As a corollary of the proof it is not difficult to show that if $M =_a N$ then for any cardinal $a$, $\mathcal{X}_a M =_a \mathcal{X}_a N$. Further, it follows that if $M$ and $N$ are $L_{\alpha \in \mathbb{N}}$-equivalent then $\exists \alpha M = \exists \alpha N$.

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A minimal model for strong analysis

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In [6] it is shown that axiomatic second order arithmetic does not possess a minimal $\omega$-model. Here we extend that result to general models of the full second order theory of $(\omega, +, \cdot)$ and show that various model theoretic concepts, e.g., the existence of prime models, minimal $\omega$-models, etc., all coincide, but are independent of Zermelo Fraenkel set theory and some of its extensions. These results are then applied to the weak second order theory of real numbers.

Let $\mathfrak{F} = (\mathbb{F}, \omega, +, \cdot)$ where $\mathbb{F}$ is the set of all functions mapping $\omega$ into $\omega$. Consider a two sorted language $\mathcal{L}$ for $\mathfrak{F}$ which contains individual variables $v_0, v_1, \ldots$ and function variables $a_0, a_1, \ldots$. Under our intended interpretation the individual variables range over $\omega$ and the function variables range over $\mathbb{F}$. This distinction between variables has been introduced for convenience. We can easily find an equivalent (though less suggestive) one sorted language for $\mathfrak{F}$. Thus we assume that all of the standard first order concepts suitably generalize to $\mathcal{L}$. In particular we shall be interested in the notions of proof $(\vdash)$, satisfaction $(\models)$, substructure $(\subseteq)$, and elementary subsystem $(\models \subseteq)$. Let $T = Th(\mathfrak{F})$ be the $\mathcal{L}$-theory of $\mathfrak{F}$. A model $\mathfrak{B}$ of $T$ is said to be prime in the sense of Vaught (cf. [16]) if $\mathfrak{B}$ is isomorphic to an elementary subsystem of every model of $T$. Let $\mathcal{A}$ be the set of functions $f \in \mathbb{F}$ which are definable in $\mathfrak{F}$ by some formula $\varphi(a_0)$ of $\mathcal{L}$ and let $\mathfrak{M} = (\mathcal{A}, \omega, +, \cdot)$. We characterize the prime models of $T$ in

Theorem 1. $\mathfrak{B}$ is a prime model of $T$ in the sense of Vaught if and only if $\mathfrak{B}$ is isomorphic to $\mathfrak{M}$ and $\mathfrak{B}$ is a model of $T$.

Proof. We use theorem 3.4 of [16] that a model is prime if and only if it is a denumerable atomic model. See [16] for an explanation of our terminology. For $n > \omega$ let $\varphi_n$ be a purely existential formula with $v_n$ as its free variable and containing no function variables which defines $v_n$ in $\mathfrak{F}$. If $\mathfrak{B} = (\mathbb{F}, \mathcal{X} \mathcal{Y}, \exists \mathcal{Z})$ is a prime model of $T$, we construct an iso-

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with $T$. If $\psi(a_1, a_2) \land \lambda(a_1, a_2)$ is consistent with $T$, then $\exists \psi(a_1, a_2) \times (\psi(a_1, a_2) \land \lambda(a_1, a_2))$. But $\exists \psi(a_1, a_2) \land \lambda(a_1, a_2)$ so that

$$\exists \psi(a_1, a_2) \land \lambda(a_1, a_2), \psi(a_1, a_2) \rightarrow \lambda(a_1, a_2) \land \lambda(a_1, a_2),$$

which implies that $\psi(a_1, a_2)$ is indeed an atom of $T$. Now let $\langle f, n \rangle$ satisfies $\psi(a_1, a_2)$ in $\mathfrak{B}$. Since $n(a_1, a_2)$ contains no function variables and $n$ satisfies $n(a_1, a_2)$ in $\mathfrak{B}$ it must do the same in $\mathfrak{A}$. Also $f$ uniquely satisfies $\psi(a_1, a_2)$ in $\mathfrak{B}$. Hence $\mathfrak{A} \models \exists \psi(a_1, a_2) \land \lambda(a_1, a_2)$. If $\psi(a_1, a_2)$ is satisfied in $\mathfrak{A}$ by some $g \neq f$, then there are $n, n_1, n_2$ such that $\psi(n, n_1) = n_1 \neq f(n_2)$ and consequently $\psi(n_1, n_2) \neq \psi(n, n_1) \land \psi(n_2, n_1)$ holds in $\mathfrak{A}$ and therefore also holds in $\mathfrak{B}$ since $\mathfrak{B}$ is a model of $T$. But this implies that $f(n_1) = n_1$, a contradiction. Thus $\exists \psi(a_1, a_2) \land \lambda(a_1, a_2)$ and $\langle f, n \rangle$ satisfies the atom $\psi(a_1, a_2)$ in $\mathfrak{B}$, q.e.d.

