Homotopically labile points of locally compact metric spaces

by

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§ 1. Introduction. A point a of a space M is called *homotopically labile* (or *unstable*) (see [1]) if for each neighbourhood U of a there exists a continuous function $h\colon M\times[0,1]\to M$ (a homotopy) which satisfies the following four conditions:

(1) $h(x,0)=x$ for all $x\in M$,

(2) $h(x,t)=x$ for all $x\notin U$, $t\in[0,1]$,

(3) $h(x,t)\in U$ for all $x\in U$, $t\in[0,1]$,

(4) $h(x,1)\neq a$ for all $x\in M$.

A point a is called *stable* if it is not homotopically labile.

In this paper is shown a connection between the concept of homotopically labile point on the one hand, and the concepts of the retraction and of the extension of a map, on the other hand, for locally compact metric spaces. This connection is established by means of the following characterization of unstable points of locally compact metric spaces:

THEOREM 1. *A point a of a locally compact metric space M is unstable if and only if there exists a homotopy $H\colon M\times[0,1]\to M$ which satisfies the following two conditions*:

(i) $H(x,0)=x$ *for all* $x\in M$,

(ii) $H(x,t)\neq a$ *for all* $x\in M$, $t>0$.

In other words, Theorem 1 states that $a\in M$ is unstable iff there exists a retraction $r\colon M\times[0,1]\to M\times\{0\}$ such that $r^{-1}(a,0)=\{(a,0)\}$.

Let Y_0 be a closed subset of a space Y. A mapping $f\colon Y_0\to M$ is said to be *extendable* if there exists an extension $\bar{f}\colon Y\to M$ of f. The following theorem establishes a connection between the unstability of a point $a\in M$ and the existence of a special extension of any extendable mapping : $Y_0\to M$.

THEOREM 2. *If a is an unstable point of a locally compact metric space M, then for any extendable mapping $f\colon Y_0\to M$, where Y_0 is a closed* *subset of a metric space Y, there exists an extension $F\colon Y \to M$ such that* $F^{-1}(a) = f^{-1}(a)$.

In a special case, where the space Y contains M as a retract, we obtain as a corollary:

THEOREM 3. *If a is an unstable point of a locally compact metric space M, then for any metric space X which contains M as a retract there exists a retraction $r\colon X \to M$ such that $r^{-1}(a) = \{a\}$.*

On the other hand, it is known (see [2], p. 562) that if $a \in Y \subset X$ is an unstable point in X and if there exists a retraction $r\colon X \to Y$ such that $r^{-1}(a) = \{a\}$, then a is unstable in Y, too. Together with Theorem 3 this implies the following

THEOREM 4. *If M is a locally compact retract of a metric space X and if $a \in M$ is unstable in X, then a is unstable in M if and only if there exists a retraction $r\colon X \to M$ such that $r^{-1}(a) = \{a\}$.*

A homotopy H, which one can obtain by Theorem 1 for an unstable point $a \in M$, can be "localised" with respect to a; more precisely:

THEOREM 5. *If a is an unstable point of a locally compact metric space M, then there exists a homotopy $h\colon M \times [0,1] \to M$ which satisfies both* (i) *and* (ii) *(from Theorem* 1) *and, moreover,*

(iii) *For any neighbourhood U of a there exists a number $t_U \in (0,1]$ such that for each $s < t_U$ $h(x,s) = x$ if $x \notin U$ and $h(x,s) \in U$ if $x \in U$.*

§ 2. Proofs.

