Direct multiples and powers of modules *

by

Philip Olin (Montreal)

In this paper we continue the study of first order properties of products of modules, begun in Volvačev [6] (see Eršov [2]) and in [4]. It seems desirable to analize the usual direct sum and product operations on modules in order to reduce the truth of an elementary statement in the product to truth in the factor modules, hoping to take advantage of methods developed by Ehrenfeucht [1] and Feferman and Vaught [3]. Furthermore the first order language used should be as strong as possible.

There are two natural first order languages which could be used to discuss modules. The first would put the scalars into the language as operations, and the second is a two-sorted language employing two kinds of variables (module element and scalar), thus allowing quantification over the scalars. This second method is equivalent to having a relativized one-sorted language. It is the second approach that we shall adopt. This two-sorted logic has the advantage that with it we can compare modules over different rings and also that it is stronger. For example, with a first order statement in this language we can state that a module is torsion, torsion-free, divisible, or n-generated for n finite. So a module, as a relational system, has both module elements and scalars in its universe, and has the usual finite number of relations.

In what follows, we first give some examples (due jointly to P. Eklof and the author) which show that elementary equivalence is not preserved even by the finite direct sum operation on modules. Other examples show that tensor product does not preserve elementary equivalence, and that elementary equivalence as Abelian groups does not imply elementary equivalence as Z-modules. Theorems 1 and 2 show that the power and multiple operations on finite modules (finite number of module elements; the ring may be infinite) do preserve elementary equivalence; for direct powers we get a strong reducibility result using methods of Ehrenfeucht [1], and for direct multiples we get a result similar to that of Feferman and Vaught [3]. Finally we show that the direct multiple operation preserves

* This work was partially supported by the National Research Council of Canada.
Present address: York University, Toronto, Canada.

equivalence with respect to sentences with at most one alternation in quantifiers. The major unanswered question is whether the infinite direct multiple operation preserves elementary equivalence.

The two-sorted language L_π which we use has module element variables $X_0, X_1, X_2, \ldots$ and scalar variables $x_0, x_1, x_2, \ldots$ The atomic well-formed formulas are of the form $x_i = x_j$, $X_i = X_j$, $x_i + x_j = x_k$, $x_i \cdot x_j = x_k$, $X_i + X_j = X_k$, and $x_i X_j = X_k$. The language has two kinds of existential and universal quantifiers. As a general rule throughout, module element variables and constants will be denoted by upper case letters, and scalars by lower case letters. We assume a fixed Gödel-numbering of L_π. If M is a structure and φ a sentence of L_π then $M \models \varphi$ states that φ is true in M. $\mathrm{Th}(M)$ is the set of all sentences of L_π true in M. $M_1 \equiv M_2$ means $\mathrm{Th}(M_1) = \mathrm{Th}(M_2)$, and we say M_1 and M_2 are elementarily equivalent. We let $\leqslant_T$ denote Turing reducibility.

If $\{M_i\}_{i\in I}$ is a collection of modules over the ring R then $\bigoplus_{i\in I} M_i$ denotes the R-module which is the direct sum of the modules in the collection. So A is a *module element in* $\bigoplus_{i\in I} M_i$ if A is a function on I such that $A(i)$ is a module element in M_i and, for all but a finite number of i's, $A(i) = 0$. If M is a module and α a cardinal then $\bigoplus_\alpha M$ is the direct sum of α copies of M; we call it a *direct multiple of* M. For the collection $\{M_i\}_{i\in I}$ we let $X_{i\in I} M_i$ denote the direct product of the modules in the collection, this being defined in the same way as the direct sum except that we remove the finitely-nonzero condition on the elements A. If M is a module and α a cardinal then $X_\alpha M$ is the direct product of α copies of M; we call it a *direct power of* M.

1. Some examples. The examples of this section are due to P. Eklof and the author.

Example 1. Let F be a field and let R be the polynomial ring $F[y]$ in one variable over F. Let Ry be the prime ideal in R consisting of the polynomials without constant term. Then R/Ry can be considered as an R-module. Similarly, if 1 is the multiplicative unit in F, $R/R(y-1)$ is an R-module. Moreover, R/Ry and $R/R(y-1)$ are isomorphic as modules, in the sense that there is a one-one function which maps R onto R and the module elements of R/Ry onto the module elements of $R/R(y-1)$, preserving all of the module structure. This function is induced by the map from R onto R which sends y to $y-1$. So the module isomorphism is not the identity on the ring of scalars. So we get $R/Ry \equiv R/R(y-1)$. Now consider the R-modules $R/Ry \oplus R/Ry$ and $R/Ry \oplus R/R(y-1)$. The universal sentence

$$(x)(X_1)(X_2)[x \neq 0 \wedge X_1 \neq 0 \wedge xX_1 = 0 \rightarrow xX_2 = 0]$$

is true in the former (because if $f(g/Ry, h/Ry) = 0$ with $f \neq 0$ and either $g \notin Ry$ or $h \notin Ry$ then $fg \in Ry$ and $fh \in Ry$ and hence $f \in Ry$, which implies $fk \in Ry$ for every $k \in R$) but false in the latter $\big(y(1/Ry, 0) = 0$ and yet $y(0, 1/R(y-1)) \neq 0\big)$. Thus direct sum, even of two modules, does not preserve elementary equivalence, even if the factor modules are isomorphic in the sense described above. In particular, the direct sum operation on modules (as considered here) cannot be a generalized product as in Feferman and Vaught [3].

