Homotopically labile points of locally compact metric spaces

by

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§ 1. Introduction. A point $a$ of a space $M$ is called homotopically labile (or unstable) (see [1]) if for each neighbourhood $U$ of $a$ there exists a continuous function $h: M \times [0, 1] \to M$ (a homotopy) which satisfies the following four conditions:

1. $h(x, 0) = a$ for all $x \in M$,
2. $h(x, t) = a$ for all $x \in U, t \in [0, 1]$,
3. $h(x, t) \in U$ for all $x \in U, t \in [0, 1]$,
4. $h(x, 1) \neq a$ for all $x \in M$.

A point $a$ is called stable if it is not homotopically labile.

In this paper is shown a connection between the concept of homotopically labile point on the one hand, and the concepts of the retraction and of the extension of a map, on the other hand, for locally compact metric spaces. This connection is established by means of the following characterization of unstable points of locally compact metric spaces:

**Theorem 1.** A point $a$ of a locally compact metric space $M$ is unstable if and only if there exists a homotopy $H: M \times [0, 1] \to M$ which satisfies the following two conditions:

1. $H(x, 0) = x$ for all $x \in M$,
2. $H(x, t) \neq a$ for all $x \in M, t > 0$.

In other words, Theorem 1 states that a $a \in M$ is unstable iff there exists a retraction $r: M \times [0, 1] \to M \times \{0\}$ such that $r^{-1}(a, 0) = \{(a, 0)\}$.

Let $Y_n$ be a closed subset of a space $Y$. A mapping $f: Y_n \to M$ is said to be extendable if there exists an extension $\tilde{f}: Y \to M$ of $f$. The following theorem establishes a connection between the unstability of a point $a \in M$ and the existence of a special extension of any extendable mapping $f: Y_n \to M$.

**Theorem 2.** If $a$ is an unstable point of a locally compact metric space $M$, then for any extendable mapping $f: Y_n \to M$, where $Y_n$ is a closed
subset of a metric space \( Y \), there exists an extension \( E: Y \to M \) such that \( E^{-1}(a) = f^{-1}(a) \).

In a special case, where the space \( Y \) contains \( M \) as a retract, we obtain as a corollary:

**Theorem 3.** If \( a \) is an unstable point of a locally compact metric space \( M \), then for any metric space \( X \) which contains \( M \) as a retract there exists a retraction \( r: X \to M \) such that \( r^{-1}(a) = \{a\} \).

On the other hand, it is known (see [2], p. 562) that if \( a \in Y \subseteq X \) is an unstable point in \( X \) and if there exists a retraction \( r: X \to M \) such that \( r^{-1}(a) = \{a\} \), then \( a \) is unstable in \( Y \), too. Together with Theorem 3 this implies the following:

**Theorem 4.** If \( M \) is a locally compact retract of a metric space \( X \) and if \( a \in M \) is unstable in \( X \), then \( a \) is unstable in \( M \) if and only if there exists a retraction \( r: X \to M \) such that \( r^{-1}(a) = \{a\} \).

A homotopy \( H \), which one can obtain by Theorem 1 for an unstable point \( a \in M \), can be "localised" with respect to \( a \); more precisely:

**Theorem 5.** If \( a \) is an unstable point of a locally compact metric space \( M \), then there exists a homotopy \( h: M \times [0, 1] \to M \) which satisfies both (i) and (ii) (from Theorem 1) and, moreover,

(iii) For any neighbourhood \( U \) of \( a \) there exists a number \( \epsilon > 0 \) such that for each \( s \in \epsilon \) \( h(x, s) = x \) if \( x \not\in U \) and \( h(x, s) \not\in U \) if \( x \in U \).

§ 2. Proofs.

**Lemma 1** (see [2], p. 562, Proposition 3). Let \( a \in X \subseteq X \) and let \( r: X \to M \) be a retraction such that \( r^{-1}(a) = \{a\} \). Then if \( a \) is unstable in \( X \), then it is unstable in \( Y \), too.

**Lemma 2** (compare [1], p. 163). Let \( a \) be an unstable point of a space \( X \) and let \( Y \) be a Tychonoff space. Then for any \( y \) \( \in Y \) the pair \( (a, y) \) is an unstable point of the Cartesian product \( X \times Y \).

**Proof.** Let \( W \) be a neighbourhood of \( (a, y) \). The point \( a \) has a neighbourhood \( U \subseteq Y \) and \( y \) has a neighbourhood \( V \subseteq V \) such that \( U \times V \subseteq W \). Let \( f: X \to [0, 1] \) be a continuous function such that \( f(y_0) = 1 \), and \( f(y) = 0 \) for all \( y \not\in V \), and let \( h: X \times [0, 1] \to X \) be a homotopy which satisfies (1)-(4). Then the homotopy \( H: X \times X \times [0, 1] \to X \times X \) defined by \( H(x, y, t) = h(x, f(y), t) \) satisfies conditions (1)-(4) with respect to \( a \) and \( y \).

**Proof of Theorem 1. 1st Necessity.** Denote the Cartesian product \( M \times [0, 1] \) by \( Z \), and assume that \( a \in M \) is unstable. Thus, by Lemma 2, each point \( a_n = (a, 1/n) \) \( (n = 1, 2, \ldots) \) is unstable in \( Z \). Let \( U_n \) be a compact neighbourhood of \( a_n \) such that \( \operatorname{diam} U_n < 1/2n(n+1) \). Observe that \( U_n \cap U_m = \emptyset \) if \( m \neq n \), and each \( U_n \) is disjoint with the set \( M \times \{0\} \).

