Proof of Theorem 2. Let $f\colon Y_0 \to M$ be an extendable mapping, denote by $\bar{f}\colon Y \to M$ an extension of f, and let a be an unstable point of M. By Theorem 1 there exists a homotopy $H\colon M \times [0,1] \to M$ which satisfies (i) and (ii). The required extension F of f can be obtained by the following formula:

$$F(y) = H\big(\bar{f}(y), \min\{1, \varrho(y, Y_0)\}\big) .$$

It is easy to see that F is an extension of f and $F^{-1}(a) = f^{-1}(a)$.

Theorems 3 and 4 are proved in the introduction.

Proof of Theorem 5. By Theorem 1 there exists a homotopy $H\colon M \times [0,1] \to M$ which satisfies (i) and (ii). The mapping $h\colon M \times [0,1] \to M$ defined by $h(x,t) = H\big(x, \max[0, t - \varrho(a,x)]\big)$ is the required homotopy. Indeed:

$h(x, 0) = H(x, 0) = x$ for any $x \in M$;

$h(x, t) = a$ iff $x = a$ and $t = 0$;

if $\varrho(a, x) \geqslant t$, then $h(x,t) = x$, which implies (iii), since h is continuous.

§ 3. Two problems.

PROBLEM 1. Is the assumption of the local compactness of M necessary in Theorem 1?

PROBLEM 2. Let a subset A of a space X be called unstable if there exists a homotopy $h\colon X \times [0,1] \to X$ such that $h(x, 0) = x$ for all $x \in X$ and $h(x,t) \notin A$ for all $x \in X$ and $t > 0$. Is it true that the set of all unstable points of a finite-dimensional compact metric space is an unstable set?

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Transfinite metrics, sequences and topological properties

by

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The concepts of sequential, Fréchet, and first-countable space have natural generalizations to higher cardinality, given by replacing the use of sequences in their definitions with the use of transfinite sequences of appropriate type. It is also known that sequential, Fréchet, and first-countable spaces are the images of metric (or pseudometric) spaces under maps that are respectively quotient, hereditarily quotient or pseudo-open, and open. We shall use the theory of transfinite metrics to extend these characterizations to generalized sequential, Fréchet, and first-countable spaces.

The characterizations of sequential, Fréchet, and first-countable spaces in terms of metric spaces are due respectively to Franklin [3], Arhangel'skii [1], and Ponamarev [4]. The paper by Franklin contains a complete discussion of these results.

1. ω_μ-metrics. The most extensive work on transfinite metrics has been done by Sikorski ([5] and [6]). The recent paper [7] by Stevenson and Thron contains a summary of work done in this field. We shall adopt the terminology of [7], and refer to that paper for terms not defined herein.

In the remainder of this paper the symbol ω_μ will refer to an arbitrary but fixed regular initial ordinal number. We shall not distinguish between initial ordinals and cardinals. The symbol W_μ represents the least algebraic field containing the set of all ordinals less than ω_μ, as constructed in [6].

2. Transfinite sequential properties. The definitions of these properties are natural extensions of the usual definitions in terms of ordinary sequences.

An *ω_μ-sequence* in X is a collection of points of X indexed by the ordinals less than ω_μ. It is *frequently* in the set A if for each $\delta < \omega_\mu$ there is $\gamma < \omega_\mu$ such that $\gamma > \delta$ and $x_\gamma \in A$; it is *eventually* in the set A if the tail of some element is a subset of A; it *converges* to the point $x \in X$ if it is eventually in every neighborhood of x.

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The subset A of X is *ω_μ-sequentially open* if every ω_μ-sequence that converges to a point in A is eventually in A, and the space X is *ω_μ-sequential* if every ω_μ-sequentially open subset of X is open. We can also examine the closed sets; a set A is *ω_μ-sequentially closed* if no ω_μ-sequence that is frequently in A converges to a point not in A, and clearly this is the case if and only if $X-A$ is ω_μ-sequentially open.

The ω_μ-sequentially closed subsets of a space can also be found as the fixed points of the following closure operator (in the sense of Čech ([2], p. 237]): $\omega_\mu\text{-cl}A = \{x\colon$ some ω_μ-sequence that is frequently in A converges to $x\}$. This operator satisfies three of the conditions for a topological closure operator, but we need not have $\omega_\mu\text{-cl}(\omega_\mu\text{-cl}A) = \omega_\mu\text{-cl}A$. The ω_μ-sequentially closed subsets of the space are those for which $\omega_\mu\text{-cl}A = A$.