LEMMA 1 (see [2], p. 562, Proposition 3). *Let $a \in Y \subset X$ and let $r\colon X \to Y$ be a retraction such that $r^{-1}(a) = \{a\}$. Then if a is unstable in X, then it is unstable in Y, too.*

LEMMA 2 (compare [1], p. 163). *Let a be an unstable point of a space X and let Y be a Tychonoff space. Then for any $y_0 \in Y$ the pair (a, y_0) is an unstable point of the Cartesian product $X \times Y$.*

Proof. Let W be a neighbourhood of (a, y_0). The point a has a neighbourhood $U \subset X$ and y_0 has a neighbourhood $V \subset Y$ such that $U \times V \subset W$. Let $f\colon Y \to [0,1]$ be a continuous function such that $f(y_0) = 1$, and $f(y) = 0$ for all $y \notin V$, and let $h\colon X \times [0,1] \to X$ be a homotopy which satisfies (1)–(4). Then the homotopy $H\colon X \times Y \times [0,1] \to X \times Y$ defined by $H(x, y, t) = (h[x, tf(y)], y)$ satisfies conditions (1)–(4) with respect to (a, y_0) and W.

Proof of Theorem 1. 1° Necessity. Denote the Cartesian product $M \times [0,1]$ by Z, and assume that $a \in M$ is unstable. Thus, by Lemma 2, each point $a_n = (a, 1/n)$ $(n = 1, 2, \ldots)$ is unstable in Z. Let U_n be a compact neighbourhood of a_n such that $\operatorname{diam} U_n < 1/(2n(n+1))$. Observe that $U_n \cap U_m = \emptyset$ if $m \neq n$, and each U_n is disjoint with the set $M \times \{0\}$.

For each n there exists a homotopy $h^{(n)}\colon Z \times [0,1] \to Z$ which satisfies (1)–(4) with respect to a_n and U_n. Now, define a continuous function $f\colon Z \to Z$ by the formula

$$f(z) = \begin{cases} h^{(n)}(z, 1) & \text{if } z \in U_n \text{ for some } n, \\ z & \text{if } z \notin \bigcup_{n=1}^{\infty} U_n. \end{cases}$$

It is easy to see that $f(z) \neq a_n$ for each $z \in Z$ and $n = 1, 2, \ldots$, moreover:

(a) the set $Z \setminus f(Z)$ is an open neighbourhood of any a_n (since any U_n is compact),

(b) $f^{-1}(a, 0) = \{(a, 0)\}$,

(c) the restriction $f|_{M \times \{0\}}$ is an identity map.

By (a), for each n, the set $Z \setminus f(Z)$ contains a neighbourhood $V_n = W_n \times [p_n, q_n]$ of a_n, where $W_n \subset X$ is a neighbourhood of the point a, and $[p_n, q_n]$ is an interval-neighbourhood of the point $1/n$ in $[0,1]$, such that $q_{n+1} < p_n$. We can assume that $W_{n+1} \subset W_n$.

The point a is unstable in M; therefore, for each n there exists a homotopy $g^{(n)}\colon M \times [0,1] \to M$ which satisfies conditions (1)–(4) with respect to a and W_n. Define a continuous function $g\colon Z \setminus \bigcup_{n=1}^{\infty} V_n \to Z$ by the following formula:

$$g(x, t) = \begin{cases} \big(g^{(n)}(x, 1), t\big) & \text{if } q_{n+1} \leqslant t \leqslant p_n \text{ for some } n, \\ (x, t) & \text{otherwise}. \end{cases}$$

Observe that g has the following properties:

(d) the set $g(Z \setminus \bigcup_{n=1}^{\infty} V_n)$ contains no point of the form (a, s) with $s > 0$,

(e) $g^{-1}(a, 0) = \{(a, 0)\}$,

(f) the restriction $g|_{M \times \{0\}}$ is an identity map.

Let $\pi\colon Z \to M$ be the canonical projection, $\pi(x, t) = x$. Then the composition $H = \pi \circ g \circ f\colon M \times [0,1] \to M$ is the required homotopy. Indeed: (c) and (f) imply that (i) is satisfied, and (ii) follows by (a), (b), (d), (e).

2° Sufficiency. By Lemma 2 the point $(a, 0)$ is unstable in the Cartesian product $M \times [0,1]$. The mapping $r\colon M \times [0,1] \to M \times \{0\}$ defined by $r(x, t) = \big(H(x, t), 0\big)$ is a retraction (by (i)) and $r^{-1}(a, 0) = \{(a, 0)\}$ (by (ii)), and so assumptions of Lemma 1 are satisfied. Thus $(a, 0)$ is an unstable point in $M \times \{0\}$, which means the same as that the point a is unstable in M.

