \Omega Y, v_i = v_i', i = 1, 2$, and finally setting $Z = Z''(k)$ if $X$ is $k$-dimensional and $l = \max (\dim Z_0, \dim Z_1, k+1)$. With this choice of $Z$ we may suppose $v_i'$ maps $Z_i$ into $Z$ and $v_i' \overline{\varepsilon} = \varepsilon Z_i$, as maps $X \to Z$. The reader should easily be able to fill in the gaps in our description of this argument.

This example figures very prominently in the next section. We recall from [5] the remark that the restriction to 1-connected complexes in this, and other examples, has no restricting effect on the scope of the extended theory $h_1$, since any theory defined on 1-connected complexes extends to all complexes, together with the full apparatus of additional algebraic structure, by passing to the double suspension.

4. The main theorem. Let $P$ be a family of prime numbers and let $C_P$ be the Serre class consisting of torsion abelian groups $A$ such that the order of any $a \in A$ is a product of members of $P$. Let $Q_0$ be the group of rationals mod 1 and let $Z_{20}$ be the subgroup of $Q_0$, which is the direct sum of its $p$-components, $p \in P$. Let $h$ be a cohomology theory and let $h(\cdot; Z_{20})$ be the theory obtained from $h$ by introducing the coefficient group $Z_{20}$ as in [9]. We specialize Example 3.12 by taking $C = C_P$ and prove

\footnote{Of course, $\delta_1$ is right adjoint to $\pi_1$ on the category of 1-connected CW-complexes.}

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Theorem 4.1. Let $h$ be a representable cohomology theory with finitely-generated coefficients $(\text{*)}$, and let $h_0$ be its restriction to the category $J$ of 1-connected finite CW-complexes belonging to $FP$. Then the Kan extension $h_0$ of $h_0$ to the category $J$ of 1-connected finite-dimensional CW-complexes is given by

\[(4.2) \quad h_0'(X) \cong h^{n-1}(X; Z_{p_0}).\]

The proof of this theorem will be achieved with the help of a series of lemmas; the lemmas themselves may well have an independent interest.

Lemma 4.3. Let $Y$ be a finite CW-complex belonging to $FP$ and let $h$ be a cohomology theory. Then $h^*(Y) \in FP$.

Proof. We apply the generalized Atiyah–Hirzebruch spectral sequence [1], [7]. Then

\[E'^{pq}_2 \cong H^p(Y; \mathbb{Z}/p) \otimes \text{Hom}(H_q(Y; \mathbb{Z}/p), \mathbb{Z}/p) \otimes \text{Ext}(H_{p-q}(Y; \mathbb{Z}/p), \mathbb{Z}/p),\]

so that $E'^{pq}_2 \in FP$, since $i_r(Y)$ is of finite type. (Notice that $E'^{pq}_2 = 0$ if $p = 0$ since we are using reduced cohomology and $Y$ is connected.) It follows that $E'^{pq}_2 \in FP$ and since $h^*(Y)$ has a finite filtration with quotients belonging to $FP$, we deduce that $h^*(Y)$ itself belongs to $FP$.

Lemma 4.4. Let $G \in FP$. Then $G \otimes Z_{p_0} = 0$, and there is a natural isomorphism

\[\text{Tor}(G, Z_{p_0}) \cong G.\]

Proof. Since $G$ is a torsion group and $Z_{p_0}$ is divisible, it follows that $G \otimes Z_{p_0} = 0$. To prove the second assertion it is plainly legitimate to replace $Z_{p_0}$ by $Q_1$. Then consider the sequence

\[0 \to Z \to Q \to Q_1 \to 0;\]

this gives rise to the exact sequence, natural in $G$,

\[\text{Tor}(G, Q) \to \text{Tor}(G, Q_1) \to G \otimes Z \to G \otimes Q.\]

Now $G \otimes Q = 0$ by the previous argument and $\text{Tor}(G, Q) = 0$ since $Q$ is torsion-free. Since $G \otimes Z \cong G$, the lemma is proved.

Lemma 4.5. Let $Y$ be a 1-connected finite CW-complex belonging to $FP$ and let $h$ be a cohomology theory. There is then a natural equivalence of cohomology theories on the category $J$ of such complexes $Y$, given by

\[h^*(Y) \cong h^{n-1}(Y; Z_{p_0}).\]

(\text{*) We must distinguish between the coefficients of a theory $h$, that is, the graded group $h^k$, and the introduction of a group $G$ as a coefficient group into a theory $h$, as in (9).}

Proof. We have (see [9]) a short exact sequence

\[(4.6) \quad 0 \to h^{n-1}(X; Z_{p_0}) \to h^{n-1}(X; Z_{p_0}) \to \text{Tor}(h^*(Y), Z_{p_0}) \to 0,
\]

which is natural in $Y$ and commutes with suspension. By Lemma 4.3, $h^{n-1}(Y)$ and $h^*(Y)$ belong to $FP$, thus by Lemma 4.1 and (4.6) we have isomorphisms

\[h^{n-1}(Y; Z_{p_0}) \cong \text{Tor}(h^*(Y), Z_{p_0}) \cong h^*(Y),\]

which are natural in $Y$ and commute with suspension.