If the above operator is indeed topological, that is, it satisfies $\omega_\mu\text{-cl}A = \omega_\mu\text{-cl}(\omega_\mu\text{-cl}A)$, then we say that X is an *ω_μ-Fréchet* space. Clearly every ω_μ-Fréchet space is ω_μ-sequential.

Ordinary sequential and Fréchet spaces are closely associated with spaces in which each point has a countable base; equivalently for such spaces each point has a descendingly ordered base of type ω_0. Given any initial ordinal ω_μ, an *ω_μ-chained base* for the point $x \in X$ is a descendingly ordered base of type ω_μ; the space X is called an *ω_μ-chainable space* if every point has an ω_μ-chained base. It follows in the usual manner that every ω_μ-chainable space is an ω_μ-Fréchet space, hence also an ω_μ-sequential space. An ω_μ-pseudometrizable space is clearly ω_μ-chainable, and thus ω^μ-Fréchet and ω_μ-sequential.

A space is said to be *ω_μ-additive* [6] if any family of open sets of cardinal less than the cardinal ω_μ has open intersection, and a space will be said to have *local character* ω_μ if every point has a neighborhood base of cardinal no larger than ω_μ. Using the assumed regularity of ω_μ one can readily establish that a space is ω_μ-chainable if and only if it is ω_μ-additive and has local character ω_μ.

3. Convergent ω_μ-sequences. The space to be examined next may be thought of as the prototypical convergent ω_μ-sequence, in the same sense as the one-point compactification N^* of the countable discrete space N is the prototypical convergent sequence.

Let D_μ be the collection of ordinals less than ω_μ taken with the discrete topology. Let D_μ^* be the collection of ordinals less than or equal to ω_μ, with each point less than $\omega_\mu \in D_\mu^*$ isolated and basic neighborhoods of ω_μ the tails $T_\tau = \{\sigma\colon \sigma < \tau \leqslant \omega_\mu\}$. Define $\varphi\colon D_\mu^* \to W_\mu$ by $\varphi(\sigma) = 1/\sigma$ when $\sigma \neq \omega_\mu$, $\varphi(\omega_\mu) = 0$ (we do not consider the ordinal 0 as belonging to D_μ^*). It is readily shown that the function $d\colon D_\mu^* \times D_\mu^* \to W_\mu$ defined by $d(x, y) = \text{abs}(\varphi(x) - \varphi(y))$ is an ω_μ-metric for the topology of D_μ^*.

The ω_μ-metric space (D_μ^*, d) can be shown to be ω_μ-complete so that the topological space D_μ^* is completely ω_μ-metrizable. Clearly it is zero-dimensional, and it is easily shown to be ω_μ-compact (using the regularity of ω_μ). Thus any topological sum of copies of D_μ^* is locally ω_μ-compact, completely ω_μ-metrizable (using the same method for constructing a complete metric as is used for D_0^*), and zero-dimensional.

Although the space D_0^* is the unique one point ω_0-compactification of D_0 it is false that for $\mu > 0$ the space D_μ^* is the unique one point ω_μ-compactification of D_μ; in fact, the one point compactification of the discrete space D_μ is a one-point ω_μ-compactification that is not equal to D_μ^* when $\mu > 0$.

4. D_μ^*-rosters. The connection between the topology of a space and its convergent transfinite sequences can now be examined using the same techniques as are used for ordinary sequences.

Any topological sum of copies of D_μ^* is called a *D_μ^*-roster*. Given a topological space X we can form a space $D_\mu^*(X)$, called the *D_μ^*-roster* of X, by taking the topological sum $+\{D_{\mu f}^*\colon f \in C(D_\mu^*(X)\}$ of one copy of D_μ^* for each continuous function from D_μ^* into X. The function $r\colon D_\mu^*(X) \to X$ given by $r(\lambda) = f(\lambda)$ when $\lambda \in D_{\mu f}^*$ is called the *D_μ^*-roster map* of X. It is obviously continuous, and onto in view of the constant functions.

There is clearly a bijective correspondence between functions $f \in C(D_\mu^*, X)$ and pairs consisting of a ω_μ-sequence $\{x_\varkappa\}$ and a point $x \in X$ to which the sequence converges. In fact, a ω_μ-sequence converges if and only if there is a continuous extension of it as a function from D_μ into X to a function from D_μ^* into X.