For each \( n \) there exists a homotopy \( h_n: Z \to [0, 1] \to Z \) which satisfies (1)-(4) with respect to \( a_n \) and \( U_n \). Now, define a continuous function \( f: Z \to [0, 1] \) by the formula:

\[
f(z) = \begin{cases} h_n(z, 1) & \text{if } z \in U_n \text{ for some } n, \\
0 & \text{if } z \not\in \bigcup_{n=1}^\infty U_n.
\end{cases}
\]

It is easy to see that \( f(z) \neq a_n \) for each \( z \in \mathbb{Z} \) and \( n = 1, 2, \ldots \), moreover:

(a) the set \( Z \setminus f(Z) \) is an open neighbourhood of any \( a_n \) (since any \( U_n \) is compact),
(b) \( f^{-1}(a, 0) = \{(a, 0)\} \),
(c) the restriction \( f \mid_{M \times \{0\}} \) is an identity map.

By (a), for each \( n \), the set \( Z \setminus f(Z) \) contains a neighbourhood \( V_n = W_n \times \{y_n, y_0\} \) of \( a_n \), where \( W_n \subseteq X \) is a neighbourhood of the point \( a_n \), and \( \{y_n, y_0\} \) is an interval-neighbourhood of the point \( 1/n \) in \( [0, 1] \), such that \( y_n + 1 < y_0 \). We can assume that \( W_n \subseteq X \).

Then \( a \) is unstable in \( M \); therefore, for each \( n \) there exists a homotopy \( h_n: M \times [0, 1] \to M \) which satisfies conditions (1)-(4) with respect to \( a \) and \( W_n \). Define a continuous function \( g: Z \setminus \bigcup_{n=1}^\infty V_n \to Z \) by the following formula:

\[
g(x, t) = \begin{cases} g_n(x, 1, t) & \text{if } y_{n+1} = t \leq y_n \text{ for some } n, \\
(x, t) & \text{otherwise}.
\end{cases}
\]

Observe that \( g \) has the following properties:

(d) the set \( g(Z \setminus \bigcup_{n=1}^\infty V_n) \) contains no point of the form \( (a, s) \) with \( s > 0 \),
(e) \( g^{-1}(a, 0) = \{(a, 0)\} \),
(f) the restriction \( g \big|_{M \times \{0\}} \) is an identity map.

Let \( \pi: Z \to M \) be the canonical projection, \( \pi(x, t) = x \). Then the composition \( \bar{F} = \pi \circ g : M \times [0, 1] \to M \) is the required homotopy. Indeed: (e) and (f) imply that (i) is satisfied, and (ii) follows by (a), (b), (d), (e).

2nd Sufficiency. By Lemma 2 the point \( (a, 0) \) is unstable in the Cartesian product \( M \times [0, 1] \). The mapping \( r: M \times [0, 1] \to M \times \{0\} \) defined by \( (x, t) = h(x, t, 0) \) is a retraction (by (i)) and \( r^{-1}(a, 0) = \{(a, 0)\} \) (by (ii)), and so assumptions of Lemma 1 are satisfied. Thus \( (a, 0) \) is an unstable point in \( M \times \{0\} \), which means the same as that the point \( a \) is unstable in \( M \).
Proof of Theorem 2. Let \( f : Y_0 \to M \) be an extendable mapping, denote by \( f^* : Y \to M \) an extension of \( f \), and let \( a \) be an unstable point of \( M \).

By Theorem 1 there exists a homotopy \( H : M \times [0, 1] \to M \) which satisfies (i) and (ii). The required extension \( F \) of \( f \) can be obtained by the following formula:

\[
F(y) = H \left( f(y), \min \{ 1, \varphi(y, Y_0) \} \right).
\]

It is easy to see that \( F \) is an extension of \( f \) and \( F^{-1}(a) = f^{-1}(a) \).

Theorems 3 and 4 are proved in the introduction.

Proof of Theorem 5. By Theorem 1 there exists a homotopy \( H : M \times [0, 1] \to M \) which satisfies (i) and (ii). The mapping \( h : M \times [0, 1] \to M \) defined by \( h(x, t) = H(x, \max \{ 0, t - \varphi(x, x) \}) \) is the required homotopy. Indeed:

- \( h(x, 0) = H(x, 0) = x \) for any \( x \in M \);
- \( h(x, t) = a \) if \( x = a \) and \( t = 0 \);
- if \( \varphi(x, x) \geq t \), then \( h(x, t) = x \), which implies (iii), since \( h \) is continuous.

§ 3. Two problems.

Problem 1. Is the assumption of the local compactness of \( M \) necessary in Theorem 1?

Problem 2. Let a subset \( A \) of a space \( X \) be called unstable if there exists a homotopy \( h : X \times [0, 1] \to X \) such that \( h(x, 0) = x \) for all \( x \in X \) and \( h(x, t) \notin A \) for all \( x \in X \) and \( t > 0 \). Is it true that the set of all unstable points of a finite-dimensional compact metric space is an unstable set?

References


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