Example 2. A similar example can be obtained with the ring of scalars non-commutative. Let F be a field, let $R = F[y, z]$ be the non-commutative polynomial ring in two variables, and let R/Ryz and R/Rzy be considered as R-modules. As above, they are isomorphic (the isomorphism being induced by the map on R which interchanges y and z). But the universal sentence

$$(x_1)(x_2)(X_1)(X_2)[x_1X_1 \neq 0 \vee x_1X_2 \neq 0 \vee x_2X_1 \neq 0 \vee x_2X_2 = 0 \vee X_1 = 0 \vee x_1 = 0]$$

is true in the R-module $R/Ryz \oplus R/Ryz$ and false in the R-module $R/Ryz \oplus R/Rzy$.

Example 3. Let F be a field, let $R = F[y, z]$ be the commutative polynomial ring in y and z over F, and let $I = Ry$, $J = Rz$. Then, as above, $R/I \cong R/J$ as modules. Consider the tensor products $R/I \otimes R/J$ and $R/I \otimes R/I$. The first is isomorphic to $R/I+J$ and the second to R/I. Now in R/I there is a module element X and a non-unit x in the ring such that for every c in the ring which is either a unit or zero, $(x-c)X \neq 0$. However in $R/I+J$ this first order sentence is false. Hence the tensor product operation on modules does not preserve elementary equivalence, even if the factor modules are isomorphic in the sense described above. S. Feferman has informed the author that an example showing this was known earlier to Ju. L. Eršov.

Example 4. Let Z^* denote Z^ω/D, where Z is the ring of integers and D is a nonprincipal ultrafilter on ω. Of course as Abelian groups (and as rings), Z and Z^* are elementarily equivalent, by Łoś theorem. But consider them as Z-modules. Let $f \in Z^\omega$ be defined by $f(n) = n!$. Then f/D is divisible by every member of Z other than zero. So the sentence $(EX)(x)(EY)[x = 0 \vee xY = X]$ is true in the Z-module Z^* but is clearly false in the Z-module Z. Thus, as Z-modules, Z and Z^* are not elementarily equivalent.

2. Direct powers of finite modules. We say M is a *finite module* if M has only a finite number of module elements, although the ring of scalars may be infinite. So every finite Abelian group considered as a Z-module is a finite module. And, as in Example 1 above, if the field F is a finite field then, with $R = F[y]$, R/Ry as an R-module is a finite module. So as a result of Example 1, even the finite direct sum (or product) operation on finite modules does not preserve elementary equivalence. However the direct power and multiple operations on finite modules do preserve elementary equivalence and in fact in Theorems 1 and 2 below we get stronger results; similar to those obtained for vector spaces in [4] (following Ehrenfeucht [1] and Scott [5]) in the case of direct powers, and an elimination-of-quantifiers result similar to Theorem 3.1 of Feferman and Vaught [3] in the case of direct multiples.

THEOREM 1. *Let φ be a sentence of L_π with n module element variables and let p be a positive integer. Then for any module M having exactly p module elements and for any cardinal $\lambda \geqslant p^n$,*

$$X_{p^n} M \models \varphi \quad \text{iff} \quad X_\lambda M \models \varphi .$$

Proof. Suppose φ contains m scalar variables. Following results of Ehrenfeucht [1], it suffices to show that player II has a winning strategy in the game G_{n+m}, using $X_{p^n} M$ and $X_\lambda M$, where each player picks (one at a time) n module elements and m scalars. Let the module elements of M be $C_1, \ldots, C_p$, let $U = \{C_1, \ldots, C_p\}$, and let U^t be the set of t-tuples of members of U. The strategy is as follows. At any stage if player I chooses a scalar then player II chooses that same scalar. If player I chooses $B_1 \in X_\lambda M$, we define $A_1 \in X_{p^n} M$ so that, for all but one k, the sets

$$D_k = \{i|\ B_1(i) = C_k\} \quad \text{and} \quad E_k = \{i|\ A_1(i) = C_k\}$$

either have equal cardinality $< p^{n-1}$ or D_k has cardinality $\geqslant p^{n-1}$ and E_k has cardinality $= p^{n-1}$. Since U has p members and since $\lambda \geqslant p^n$, we know that there is at least one k, call it k_0, such that D_{k_0} has cardinality $\geqslant p^{n-1}$. After A_1 has been defined to satisfy the conditions above, let $A_1(i) = C_{k_0}$ for all remaining coordinates i. If player I chooses $A_1 \in X_{p^n} M$ then $B_1 \in X_\lambda M$ can be defined in exactly the same way.