Lemma 4.7. Let $J_0$ be $J_0$-adapted, let $F_0 : F_{\text{top}} \to G$ be a contravariant functor from $J_0$ to the category of sets $\mathcal{S}$, and let $F_0 : F_{\text{top}} \to G$ be an extension $F_0$ to $G_{\text{top}}$ such that

(i) every $f \in F_{\text{top}}X$ is expressible as $f = (F_0(a)_f, a \in F_0X, X \cong J_0$;

(ii) if $(F_0(a)_f, a \in F_0X, X \cong J_0$,

there exists a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & Y' \\
\downarrow{g_1} & & \downarrow{g_2} \\
X & \xrightarrow{g} & Y \\
\end{array}
\]

with $(F_0(a)_g, a \in F_0X, X \cong J_0$.

Then $F_0$ is the Kan extension of $F_0$.

(Compare the Proposition on p. 431 of [11]).

Proof. If $F$ is the Kan extension of $F_0$, there is a canonical natural transformation $\omega : F \to F_0$, which is the identity on $J_0$. Now the theory of Section 2 applies to functors to sets as well as to functors to groups — and is, indeed, more elementary in this case (9). Thus we have the representation of $F_{\text{top}}$ given by (2.16), and condition (i) asserts then that $\omega$ is surjective, while condition (ii) asserts that $\omega$ is injective.

Lemma 4.8. Let $J_0$ be $J_0$-adapted. If $F : F_{\text{top}} \to G_{\text{top}}$ is a direct limit of functors representable in $J_0$, then $F$ is the Kan extension of $F_{\text{top}}$.

Proof. We have $F(X) = \lim_{\uparrow X} [X, T], T \in \mathcal{T}_0$. We proceed to verify

(i) and (ii) of Lemma 4.7 for $F_1 = F$, $F_0 = F_{\text{top}}$, Set $T_0 = \{[\text{ }, T], T \in \mathcal{T}_0$ and

(*) The index category $I$ need only satisfy condition (a) at the beginning of Section 2 in order to be adapted for colimits of functors to sets. The construction of the colimit set is then exactly as in Section 2, except that we do not have to worry about group structure.
let \( \eta \in F_p(T_p) \) be the class of the identity map of \( T_p \). Let \( \xi \in P(X) \) be represented by \( f : X \to T_p \). Then \( \eta = (F_p f)(\eta_p) \), so that \( \xi = (F_p f)(\eta_p) \) where \( (\eta_p) \in FT_p \) is represented by \( \eta_p \). This proves (i).

To verify (ii), we have

\[
F_p(T_p) \ni \xi \mapsto \eta_p
\]

where \( \eta_i \in P(T_i) \) is represented by \( g_i \in P(T_i) \), \( i = 1, 2 \). We may find \( T_i \) and maps \( \tau_i : T_i \to T_j \), \( i = 1, 2 \), such that \( \tau_1 \eta \tau_2 \eta = \tau_2 \eta \tau_1 \eta 
\)

since \( (T_p)_{\eta_p} = (T_i)_{\eta_i} \).

Then \( F_p(\tau_1 \eta)(\eta_i) = \eta_i, i = 1, 2 \), and the lemma is proved. Such a functor \( F \) we may describe as \( J \)-prorepresentable [3].

**Lemma 4.9.** If \( h \) is a representable cohomology theory with finitely-generated coefficients, then the functor \( h_*(\ ; Z_p) : \mathbb{J}^{\mathrm{op}} \to \mathbb{S} \) is \( J \)-prorepresentable, where \( J, J_1 \) are as in Theorem 4.1 and \( p \in P \).

Proof. Let \( h^* \) be represented by \( M_\bullet \), \( h^*(X) = [X_\bullet, M_{\bullet}] \).