When X is Hausdorff the correspondence above is of course one-one between convergent sequences and functions $f \in C(D_\mu^*, X)$, since limits are unique.

Associated with the D_μ^*-roster and the roster map r there is the *reduced D_μ^*-roster* $R_\mu^*(X)$ and the *reduced roster map* p: the space $R_\mu^*(X)$ is the quotient space obtained from the equivalence relation $r(a) = r(b)$ on $D_\mu^*(X)$, and p is the map such that $pq = r$, where q is the quotient map of $D_\mu^*(X)$ onto $R_\mu^*(X)$. The functions p and q are of course continuous, and p is a contraction.

5. ω_μ-sequential spaces. The following characterization of ω_μ-sequential spaces generalizes the characterization of sequential spaces given by Franklin [3].

THEOREM D. *The following are equivalent for a space X.*

(i) *X is ω_μ-sequential.*

(ii) *X is the quotient of an ω_μ-sequential space.*

(iii) *X is the quotient of an ω_μ-Fréchet space.*

(iv) *X is the quotient of an ω_μ-metrizable space.*

(v) *X is the quotient of a zero dimensional locally ω_μ-compact completely ω_μ-metrizable space.*

(vi) *X is the quotient of a D^*_μ-roster.*

(vii) *The D^*_μ-roster map of X is a quotient.*

(viii) *The reduced D^*_μ-roster map is a homeomorphism.*

Proof. In view of what has already been shown, we need establish only that (i) implies (viii), (vi) implies (v), and (ii) implies (i).

That (ii) implies (i) is shown just as the corresponding result in Franklin's paper. That (vi) implies (v) is shown by observing that the appropriate properties are preserved under taking of topological sums.

Since (vii) and (viii) are clearly equivalent, we show that (i) implies (vii). Now if $r^{\leftarrow}[A]$ is open in $D^*_\mu(X)$, $x \in A$, $\{x_\varkappa\}$ is an ω_μ-sequence converging to x, and f is the function in $C(D^*_\mu, X)$ associated with x and $\{x_\varkappa\}$, then $r^{\leftarrow}[A]$ is a neighborhood in $D^*_\mu(X)$ of $\omega_\mu \in D^*_{\mu f}$, from which it follows that $\{x_\varkappa\}$ is eventually in A. Thus A is sequentially open, and therefore open, in X.

6. ω_μ-Fréchet spaces. The following characterization of ω_μ-Fréchet spaces generalizes the characterization of Fréchet spaces given by Arhangel'skii [1].

The characterization is in terms of *hereditarily quotient* maps, also called *pseudo-open*: a map $f\colon Y \to X$ is hereditarily quotient if it is continuous and for each $A \subset X$ the reduced map $f_A\colon f^{\leftarrow}[A] \to A$ is quotient. It is known that this is equivalent to the condition that for each $x \in X$ and each neighborhood U of $f^{\leftarrow}[x]$ we have $x \in \operatorname{int} f[U]$. Another equivalent condition is that for each $A \subset X$ and each $x \in \operatorname{cl} A$ there is $y \in Y$ with $f(y) = x$ and $y \in \operatorname{cl} f^{\leftarrow}[A]$.

THEOREM E. *The following are equivalent for a space X.*

(i) *X is ω_μ-Fréchet.*

(ii) *X is the hereditarily quotient image of an ω_μ-Fréchet space.*

(iii) *X is the hereditarily quotient image of an ω_μ-metrizable space.*

(iv) *X is the hereditarily quotient image of a zero-dimensional locally ω_μ-compact completely ω_μ-metrizable space.*

(v) *X is the hereditarily quotient image of a D^*_μ-roster.*

(vi) *The D^*_μ-roster map of X is hereditarily quotient.*

Proof. To show that (ii) implies (i) is straightforward using the definitions. The only other implication that requires proof is that (i) implies (vi). Now if $A \subset X$ and $x \in \operatorname{cl} A$ there is by (i) a ω_μ-sequence lying in A and converging to x. If f is the associated function from D^*_μ into X and y is the limit point $\omega_\mu \in D^*_{\mu f}$ we have $f(y) = x$ and $y \in \operatorname{cl} r^{\leftarrow}[A]$. Thus r is hereditarily quotient.

7. ω_μ-chainable spaces. The following characterization of ω_μ-chainable spaces generalizes the characterization of first-countable spaces given by Ponamarev [4]. As noted by Franklin [3] the chain of characterizations using the D^*_μ-roster does not extend to this case, since the rational space Q_0 is not the open image of its D^*_0-roster.