Now suppose players I and II have each choosen r $(1 \leqslant r < n)$ module elements; $A_1, \ldots, A_r$ from $X_{p^n} M$ and $B_1, \ldots, B_r$ from $X_\lambda M$. For each coordinate j, $(A_1(j), \ldots, A_r(j))$ is called an A^r-*column*. The columns are thus members of U^r. Similar notation is adopted for the B_i's. Now as r-hypothesis, suppose we have the following. Let γ be any member of U^r, let α be the number of A^r-columns which equal γ and β the number of B^r-columns which equal γ. Then $\alpha < p^{n-r}$ iff $\beta < p^{n-r}$, and in this case $\alpha = \beta$. Now suppose player I chooses $B_{r+1} \in X_\lambda M$. We wish to define $A_{r+1} \in X_{p^n} M$. Let $\gamma \in U^r$ be such that $\alpha = \beta < p^{n-r}$. Let C_k be any member of U and let β_k be the number of B^{r+1}-columns with last member C_k and with the first r members equal to γ. So $\beta_k \leqslant \beta$. We choose any β_k A^r-columns which are equal to γ and for each such coordinate j

we define $A_{r+1}(j) = C_k$. Repeat for all $C_k \in U$. This construction ensures that, for every $\gamma \in U^r$ such that $\alpha = \beta < p^{n-r}$, if $\gamma' \in U^{r+1}$ and γ' extends γ then the number of B^{r+1}-columns equal to γ' equals the number of A^{r+1}-columns equal to γ'. Hence for such γ' the $(r+1)$-hypothesis will be true. Now suppose $\gamma \in U^r$ is such that $\beta \geqslant p^{n-r}$ (hence $\alpha \geqslant p^{n-r}$), and suppose C_k and β_k are as above with $\beta_k < p^{n-(r+1)} = p^{n-r-1}$. In this case we define A_{r+1} at a further β_k coordinates whose A^r-columns equal γ to be C_k. Because $\beta \geqslant p^{n-r}$ and because U has p members, not every C_k has $\beta_k < p^{n-r-1}$. Without loss of generality, assume $\beta_0 \geqslant p^{n-r-1}$. Now if $k \neq 0$ and $\beta_k \geqslant p^{n-r-1}$, define A_{r+1} at a further p^{n-r-1} coordinates whose A^r-columns equal γ to be C_k. So for this γ we have defined so far at most $(p-1)p^{n-r-1}$ coordinates of A_{r+1}. Since $\alpha \geqslant p^{n-r}$, there are at least p^{n-r-1} A^r-columns equal to γ which have not yet been considered. For all of these we define $A_{r+1}(j)$ to be C_0. The construction ensures that the $(r+1)$-hypothesis is satisfied.

If player I chooses A_{r+1} in $X_{p^n} M$, the procedure for constructing B_{r+1} in $X_\lambda M$ is the same as that given above. In this case, for $\gamma \in U^r$ such that $\alpha \geqslant p^{n-r}$ and $\alpha_0 \geqslant p^{n-r-1}$, we will be defining $B_{r+1}(j)$ to be C_0 at an infinite number of coordinates j if λ is infinite.

This completes the strategy of player II. After all the choices have been made in the game G_{n+m} we have the r-hypothesis satisfied with $r = n$. This guarantees that if $\gamma \in U^n$ then some A^n-column equals γ iff some B^n-column equals γ; i.e. they "realize" exactly the same columns. This implies at once that the map which is the identity map on the scalars and which maps A_i to B_i is an isomorphism. Hence player II has a winning strategy, completing the proof.

COROLLARY 1. *If φ is a sentence of L_π, p a positive integer, and λ any cardinal > 0 then we can effectively find a sentence ψ in L_π such that for any module M with exactly p module elements*

$$X_\lambda M \models \varphi \quad \text{iff} \quad M \models \psi .$$

Proof. Suppose φ has n module element variables in it. Theorem 1.1 of [4] contains the following result: if θ is a sentence of L_π and r a positive integer then we can effectively find a sentence ψ of L_π such that for any module M, $X_r M \models \theta$ iff $M \models \psi$. Now if $\lambda \geqslant p^n$ then by the theorem, $X_\lambda M \models \varphi$ iff $X_{p^n} M \models \varphi$. Applying the above result with $\theta = \varphi$ and $r = p^n$ we effectively find a sentence ψ such that $X_{p^n} M \models \varphi$ iff $M \models \psi$. If $\lambda < p^n$ then just apply the above result directly with $\theta = \varphi$ and $r = \lambda$.

COROLLARY 2. *If M has a finite number of module elements then* (a) *for λ, ν infinite cardinals, $X_\lambda M \equiv X_\nu M$ and* $\mathrm{Th}(X_\lambda M) = \bigcup_{m<\omega} \bigcap_{m \leqslant n < \omega} \mathrm{Th}(X_n M)$ *and* (b) *for any cardinal λ,* $\mathrm{Th}(X_\lambda M) \leqslant_T \mathrm{Th}(M)$.

Hence any direct power of a decidable finite module is decidable, the theory of the class of all direct powers of a decidable finite module is decidable, and the theory of the class of all finite direct powers (equals direct multiples) of a decidable finite module is decidable.

3. Direct multiples of finite modules. We now wish to consider direct multiples of finite modules. Let p, r, n be positive integers with $n > r$. Let M be a module with p module elements $C_1, \ldots, C_p$. Let $U = \{C_1, \ldots, C_p\}$ and U^r the set of r-tuples of members of U. Suppose $\{B_k^h\}$, $1 \leqslant k \leqslant r+1$, $1 \leqslant h \leqslant p^n$, is a matrix with $r+1$ rows and p^n columns, the entry B_k^h in the kth row and hth column being a member of U. We call the columns of this matrix *h-columns*. The zero column is the column all of whose entries are the zero module element. Similar notation would apply to the matrix $\{A_i^j\}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant p^n$.