There is another notion of prime current model in current usage. A model of $\mathfrak{B}$ of $T$ is said to be prime in the sense of Robinson (cf. [11]) if $\mathfrak{B}$ is isomorphic to a subsystem of every model of $T$. $\mathfrak{B}$ is called an $\omega$-model if it has the form $\mathfrak{B} = \langle \mathfrak{B}, \cdot, +, \cdot \rangle$, $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B}$ is a model of $T$. We characterize this notion of prime model in

**Theorem 2.** $\mathfrak{B}$ is a prime model of $T$ in the sense of Robinson if and only if $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$ and $\mathfrak{B}$ is a model of $T$.

*Proof.** If $\mathfrak{B}$ is a model of $T$, then by theorem 1 it is prime in the sense of Vauhaft, a fortiori, prime in the sense of Robinson. Conversely suppose that $\mathfrak{B}$ is a prime model of $T$ in the sense of Robinson. Let $\mathfrak{M}$ be a subsystem of $\mathfrak{B}$ which is isomorphic to $\mathfrak{B}$. Since $\mathfrak{B}$ is also a model of $T$ it must be an $\omega$-model of the form $\mathfrak{M} = \langle \mathfrak{M}, \cdot, +, \cdot \rangle$. Let $\mathfrak{M} = \langle \mathfrak{M}, \cdot, +, \cdot \rangle$ be an arbitrary $\omega$-model of $T$ and let $H$ be an embedding of $\mathfrak{M}$ onto a subsystem $\mathfrak{M}$ of $\mathfrak{B}$. Since $n(a_1, a_2)$ contains no function variables it is uniquely satisfied in any $\omega$-model by the number $n < \omega$. We will show that $H$ is an identity function by using the fact that embeddings preserve purely existential formulas. Each $n < \omega$ uniquely satisfies $n(a_1, a_2)$ in $\mathfrak{B}$. Since $n(a_1, a_2)$ is purely existential $H(n)$ satisfies $n(a_1, a_2)$ in $\mathfrak{B}$ giving $H(n) = n$. Let $f \in \mathfrak{M}$ and $n_1, n < \omega$ such that $f(n_1) = n_1 \neq f(n)$. Let $\mathfrak{M} = \langle \mathfrak{M}, \cdot, +, \cdot \rangle$ satisfy $a_1(n_1) = n_1$. Let $H(n) = f(n_1)$ in $\mathfrak{M}$ and consequently $(H(f), H(n_1), H(n_1))$ satisfies $\psi(a_1, a_2) = n_1$. Since $H$ is an identity on $a_1$, $H(f) = f(n_1)$ for every $n_1 < \omega$ giving $H(f) = f$. Thus $\mathfrak{M}$ is a subsystem of every $\omega$-model of $T$, i.e., it is a minimal $\omega$-model of $T$. We determine $H$ as follows. Let $g \in \mathfrak{A}$ and let $\psi(a_1, a_2) \land \lambda(a_1, a_2)$ sufficiently determines some function $f \in \mathfrak{M}$. If $\psi(a_1, a_2) = n_1$, then

$$\psi(a_1, a_2) \land \lambda(a_1, a_2) \rightarrow \lambda(a_1, a_2) \land \lambda(a_1, a_2),$$