THEOREM C. *The space X is ω_μ-chainable if and only if it is the open image of an ω_μ-pseudometrizable space, and if it is T_0 it is the open image of an ω_μ-metrizable space.*

Proof. It is straightforward to show that the open image of an ω_μ-pseudometrizable space is ω_μ-chainable.

To establish the converse we generalize the method of proof used in the first countable case.

Consider the collection of ω_μ-sequences (ordered by inclusion) of open subsets of X that form an ω_μ-chained base for some point $x \in X$; such a sequence may be a base for several points of X (if X is not T_0) and we repeat a given sequence once for each point for which it forms a base, using the particular point as an additional index. Let Z be the set of such sequence-point pairs.

Define a function $\delta(p, q)$ on $Z \times Z$ by setting $\delta(p, q) = \omega_\mu$ if the sequences represented by p and q are identical, and otherwise letting $\delta(p, q)$ be the first ordinal for which the sequences differ. It follows that for $p, q, r \in Z$ we have $\delta(p, q) \geqslant \min\{\delta(p, r), \delta(q, r)\}$. Setting $d(p, q) = 0$ if $\delta(p, q) = \omega_a$ and $d(p, q) = 1/\delta(p, q) \in W_\mu$ if not, it is straightforward to show that d is an ω_μ-pseudometric on the set Z.

A function h from Z into X is defined by letting $h(p)$ be the point $x \in X$ represented by $p \in Z$. We shall show that h is a continuous open function from the space Z with the ω_μ-pseudometric topology onto X.

For each $p \in Z$ and each $\lambda < \omega_\mu$ let p_λ be the open subset of X which is the λth member of the sequence represented by $p \in Z$. It is readily shown that p_λ is the image under h of the ball about $p \in Z$ of radius $1/\lambda$. Therefore h is an open function. To show that h is continuous we observe first that if V is a nonempty open subset of X then there is $p \in Z$ with $V = p_1$, from which it follows that $h^{\leftarrow}[V]$ is the union of balls of radius 1 about points $p \in Z$ for which $V = p_1$. This argument also shows that h is onto. Thus the proof of Theorem F is complete.

In the case of first countable spaces it is known that the space Z can be taken to have the same total character (minimum basis cardinal) as X. This can also be done in the present instance. One need only show that if $\mathfrak{B}$ is any basis for X then for each point there is an ω_μ-chained basis of members of $\mathfrak{B}$.

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On Kan extensions of cohomology theories and Serre classes of groups

by

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1. Introduction. This paper constitutes a continuation of the investigation initiated in [4], [5]. In those papers we introduced a process, involving the Kan extension of a functor, for extending a cohomology theory from a category J_0 of based topological spaces to a larger category J_1. This process generalizes a characterization of Čech cohomology which has been noted by Eilenberg and Steenrod and studied by Dold. However, the process partakes far more of the spirit of Kan's work on extending functors than of the original description of Čech cohomology, so that the examples of cohomology theories expressible as Kan extensions take one very far from Čech cohomology, while retaining a certain generalized continuity property. We should mention here that Lee and Raymond [11] have studied generalized Čech theories in a somewhat different sense, more strongly motivated by the classical description of Čech theory. There is some small overlap with the present authors' work, and a comparison of the two approaches will form the subject of a later paper (1).

A principal concern in [4], [5] is that of deciding under what conditions the Kan extension (2), h_1, to J_1 of a cohomology theory h on J_0 (or *maximal* extension in the terminology of [2]) is itself a cohomology theory. We always require that the categories J considered suitable for supporting a cohomology theory be admissible; that is, they should be non-empty full subcategories of the category of based spaces and based maps, they should admit mapping cones, and should contain entire homotopy types. We can state the axioms for h or, more precisely, (h, σ), where σ is the suspension transformation $h^n \to h^{n+1}\Sigma$, to be a cohomology theory in any admissible category; but the Kan extension of a cohomology theory from an admissible category J_0 to an admissible category J_1 may well fail to be a cohomology theory. After some preliminary algebraic argument in Section 2, we formulate a criterion for the Kan extension

(1) *Remark on Čech extensions of cohomology functors*, Proc. Adv. St. Inst. Aarhus (1970), pp. 44–66.

(2) We use here the notation h_1 in preference to the ${}_0h$ notation of [4], [5].