We wish to define a first order predicate $H_{r,n}^p(Y_k^h, X_i^j)$ of $(r+1)p^n + rp^n$ module element variables. A detailed first order writing of this predicate would be too obscure and so we give an informal definition. It will be clear that the predicate is first order. For $\{B_k^h\}$ and $\{A_i^j\}$ as above, $H_{r,n}^p(B_k^h, A_i^j)$ states that:

(1) there can be at most $p^{n-(r+1)}$ h-columns which are identical and not the zero column and

(2) if γ in U^r is not the zero column, β is the number of j-columns which are equal to γ, $\gamma_1, \ldots, \gamma_p$ are the p possible ways of extending γ to be a member of U^{r+1}, and β_i is the number of h-columns which are equal to γ_i $(1 \leqslant i \leqslant p)$, then either (i) $\beta_1 + \ldots + \beta_p = \beta$ or (ii) $\beta_1 + \ldots + \beta_p < \beta$ and there is at least one i such that $\beta_i = p^{n-(r+1)}$.

Assume $r \leqslant n$ and $A_1, \ldots, A_r$ are module elements in $\bigoplus_\omega M$ (the countable direct multiple of M). We define $K_{r,n}^p(A_1, \ldots, A_r)$ to be a matrix $\{A_i^j\}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant p^n$, with r rows and p^n columns, unique up to permutation of the columns, and with $A_i^j \in U$. The definition is as follows. If γ in U^r is not the zero column and δ is the cardinality of $\{j|\ (A_1(j), \ldots, A_r(j)) = \gamma\}$ then

(1) if $\delta < p^{n-r}$, then there are exactly δ j-columns in $\{A_i^j\}$ which equal γ and

(2) if $\delta \geqslant p^{n-r}$ then there are exactly p^{n-r} j-columns in $\{A_i^j\}$ which equal γ.

All the other columns in the matrix are defined to be the zero column. The matrix is well-defined since U^r has p^r members.

LEMMA. *If $A_1, \ldots, A_r, A_{r+1} \in \bigoplus_\omega M$ $(r+1 \leqslant n)$, $K_{r,n}^p(A_1, \ldots, A_r) = \{A_i^j\}$ and $K_{r+1,n}^p(A_1, \ldots, A_{r+1}) = \{B_k^h\}$ then $H_{r,n}^p(B_k^h, A_i^j)$.*

Proof. By the definition of $K_{r,n}^p$ and $K_{r+1,n}^p$, $\{B_k^h\}$ is an $r+1$ by p^n matrix and $\{A_i^j\}$ is an r by p^n matrix. Condition (1) in the definition of $H_{r,n}^p$ follows at once from the definition of $K_{r+1,n}^p$. Assume the hypothesis of condition (2) of $H_{r,n}^p$. Let δ_i be the cardinality of $\{h|\ (A_1(h), \ldots, A_{r+1}(h)) = \gamma_i\}$. If, for each i, $\delta_i < p^{n-(r+1)}$ then $\beta_i = \delta_i$ and $\beta_1 + \ldots + \beta_p = \beta < p^{n-r}$. Now suppose some $\delta_i \geqslant p^{n-(r+1)}$. Let $\nu_1, \ldots, \nu_s$ be those δ_i's such that $\delta_i \geqslant p^{n-(r+1)}$ and $\mu_1, \ldots, \mu_t$ be those δ_i's such that $\delta_i < p^{n-(r+1)}$. (Thus $s+t = p$ and $s \geqslant 1$). If $\mu_j = \delta_i$ then $\beta_i = \delta_i$. If $\nu_j = \delta_i$ then $\beta_i = p^{n-(r+1)}$. If $t = 0$ then $\beta_1 + \ldots + \beta_p = p(p^{n-(r+1)}) = p^{n-r}$ and $\beta = p^{n-r}$. If $t \geqslant 1$ then $\beta_1 + \ldots + \beta_p = \mu_1 + \ldots + \mu_t + (p-t)p^{n-(r+1)}$. This last number will be $< \beta$ if some $\nu_j > p^{n-(r+1)}$ and it will be equal to β if every $\nu_j = p^{n-(r+1)}$.

We can now prove a theorem similar to Theorem 3.1 of Feferman and Vaught [3].

THEOREM 2. *Let p be a positive integer and $\varphi(X_1, \ldots, X_r, x_1, \ldots, x_s)$ a formula in L_π whose free variables are among $X_1, \ldots, X_r, x_1, \ldots, x_s$. Suppose at most n module element variables $(n \geqslant r)$ are mentioned in φ. Then we can effectively find a formula $\psi(X_i^j, x_1, \ldots, x_s)$ in L_π, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant p^n$, of rp^n free module element variables and s free scalar variables such that if M is a module over R with p module elements and if $A_1, \ldots, A_r$ are in $\bigoplus_\omega M$, $a_1, \ldots, a_s \in R$ and $K_{r,n}^p(A_1, \ldots, A_r) = \{A_i^j\}$ then*

$$\bigoplus_\omega M \models \varphi(A_1, \ldots, A_r, a_1, \ldots, a_s) \quad \text{iff} \quad M \models \psi(A_i^j, a_1, \ldots, a_s)\,.$$