and consequently $f(n_1) = n_1$. Since $\mathfrak{M}$ is a model of $T$, $\mathfrak{M} = \langle \mathfrak{M}, a_1(n), \psi(a_1, a_2) \rangle$ so that $\psi(a_1, a_2)$ uniquely determines some function $f \in \mathfrak{M}$.
must hold in $\mathcal{F}$ and consequently must also hold in $\mathcal{M}$. But this can only happen if $\phi(\alpha) = \eta$. Thus $\psi = \varphi$ and $\mathcal{A}$ is a subsystem of $M$. We show that $M = \mathcal{M}$ by finding an $\mathcal{A}$-model $\mathcal{E}$ of $T$ which omits any given function $f \in F - A$. Although this could be done by the methods of [16], it is more convenient to use theorem 2.1 of [5]. This asserts that if $T$ is a consistent theory in a countable logic and $\mathcal{A}$ is a finite or countable set of formulas $\sigma(\alpha)$ such that each $\sigma(\alpha) \vdash T$ has the property (o) for each formula $\varphi(\alpha)$ which is consistent with $T$, there exists a formula $\psi(\alpha) \in \Sigma$ such that $\varphi(\alpha) \vdash \psi(\alpha)$ is consistent with $T$, then $T$ has a countable model which omits each $\sigma(\alpha) \in \Sigma$. There is no difficulty in applying this result to the two sorted logic $C$. For $\Sigma$, take the set $\{ \neg \psi(\alpha) \mid \alpha \in \omega \}$. If $\psi(\alpha)$ is a formula consistent with $T$, then $\exists \mathcal{F}(\alpha)$ and we can find $\alpha \in \omega$ which satisfies $\psi(\alpha)$ in $\mathcal{F}$. Hence $\exists \mathcal{F}(\alpha)$ shows that $\varphi(\alpha) \vdash \psi(\alpha)$ is consistent with $T$. Thus $\Sigma$ has the property (o). If $f \in F - A$, then for $\Sigma$, take the set

$$
\{(\forall \alpha \in \omega)[(\exists \beta \in \omega)(\exists \gamma \in \omega)(\beta < \gamma) \rightarrow \alpha(\gamma) = \eta(\beta)] : f(\gamma) = \eta(\beta) \}.
$$
If $\psi(\alpha)$ is a formula consistent with $T$, then $\exists \mathcal{F}(\alpha)$ so that some function $g \in F$ satisfies $\psi(\alpha)$ in $\mathcal{F}$. Since $\alpha$ is not definable in $\mathcal{F}$ we may take $g \neq f$, i.e., there are $\alpha_1, \alpha_2, \alpha_3 < \omega$ such that $\alpha(\alpha_1) \neq \alpha_1 = f(\alpha_2)$. Hence

$$
\exists \mathcal{F}(\alpha)(\varphi(\alpha) \land (\exists \beta \in \omega)(\exists \gamma \in \omega)(\beta < \gamma) \rightarrow \alpha(\gamma) = \eta(\beta))
$$
so that $\varphi(\alpha) \land (\exists \exists \beta \in \omega)(\exists \gamma \in \omega)(\beta < \gamma) \rightarrow \alpha(\gamma) = \eta(\beta))$ is consistent with $T$. But $f(\gamma) = \eta(\beta)$ and consequently $\Sigma$ has the property (o). Let $\mathcal{E}$ be a $\mathcal{A}$-model of $T$ which omits both $\Sigma_1$ and $\Sigma_2$. Since $\mathcal{E}$ omits $\Sigma$, we may take $\mathcal{E}$ to be an $\mathcal{A}$-model, and since $\mathcal{E}$ omits $\Sigma_1$, but $f$ satisfies $\Sigma_1$, $\Sigma_1$ will not belong to $\mathcal{E}$. Thus $M = \mathcal{A}$ and $\mathcal{F}$ is isomorphic to $\mathcal{E}$.

Thus the notions of prime models (in both senses) and minimal $\mathcal{A}$-models are coextensive for the theory $T$ and are nonaxiomatizable if and only if $\mathcal{A}$ is a model of $T$. We say that $\mathcal{E}$ satisfies an analytic basis theorem if whenever $\psi(\alpha) \in \mathcal{E}$ is a formula with one free variable and $\exists \mathcal{F}(\alpha)$, then $\psi(\alpha)$ is satisfied in $\mathcal{F}$ by some function $f \in T$. We say that $\mathcal{E}$ satisfies an analytic well-ordering theorem if there is a formula $\lambda(\alpha, \beta)$ with two free variables such that $(\exists \mathcal{F}(\alpha, \beta) : \forall \mathcal{F}(\alpha, \beta))$. Then we have the well known

**Lemma.** $\mathcal{A}$ is a model of $T$ if and only if $\mathcal{E}$ satisfies an analytic basis theorem.

**Lemma.** If $\mathcal{E}$ satisfies an analytic well-ordering theorem then $\mathcal{E}$ satisfies an analytic basis theorem.