Proof of Theorem 2. Let $f\colon Y_0 \to M$ be an extendable mapping, denote by $\bar{f}\colon Y \to M$ an extension of f, and let a be an unstable point of M. By Theorem 1 there exists a homotopy $H\colon M\times[0,1]\to M$ which satisfies (i) and (ii). The required extension F of f can be obtained by the following formula:

$$F(y) = H\big(\bar{f}(y), \min\{1, \varrho(y, Y_0)\}\big) .$$

It is easy to see that F is an extension of f and $F^{-1}(a) = f^{-1}(a)$.

Theorems 3 and 4 are proved in the introduction.

Proof of Theorem 5. By Theorem 1 there exists a homotopy $H\colon M\times[0,1]\to M$ which satisfies (i) and (ii). The mapping $h\colon M\times[0,1]\to M$ defined by $h(x,t) = H\big(x, \max[0, t-\varrho(a,x)]\big)$ is the required homotopy. Indeed:

$h(x,0) = H(x,0) = x$ for any $x\in M$;

$h(x,t) = a$ iff $x = a$ and $t = 0$;

if $\varrho(a,x)\geqslant t$, then $h(x,t) = x$, which implies (iii), since h is continuous.

§ 3. Two problems.

PROBLEM 1. Is the assumption of the local compactness of M necessary in Theorem 1?

PROBLEM 2. Let a subset A of a space X be called unstable if there exists a homotopy $h\colon X\times[0,1]\to X$ such that $h(x,0) = x$ for all $x\in X$ and $h(x,t)\notin A$ for all $x\in X$ and $t>0$. Is it true that the set of all unstable points of a finite-dimensional compact metric space is an unstable set?

References

[1] K. Borsuk and J. W. Jaworowski, *On labil and stabil points*, Fund. Math. 39 (1952), pp. 159–175.

[2] W. Holsztyński and W. Kuperberg, *Star spaces and 2-dimensional contractible polytopes*, Bull. Acad. Polon. Sci. 13 (1965), pp. 561–563.

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Transfinite metrics, sequences and topological properties

by

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The concepts of sequential, Fréchet, and first-countable space have natural generalizations to higher cardinality, given by replacing the use of sequences in their definitions with the use of transfinite sequences of appropriate type. It is also known that sequential, Fréchet, and first-countable spaces are the images of metric (or pseudometric) spaces under maps that are respectively quotient, hereditarily quotient or pseudo-open, and open. We shall use the theory of transfinite metrics to extend these characterizations to generalized sequential, Fréchet, and first-countable spaces.

The characterizations of sequential, Fréchet, and first-countable spaces in terms of metric spaces are due respectively to Franklin [3], Arhangel'skii [1], and Ponamarev [4]. The paper by Franklin contains a complete discussion of these results.

1. ω_μ-metrics. The most extensive work on transfinite metrics has been done by Sikorski ([5] and [6]). The recent paper [7] by Stevenson and Thron contains a summary of work done in this field. We shall adopt the terminology of [7], and refer to that paper for terms not defined herein.

In the remainder of this paper the symbol ω_μ will refer to an arbitrary but fixed regular initial ordinal number. We shall not distinguish between initial ordinals and cardinals. The symbol W_μ represents the least algebraic field containing the set of all ordinals less than ω_μ, as constructed in [6].

2. Transfinite sequential properties. The definitions of these properties are natural extensions of the usual definitions in terms of ordinary sequences.

An *ω_μ-sequence* in X is a collection of points of X indexed by the ordinals less than ω_μ. It is *frequently* in the set A if for each $\delta<\omega_\mu$ there is $\gamma<\omega_\mu$ such that $\gamma>\delta$ and $x_\gamma\in A$; it is *eventually* in the set A if the tail of some element is a subset of A; it *converges* to the point $x\in X$ if it is eventually in every neighborhood of x.