Proof. The proof is by induction on the number of logical connectives in φ. Suppose φ is an atomic formula. If φ is $x_1 = x_2$, $x_1 + x_2 = x_3$, or $x_1 \cdot x_2 = x_3$ then let ψ be φ. If φ is $X_1 = X_2$, $X_1 + X_2 = X_3$, or $x_1 X_2 = X_3$ then let ψ be, respectively, $\bigwedge_{j=1}^{p^2} (X_1^j = X_2^j)$, $\bigwedge_{j=1}^{p^3} (X_1^j + X_2^j = X_3^j)$, or $\bigwedge_{j=1}^{p^2} (x_1 X_2^j = X_3^j)$. The result then follows at once since we have $n = r$ and, from the definitions involved, for any $\gamma \in U^r$: there is a j such that $\gamma = (A_1(j), \ldots, A_r(j))$ iff there is exactly one j such that $\gamma = (A_1^j, \ldots, A_r^j)$. If, by the induction hypothesis, ψ corresponds to φ then $\sim\psi$ will correspond to $\sim\varphi$. Suppose each $\varphi_t(X_1, \ldots, X_r, x_1, \ldots, x_s)$, for $t = 1, 2$, has free variables among $X_1, \ldots, X_r, x_1, \ldots, x_s$, and a total of n module element variables are mentioned in φ_1 and φ_2. By the induction hypothesis, suppose $\psi_t(X_i^j, x_1, \ldots, x_s)$ corresponds to φ_t. Then we have at once that $\psi_1 \wedge \psi_2$ will correspond to $\varphi_1 \wedge \varphi_2$.

Now consider $(Ex_1)\varphi(X_1, \ldots, X_r, x_1, \ldots, x_s)$. By the induction hypothesis suppose $\psi(X_i^j, x_1, \ldots, x_s)$ corresponds to φ. Then it follows at once that $(Ex_1)\psi$ will corresponds to $(Ex_1)\varphi$.

Finally consider $(EX_{r+1})\varphi(X_1, \ldots, X_r, X_{r+1}, x_1, \ldots, x_s)$ where, by the induction hypothesis, we have $\psi(X_i^j, x_1, \ldots, x_s)$ corresponding to $\varphi(X_1, \ldots,$

$\ldots, X_r, X_{r+1}, x_1, \ldots, x_s)$, with $1 \leqslant i \leqslant r+1$, $1 \leqslant j \leqslant p^n$, and $r+1 \leqslant n$. We claim that to $(EX_{r+1})\varphi(X_1, \ldots, X_r, X_{r+1}, x_1, \ldots, x_s)$ we can correspond

$$(EY_k^h)_{1\leqslant k\leqslant r+1, 1\leqslant h\leqslant p^n}[\psi(Y_k^h, x_1, \ldots, x_s) \wedge H_{r,n}^p(Y_k^h, X_i^j)] .$$

Suppose $A_1, \ldots, A_r \in \bigoplus_\omega M$, $a_1, \ldots, a_s \in R$, and $K_{r,n}^p(A_1, \ldots, A_r) = \{A_i^j\}$. First assume that

$$\bigoplus_\omega M \models (EX_{r+1})\varphi(A_1, \ldots, A_r, X_{r+1}, a_1, \ldots, a_s) .$$

Let A_{r+1} in $\bigoplus_\omega M$ be such that $\bigoplus_\omega M \models \varphi(A_1, \ldots, A_r, A_{r+1}, a_1, \ldots, a_s)$. By the induction hypothesis, for $K_{r+1,n}^p(A_1, \ldots, A_{r+1}) = \{B_k^h\}$, $1 \leqslant k \leqslant r+1$, $1 \leqslant h \leqslant p^n$, we have $M \models \psi(B_k^h, a_1, \ldots, a_s)$. By the lemma above we have $H_{r,n}^p(B_k^h, A_i^j)$. Hence we have the result by letting Y_k^h be B_k^h.

Now assume that

$$M \models (EY_k^h)_{1\leqslant k\leqslant r+1, 1\leqslant h\leqslant p^n}[\psi(Y_k^h, a_1, \ldots, a_s) \wedge H_{r,n}^p(Y_k^h, A_i^j)] .$$

Let $\{B_k^h\}$ be the matrix whose existence is asserted. We shall define a module element $A_{r+1} \in \bigoplus_\omega M$ so that $K_{r,n}^p(A_1, \ldots, A_r, A_{r+1}) = \{B_k^h\}$. Suppose γ is in U^r and let β be the number of j-columns in $\{A_j^i\}$ which equal γ. Let $\gamma_1, \ldots, \gamma_p$ in U^{r+1} be the p ways of extending γ by one element, and let β_i be the number of h-columns in $\{B_k^h\}$ equal to γ_i. Let a be the cardinality of $E = \{j|\ (A_1(j), \ldots, A_r(j)) = \gamma\}$. We shall partition E into p disjoint sets $E_1, \ldots, E_p$. Then, in each case below, we define $A_{r+1}(j)$, for $j \in E_i$, to be the last member of the $(r+1)$-tuple γ_i.

If γ is the zero column then $a = \aleph_0$. Assume γ_p is the (extended) zero column. Then let E_i have cardinality β_i for $1 \leqslant i \leqslant p$, and let $E_p = E - (E_1 \cup \ldots \cup E_{p-1})$.