Let $\mathcal{Z}$ be Zermelo-Fraenkel set theory including the axiom of choice, $V = L$ is the axiom of constructibility, $CH$ is the continuum hypothesis, and $MC$ asserts the existence of a measurable cardinal. Then we have the independence result

**Theorem 3.** The statement "$\mathcal{A}$ is a prime model of $T$" is relatively consistent with (1) $ZF \vdash \neg L$, (2) $ZF \vdash \neg CH$, (3) $ZF \vdash CH$, (4) $ZF \vdash MC$, (5) $ZF \vdash \neg MC$. The statement "$\mathcal{A}$ is not a prime model of $T$" is relatively consistent with (6) $ZF \vdash \neg L$, (7) $ZF \vdash CH$, (8) $ZF \vdash \neg MC$, (9) $ZF \vdash MC$, (10) $ZF \vdash \neg MC$.

**Proof.** Let $\mathcal{M}$ be a countable transitive model of $ZF = L$. We know from [4] that $\mathcal{M}$ satisfies $CH$, from [12] that $\mathcal{M}$ satisfies $\neg MC$, and from [1] that $\mathcal{M}$ satisfies $\neg L$ admits an analytic well ordering (in fact a $\Delta$ well ordering). This proves (1), (3), and (5). Let $\mathcal{M}$ be obtained from $\mathcal{M}$ by adjoining a single generic function $f : \omega \rightarrow \omega$. From [3] we know that $\mathcal{M}$ contains with the constructible sets of $\mathcal{M}$, however $f \in \mathcal{M}$, from (7) that $\mathcal{M}$ has the property (o) if $g : \omega \rightarrow \omega, g \in \mathcal{M}$, and $g$ is definable in $\mathcal{M}$ from elements of $\mathcal{M}$ then $g \in \mathcal{M}$, and from (1) that the predicate $\neg \Delta \omega$ non-constructible can be expressed in $\mathcal{M}$. Form, say $\varphi(\alpha, \beta)$. Then in $\mathcal{M}$, $\varphi(\alpha, \beta)$ is a formula which is satisfiable in $\mathcal{M}$ but is not satisfiable by any element of $\mathcal{M}$. This proves (8) and since $\mathcal{M}$ satisfies $\neg MC$ (cf. [3]) we obtain (7) as well. The extension of $\mathcal{M}$ to $\mathcal{M}$ is mild in the sense of [8] so that $\mathcal{M}$ satisfies the $MC$ if and only if $\mathcal{M}$ satisfies $\neg MC$ (cf. [8]). Since $\mathcal{M}$ does not, neither does $\mathcal{M}$, and we have proved (10). By a result of Solovay (stated in [9]) a non-generic $f$ may be chosen so that $\mathcal{M}$ is a model of $ZF$, $f \in \mathcal{M}$, every element of $\mathcal{M}$ is constructible from $g$, $f \in \mathcal{M}$, and $\neg L$ in $\mathcal{M}$. Then in $\mathcal{M}$, $\mathcal{M}$ admits a well ordering which is $\Delta$ in $\mathcal{M}$, function, and hence a $\Delta$ well ordering. This proves (2). We can prove (3) in exactly the same way that we proved (2) by adjoining a single generic function $f : \omega \rightarrow \omega$. Since this extension is mild, by [8] we know that $\mathcal{M}$ uniquely extends to a normal $\omega$-complete non-principal ultrafilter $\mathcal{D}$ on $\omega$ (in the sense of $\mathcal{M}$) such that every element of $\mathcal{M}$ is constructible relative to $D$. From (13) we know that $\mathcal{M}$ satisfies $\neg L$ admits an analytic well ordering (in fact a $\Delta$ well ordering). This proves (4). Let $\mathcal{M}$ be obtained from $\mathcal{M}$ by adjoining a single generic function $f : \omega \rightarrow \omega$. Since this extension is mild, by [8] we know that $\mathcal{M}$ uniquely extends to a normal $\omega$-complete non-principal ultrafilter $\mathcal{D}$ on $\omega$ (in the sense of $\mathcal{M}$) coincides with the elements of $\mathcal{M}$ constructible relative to $D$, and $\mathcal{D} \in \mathcal{M}$. From (13) we know that the predicate $\neg \Delta \omega$ non-constructible relative to $D$ can be expressed in $\mathcal{M}$, say $\psi(\alpha, \beta) \in \mathcal{M}$, and from (13) we see that $\mathcal{M}$ has the property (o). Then in $\mathcal{M}$, $\psi(\alpha, \beta)$ is a formula which is satisfiable in $\mathcal{M}$ but is not satisfied by any element of $\mathcal{M}$. This proves (9), q.e.d.