Now (see [9])

\[
h^*(X; Z_p) = h^*(X, X_\bullet, Z_p) \]

so (10) that \( h^*(X; Z_p) \) is represented by \( M_{\bullet; p} \). We may assume \( \eta_p \) is connected (replacing it by its universal cover if necessary) since \( J_1 \) consists of 1-connected spaces. Then

\[
\pi_0(\eta_p) = \pi_0(\eta_p) = \pi_0(\eta_p) = \pi_0(\eta_p)
\]

and we have the universal coefficient sequence

\[
0 \to \pi_0(\eta) \to \pi_0(\eta) \to \pi_0(\eta) \to 0
\]

Since the homotopy groups of \( M_{\bullet; p} \) are finitely-generated, it follows that \( \pi_0(\eta) \) is a finite group belonging to \( C_p \) and hence to \( C_p \). If follows as for Example 3.12 that we may find a homology decomposition

\[
\ldots \subseteq \mathbb{Q}^{(\infty)}_{\eta} \subseteq \mathbb{Q}^{(\infty)}_{\eta} \subseteq \ldots
\]

going with each \( \mathbb{Q}^{(\infty)}_{\eta} \) a finite complex belonging to \( C_p \). Thus, \( X \) being finite-dimensional,

\[
h^*(X; Z_p) = [X_\bullet; \eta] \subseteq \mathbb{Q}^{(\infty)}_{\eta}, \quad \mathbb{Q}^{(\infty)}_{\eta} \in [J_1]
\]

and the lemma is proved.

---

**Remark.** The requirement that \( h \) be representable is not, in fact, a limitation on the scope of Theorem 4.1. For if \( h \) is a cohomology theory with finitely-generated coefficients, then its restriction to the category of 1-connected finite CW-complexes is certainly representable. If this
representable theory is called $h'$ then $h$ and $h'$ may, of course, not coincide over the whole of $J_0$; and Theorem 4.1 enables us to infer that $l^m(X; Z\mathbb{Q}) = h^{m-1}(X; Z\mathbb{Q})$.

We now introduce two variants of Theorem 4.1.

**Theorem 4.14.** Let $h$ be a representable cohomology theory, let $h_0$ be its restriction to the category of 1-connected finite CW-complexes belonging to $C_p$ and let $h_+$ be the Eilenberg extension of $h_0$ to the category of 1-connected finite complexes. Then (4.2) holds.

Notice that, compared with Theorem 4.1, we have removed a restriction from $h$, but imposed a restriction on $J_0$. Let us write $J_0$ for this new version of $J_0$; thus $J_0$ is the category of 1-connected finite CW-complexes. It is plain from a scrutiny of the proof of Theorem 4.1 that the only lemma requiring amendment in order to prove Theorem 4.14 is Lemma 4.9. Thus we must prove

**Lemma 4.9'.** If $h$ is a representable cohomology theory, then the functor $h^m(X; Z\mathbb{Q})$ is $J_0$-pro-representable, where $J_0$ is as in Theorem 4.1 and $J_0$ is as above, $p < \infty$.

We base the proof of this lemma on the following general proposition concerning Serre classes.

**Proposition 4.15.** Let $C$ be a Serre class and let $f: X \to Q$ be a map of a finite 1-connected CW-complex $X$ into a CW-complex $Q$ belonging to $C$. Then $f$ may be factored up to homotopy as

$$X \stackrel{\sim}{\to} Y \to Q,$$

where $Y$ is a finite CW-complex belonging to $C$.

**Proof.** Suppose $\pi_n(X) \in C$, $i < n$, $m > 2$. Then if $A = \ker f_2: \pi_m(X) \to \pi_m(Q)$, $A$ is a finitely generated abelian group and we may attach $(m+1)$-cells $e_i$ to $X$, corresponding to each generator $x$ of $A$ by constructing a finite CW-complex $X_\infty$ for $A$, by a map in the class $\gamma$. Then $X_\infty$ is finite and $f$ extends to $f_\infty: X_\infty \to Q$. Moreover $\pi_m(X_\infty) = \pi_m(X)A$ and $f_\infty: \pi_m(X_\infty) \to \pi_m(Q)$ is a monomorphism. Thus $\pi_m(X_\infty) \in C$. We may proceed inductively in this way and eventually arrive at a factorization

$$X \stackrel{\sim}{\to} X_\infty \to Q$$

where $X_\infty$ belongs to $C$ and is finite in each dimension. Thus the homology sections of $X_\infty$ are finite complexes belonging to $C$ and, since $X_\infty$ is finite, $i$ factors (up to homotopy) through some homology section of $X_\infty$. Thus we obtain the required factorization by restricting the domain of $f_\infty$ and the range of $i$ in (4.16) to this homology section $Y$.