Suppose γ is not the zero column. Hence a is finite.

Case 1. $\beta = \beta_1 + \ldots + \beta_p$ and $a \geqslant p^{n-r}$. In this case $\beta = p^{n-r}$ and, since $\beta_i \leqslant p^{n-(r+1)}$, we get $\beta_i = p^{n-(r+1)}$ for each i. So let E_i have at least $p^{n-(r+1)}$ members for each i.

Case 2. $\beta = \beta_1 + \ldots + \beta_p$ and $a < p^{n-r}$. In this case $\beta = a$. Let E_i have cardinality β_i.

Case 3. $\beta < \beta_1 + \ldots + \beta_p$ and $a \geqslant p^{n-r}$. In this case $\beta = p^{n-r}$. By the definition of $H_{r,n}^p$, we can assume without loss of generality that $\beta_p = p^{n-(r+1)}$. Let E_i have cardinality β_i for $1 \leqslant i < p$, and let $E_p = E - (E_1 \cup \ldots \cup E_{p-1})$.

Case 4. $\beta < \beta_1 + \ldots + \beta_p$ and $a < p^{n+r}$. In this case $a = \beta$. As in case 3, we can assume $\beta_p = p^{n-(r+1)}$. And each E_i is defined as in case 3.

The construction ensures that $K_{r+1,n}^p(A_1, \ldots, A_{r+1}) = \{B_k^h\}$. Since we have $M \models \psi(B_k^h, a_1, \ldots, a_s)$: by the induction hypothesis $\bigoplus_\omega M \models \varphi(A_1, \ldots$ $\ldots, A_{r+1}, a_1, \ldots, a_s)$. Hence $\bigoplus_\omega M \models (EX_{r+1})\varphi(A_1, \ldots, A_r, X_{r+1}, a_1, \ldots, a_s)$. This completes the proof of the theorem.

Corollary 3. *Let p be a positive integer and φ a sentence of L_π. Then we can effectively find a sentence ψ of L_π such that if M is a module with p module elements then*

$$\bigoplus_\omega M \models \varphi \quad \textit{iff} \quad M \models \psi .$$

Proof. This is just the theorem with $r = s = 0$.

Corollary 4. *For any cardinal a and any finite module M, $\mathrm{Th}(\bigoplus_a M) \leqslant_T \mathrm{Th}(M)$.*

Proof. If a is infinite, it is clear that $\bigoplus_a M \equiv \bigoplus_\omega M$. The result then follows by Corollary 3 (for a infinite) and by the result of [4] which is cited in Corollary 1 above (for a finite).

Hence any direct multiple of a decidable finite module is decidable, and the theory of the class of all direct multiples of a decidable finite module is decidable (using the remark following Corollary 2 above).

4. Further results. In Examples 1 and 2 above it was shown that not only was elementary equivalence not preserved by the finite direct sum operation on modules but that the two direct sums could differ for a universal sentence. If all the modules in question had been Z-modules we will show that this could not have happened. We write $M \equiv_A N$ if M and N have the same true universal (and hence existential) sentences.

Theorem 3. *If $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$ are Z-modules and $M_i \equiv N_i$ then $\bigoplus_{i\in I} M_i \equiv_A \bigoplus_{i\in I} N_i$ and $X_{i\in I} M_i \equiv_A X_{i\in I} N_i$.*

Proof. The proof is simple; it is based on the fact that any member of Z is definable in Z. For the first part it suffices to show that if $A_1, \ldots, A_n$ are module elements in $\bigoplus_{i\in I} M_i$ and $a_1, \ldots, a_m \in Z$ then we can find module elements $B_1, \ldots, B_n \in \bigoplus_{i\in I} N_i$ such that the correspondence which is the identity on a_j and maps A_j to B_j is an isomorphism. Let $\varphi_j(x)$ be a formula with one free variable (in the language for rings) such that $Z \models \varphi_j(b)$ iff $b = a_j$; i.e. φ_j defines a_j in Z. Fix $i \in I$. Form $\bigwedge_k P_k$, where each P_k is either an atomic relation among $A_1(i), \ldots, A_n(i), a_1, \ldots, a_m$ or the negation of such, whichever is true in M_i, and where every such atomic relation or its negation is some P_k. Now form $\bigwedge_k P_k \wedge \bigwedge_{j=1}^m \varphi_j(a_j)$. In this formula, replace $A_j(i)$ by X_j and a_j by x_j and existentially quantify the result, getting a sentence F of L_π. Note that F is not necessarily an existential sentence because of the φ_j's involved. We have $M_i \models F$. Since $M_i \equiv N_i$, we get $N_i \models F$. Let $B_1^i, \ldots, B_n^i, b_1, \ldots, b_m$ be the members of N_i whose existence is asserted. Since $N_i \models \varphi_j(b_j)$, $Z \models \varphi_j(b_j)$ and so b_j is a_j. And so the map which is the identity on a_j and which sends $A_j(i)$ to B_j^i is an isomorphism. Repeat for all $i \in I$. For $1 \leqslant j \leqslant n$, define $B_j \in \bigoplus_{i \in I} N_i$ by $B_j(i) = B_j^i$. The construction ensures that $a_j \leftrightarrow a_j$ and $A_j \leftrightarrow B_j$ is an isomorphism. Interchanging the roles of M_i and N_i, we get finally that $\bigoplus_{i \in I} M_i \equiv_A \bigoplus_{i \in I} N_i$. The proof that $X_{i \in I} M_i \equiv_A X_{i \in I} N_i$ is exactly the same.