There is one asymmetry in the statement of our theorem. We have not shown that "$\mathcal{A}$ is a prime model of $T$" is consistent with $ZF \vdash \neg CH$. This seems to be related to the open problem (summer 1967, cf. [9]) as to whether $\neg CH$ is consistent with the existence of a projective well ordering of $\mathcal{A}$.
We apply our results to certain weak second order theories. Let $\mathcal{R} = \langle R, +, \cdot \rangle$ where $R$ is the set of real numbers and $+, \cdot$ are the usual arithmetic operations. Let $T^e$ be a weak second order language for $\mathcal{R}$ and let $T^m$ be its weak second order theory. The notion of "prime model in the sense of Robinson" has an immediate generalization to the case of $T^m$ models, and so does "prime model in the sense of Vaught" once we have defined $w$-elementary subsystem to read exactly like its first order equivalent except that we require all parameters to be individual. There is a sentence in $T^e$ which guarantees that each model of $T^e$ admits an Archimedean ordering and therefore has a unique embedding into $\mathcal{R}$. Thus it is meaningful to talk about minimal models of $T^m$. Let $B = \{x \in R : 0 < x < 1 \text{ and } x \text{ is irrational} \}$ and define a function $\theta$ from $B$ onto $F$ by letting $\theta(x) = f$ where $1 + f(n)$ is the $n$th denominator in the continued fraction expansion of $x$. For each subsystem $\mathcal{S} = \langle S, +, \cdot \rangle$ of $\mathcal{R}$ let $H(\mathcal{S}) = \mathcal{S} = \langle \mathcal{S}, a, +, \cdot \rangle$ where $a = \{\theta(x) : x \in S\}$, and for each subsystem $\mathcal{S} \subseteq \mathcal{R}$ let $H(\mathcal{S}) = \mathcal{S}''$ where $\mathcal{S}''$ is the closure under rational operations of $\langle \theta(n) : f(n) \rangle$. Then we have the lemma. $\mathcal{H}$ takes models into models, is self inverse there, and preserves the notion of proper elementary subsystem.

We merely sketch a proof of this result. By [10] there is a formula $\psi(r_1, r_2, r_3)$ in $\mathcal{S}''$ with three free variables, each individual, such that if $\exists r$ is a model of $T^m$, $x \in S'$, and $n \neq a$, then $(x, n, p)$ satisfies $\psi$ in $\mathcal{S}''$ if and only if $p$ is the $n$th denominator in the continued fraction expansion of $x$. From this we immediately see that $\mathcal{H}$ takes models of $T^m$ into models of $T^e$ preserving the notion of proper elementary subsystem. Conversely it is clear that given a family of functions, we can define the field operations which give rise to these functions as continued fraction expansions in a perfectly elementary way, i.e., in the language $\mathcal{L}$. Thus $\mathcal{H}$ takes models of $T^e$ into models of $T^m$ preserving the notion of proper elementary subsystem. The self inverse property is immediate. Let $\mathcal{S}'' = H(\mathcal{S})$. Then granting our lemma all the results which are mentioned in theorems 1-3 go over for models of $T^m$ (by replacing $T$ by $T^m$ and $\mathcal{S}$ by $\mathcal{S}''$ in their statements). This is in sharp distinction to the first order case where it is known (cf. [13]) that the algebraic reals is a minimal, and prime in both senses, model of the first order theory of $\mathcal{R}$. We briefly compare these results with those concerning the weak second order theory of complex numbers. Let $\mathcal{C} = \langle C, +, \cdot \rangle$ where $C$ is the set of complex numbers and $+, \cdot$ are the usual arithmetic operations, and let $T^C_C$ be its weak second order theory. $\mathcal{S} = \langle C, a, C \rangle$ is a model of $T^C_C$ if and only if $\mathcal{S}$ is an algebraically closed field of characteristic $0$ and infinite degree of transcendence (cf. [14]). Thus every such $\mathcal{S}$ has a proper subsystem $\mathcal{S}$' which is also a model of $T^C_C$, and consequently there is no minimal model. On the other hand, given models $\mathcal{S}, \mathcal{S}'$ of $T^C_C$, where $\mathcal{S}$ has countable degree of transcendence, by purely algebraic methods, we can find an embedding $\mathcal{H}$ which maps $\mathcal{S}$ isomorphically onto a subsystem $\mathcal{S}'$ of $\mathcal{S}$. The methods of [15] then generalize so that $\mathcal{S}'$ will be a $w$-elementary subsystem of $\mathcal{S}$. Thus $T^C_C$ has prime models in both senses, just as in the first order case (cf. [15]).

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