**Proof of Lemma 4.10.** We proceed as in the proof of Lemma 4.9, obtaining a 1-connected CW-complex $Q_\infty$ in $C_p$ such that

$$h^m(X; Z\mathbb{Q}) = [X; Q_\infty].$$

Thus to prove the lemma we must show that every homotopy class $x \to Q_\infty$, may be factored as

$$X \stackrel{x}{\to} Y \to Q_\infty,$$

and, second, that given two factorizations

$$X \stackrel{\gamma}{\to} Y \to Q_\infty\quad Y \in J_0,$$

we may embed (4.17) in a commutative diagram

$$X \to Y \to Q_\infty$$

Now the first assertion follows immediately from Proposition 4.15, so it remains to prove the second. We may take the double mapping cylinder of $g_1$ and $g_2$; this amounts to replacing $g_1$ and $g_2$ by cofibration-inclusions and then taking the union $Z$ of $Y_1$ and $Y_2$ with $X$ amalgamated. We may then further suppose that $\tau_1 g_1 = \tau_2 g_2$ as maps (and not merely as homotopy classes) so that we have a commutative diagram

$$X \to Y \to Q_\infty$$

By Van Kampen's Theorem $Z$ is 1-connected and it is certainly finite. Thus we may apply Proposition 4.13 to $w: Z \to Q_\infty$, and immediately infer the existence of a diagram (4.18). This completes the proof of Lemma 4.9' and with it Theorem 4.14.
The second variant of Theorem 4.1 is concerned with ordinary cohomology \( H \). We now take \( J_0 \) to be the category of 1-connected CW-complexes whose homology groups are finite groups in \( CP \) and \( J_1 \) to be the category of 1-connected CW-complexes. It was already observed in [5] that the Kan extension, from \( J_0 \) to \( J_1 \), of any cohomology theory is again a cohomology theory. We prove

**Theorem 4.19.** If \( h_i \) is the Kan extension of \( H \) from \( J_0 \) to \( J_1 \), where \( J_0, J_1 \) are as in the paragraph above, then

\[ h_i^\ast(X) = H^{n-i}(X, \mathbb{Z} \text{mod}) \]

the latter group being taken in the sense of [9].

Notice that, compared with Theorem 4.1, we have greatly restricted \( h \), but we have also greatly enlarged \( J_0 \) and \( J_1 \) by allowing infinite-dimensional complexes.

**Proof of Theorem 4.19.** The argument is substantially easier than that of Theorem 4.1. \( H^i(Y, \mathbb{Z} \text{mod}) \) if \( Y \in J_0 \) and the analogue of Lemma 4.5 continues to hold,

\[ H^i(Y) \cong H^{n-i}(Y, \mathbb{Z} \text{mod}) \]

The analogue of Lemma 4.9 is the stronger, but trivial, statement that \( H^i( ; \mathbb{Z} \text{mod}) \) is \( J_0 \)-representable, since the Eilenberg-MacLane complex \( K(\mathbb{Z} \text{mod}, n) \) is in \( J_0 \). The rest of the argument holds and the theorem follows.

**Remarks.** (i) Notice that although \( H^i( ; \mathbb{Z} \text{mod}) \) has its usual meaning in \( J_0 \), it does not have its usual meaning in \( J_1 \). For if the homology groups of \( X \) are not of finite type, then \( \mathcal{C}(X) \otimes \mathbb{Z} \text{mod} \) and Hom(\( \mathcal{C}(X), \mathbb{Z} \text{mod} \)) are not cochain-equivalent. The cohomology groups of the former cochain complex enter into the statement of Theorem 4.19; those of the latter are the usual cohomology groups of \( X \) with values in \( \mathbb{Z} \text{mod} \).

(ii) We may elaborate Theorem 4.19 by looking to see what happens if we replace \( H \) by \( H( ; G) \). If \( G \) is a torsion group and \( G = G_F \otimes G_P \wedge \) where \( G_F \subset CP \) and \( P \) is the set of primes complementary to \( F \), then it is not difficult to see that the Kan extension of \( H( ; G) \) is \( H( ; G_P) \), where again the latter has to be understood in the sense of [9].

(iii) With regard to Theorem 4.1, 4.14 or 4.19, the result implies, of course, a natural transformation

\[ \omega: H^\ast(X, \mathbb{Z} \text{mod}) \to h_i^\ast(X) \]

of cohomology theories on \( J \), which is an equivalence on \( J_0 \). This natural transformation \( \omega \) is nothing other than the composite

\[ H^\ast(X, \mathbb{Z} \text{mod}) \to \text{Tor}(\mathbb{H}^a(X), \mathbb{Z} \text{mod}) \to \text{Tor}(\mathbb{H}^a(X), G) \]

\[ \to h_i^\ast(X) \otimes \mathbb{Z} \cong h_i^\ast(X) \].

The intermediate terms in (4.20) are not cohomology theories, but they admit suspension isomorphisms. The arrows are all compatible with suspension and hence the composite is a natural transformation of cohomology theories. Of course, if \( X \in J_0 \), then each arrow in (4.20) is an isomorphism.

**References**


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