It is clear that if R is a ring such that for any $r \in R$ there exists a formula $\varphi(t)$, with one free variable, in the language for rings, such that $R \models \varphi(a)$ iff $a = r$ then the theorem is true with Z replaced by R.

Since the direct sum operation does not preserve elementary equivalence and the direct multiple operation on finite modules does, one major open question is whether the direct multiple operation on modules preserves elementary equivalence. Theorem 4 below is a partial answer to this question.

We say that $M \equiv_{AE} N$ if M and N are modules and if φ is a sentence with at most one alternation in quantifiers then $M \models \varphi$ iff $N \models \varphi$.

THEOREM 4. *If M and N are modules (not necessarily over the same ring) and $M \equiv_{AE} N$ then for any cardinal a, $\bigoplus_a M \equiv_{AE} \bigoplus_a N$.*

Proof. If a is finite the result follows easily from the proof of Theorem 1.1 of [4]. If $a > \omega$ then for any module M, $\bigoplus_a M \equiv \bigoplus_\omega M$. So it suffices to prove the result with $a = \omega$. Using the method of Ehrenfeucht [1] the result will follow if we can show the following. Let n, m, p, q be fixed non-negative integers. Suppose $A_1, \ldots, A_n$ are module elements and $a_1, \ldots, a_m$ scalars from $\bigoplus_\omega M$. Then we can find module elements $C_1, \ldots, C_n$ and scalars $c_1, \ldots, c_m$ in $\bigoplus_\omega N$ such that for any module elements $D_1, \ldots, D_p$ and scalars $d_1, \ldots, d_q$ in $\bigoplus_\omega N$ there exist module elements $B_1, \ldots, B_p$ and scalars $b_1, \ldots, b_q$ in $\bigoplus_\omega M$ such that the mapping $A_i \leftrightarrow C_i$, $B_i \leftrightarrow D_i$, $a_i \leftrightarrow c_i$, $b_i \leftrightarrow d_i$ is an isomorphism.

So suppose now that $A_1, \ldots, A_n, a_1, \ldots, a_m$ are given. Let t be the smallest integer such that if $j > t$ then $A_1(j) = \ldots = A_n(j) = 0$.

Let S be the set of all atomic relations which are formed from $W_0 = 0$, $W_1, \ldots, W_p$, $y_1, \ldots, y_q$, $a_1, \ldots, a_m$ and which are of the form $W_i = W_j$, $W_i + W_j = W_k$, $a_i W_j = W_k$, or $y_i W_j = W_k$. Then S has finite cardinality s. Let $r = 2^s$. Let θ_i, $1 \leqslant i \leqslant r$, be a list of all formulas $P_1 \wedge \ldots \wedge P_s$ where each P_k is either the kth member of S or its negation.

Let T be the set containing zero; the constants $A_i(j)$, $0 \leqslant j \leqslant t+r$ and $1 \leqslant i \leqslant n$; a_i, $1 \leqslant i \leqslant m$; and the variables Y_i^j, $0 \leqslant j \leqslant t+r$ and $1 \leqslant i \leqslant p$; y_i, $1 \leqslant i \leqslant q$. Let $\mu_1, \ldots, \mu_v$ be a list of all the formulas $\bigwedge_k Q_k$ where each Q_k is an atomic relation among the members of T or the negation of such, and each such atomic relation or its negation is present with the following restrictions:

(i) If only constants are involved in such a relation then Q_k must be true in M.

(ii) If Q_k involves $A_i(j)$'s and/or Y_i^j's then all the numbers in the j-places must be equal; for example, Q_k might be $A_1(3) + Y_2^3 = Y_6^3$ but it could not be $A_1(3) + Y_2^4 = Y_6^3$. Let ν_i be the sentence $(EY_1^0) \ldots \ldots (EY_p^{t+r})(Ey_1) \ldots (Ey_q)\mu_i$. Form $\bigwedge_{i=1}^{v} \psi_i$ where ψ_i is ν_i if $M \models \nu_i$ and ψ_i is $\sim\nu_i$ if $M \models \sim\nu_i$. Hence $M \models \bigwedge_{i=1}^{v} \psi_i$. Now in this last formula replace a_i by x_i and $A_i(j)$ by X_i^j and existentially quantify to get a sentence ψ. Of course, $M \models \psi$. And, from its construction, ψ is an EA sentence (one alternation in quantifiers). So, since $M \equiv_{AE} N$, we get $N \models \psi$. Let C_i^j and c_i be the X_i^j and x_i whose existence in N is first asserted. Define $C_i \in \bigoplus_\omega N$ by $C_i(j) = C_i^j$ for $0 \leqslant j \leqslant t+r$, and if $j > t+r$, $C_i(j) = 0$. In fact, since $A_i(j) = 0$ for $t < j \leqslant t+r$, we have $C_i(j) = 0$ for all $j > t$. Also, the sentence gotten from ψ_i by replacing $A_i(j)$ by $C_i(j)$ and a_i by c_i is now true in N.

Now suppose $D_1, \ldots, D_p$ are module elements and $d_1, \ldots, d_q$ are scalars in $\bigoplus_\omega N$. Recall the definition of S and θ_k, $1 \leqslant k \leqslant r$, above. Without loss of generality let $\theta_1, \ldots, \theta_u$, $u \leqslant r$, be those θ_k's such that there exists a $j > t$ such that if W_i is replaced by $D_i(j)$, $1 \leqslant i \leqslant p$, y_i by d_i, and a_i by c_i then the result is true in N. Let $j_1, \ldots, j_u$ be such that j_k is a j which has this property with respect to θ_k. The definition of S and the θ_i's ensures that for every $j > t$ there is a j_k such that the mapping $d_i \leftrightarrow d_i$, $c_i \leftrightarrow c_i$, $0 \leftrightarrow 0$, and $D_i(j) \leftrightarrow D_i(j_k)$ is an isomorphism. Define D_i^j to be $D_i(j)$ if $j \leqslant t$, to be $D_i(j_k)$ for $j = t+k$, and to be 0 for $t+u < j \leqslant t+r$. Now replacing Y_i^j by D_i^j, y_i by d_i, $A_i(j)$ by $C_i(j)$ and a_i by c_i, there is a ψ_h and μ_h which are satisfied in N. So by the choice of C_i and c_i, ψ_h as originally defined is true in M. Let B_i^j and b_i be the Y_i^j and y_i whose existence in M is thus stated. Define $B_i \in \bigoplus_\omega M$ by $B_i(j) = B_i^j$ for $0 \leqslant j \leqslant t+r$ and $B_i(j) = 0$ for $j > t+r$. This construction ensures that for any j there is a j' such that $A_i(j) \leftrightarrow C_i(j')$, $a_i \leftrightarrow c_i$, $B_i(j) \leftrightarrow D_i(j')$, $b_i \leftrightarrow d_i$ is an isomorphism; and similarly for any j' there is a j such that the same map is again an isomorphism. Hence $A_i \leftrightarrow C_i$, $a_i \leftrightarrow c_i$, $B_i \leftrightarrow D_i$, $b_i \leftrightarrow d_i$ is an isomorphism and the proof is complete.

As a corollary of the proof it is not difficult to show that if $M \equiv_A N$ then for any cardinal α, $X_\alpha M \equiv_A X_\alpha N$. Further, it follows that if M and N are $L_{\omega_1,\omega}$-equivalent then $\bigoplus_\alpha M \equiv \bigoplus_\alpha N$.

References

[1] A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, Fund. Math. 49 (1961), pp. 129–141.
[2] Ju. L. Eršov, Review of [6] below, Math. Rev. 38 (1969), p. 5.
[3] S. Feferman and R. L. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. 47 (1959), pp. 57–103.
[4] P. Olin, *Some first order properties of direct sums of modules*, Zeitschr. Math. Logik und Grundlagen Math. 16 (1970), pp. 405–416.
[5] D. Scott, *Dimension in elementary Euclidean geometry*, Symposium on the axiomatic method, Amsterdam 1959, pp. 53–67.
[6] R. T. Volvačev, *The elementary theory of modules* (Russian), Vesci Akad. Nauk B.S.S.R. Ser. Fiz.-Mat. Nauk 4 (1967), pp. 7–12 (see [2] above).

MCGILL UNIVERSITY
Montreal, Canada

Reçu par la Rédaction le 10. 3. 1970

A minimal model for strong analysis

by

Erik Ellentuck * (New Brunswick, N. J.)

In [6] it is shown that axiomatic second order arithmetic does not possess a minimal ω-model. Here we extend that result to general models of the full second order theory of $\langle\omega, +, \cdot\rangle$ and show that various model theoretic concepts, e.g., the existence of prime models, minimal ω-models, etc., all coincide, but are independent of Zermelo Fraenkel set theory and some of its extensions. These results are then applied to the weak second order theory of real numbers.

Let $\mathfrak{F} = \langle F, \omega, +, \cdot\rangle$ where F is the set of all functions mapping ω into ω. Consider a two sorted language $\mathfrak{L}$ for $\mathfrak{F}$ which contains individual variables $v_0, v_1, \ldots$ and function variables $\alpha_0, \alpha_1, \ldots$ Under our intended interpretation the individual variables range over ω and the function variables range over F. This distinction between variables has been introduced for convenience. We can easily find an equivalent (though less suggestive) one sorted language for $\mathfrak{F}$. Thus we assume that all of the standard first order concepts suitably generalize to $\mathfrak{L}$. In particular we shall be interested in the notions of proof ($\vdash$), satisfaction ($\models$), subsystem ($\subseteq$), and elementary subsystem ($\prec$). Let $T = \mathrm{Th}(\mathfrak{F})$ be the $\mathfrak{L}$-theory of $\mathfrak{F}$. A model $\mathfrak{P}$ of T is said to be *prime in the sense of Vaught* (cf. [16]) if $\mathfrak{P}$ is isomorphic to an elementary subsystem of every model of T. Let A be the set of functions $f \in F$ which are definable in $\mathfrak{F}$ by some formula $\varphi(\alpha_0)$ of $\mathfrak{L}$ and let $\mathfrak{A} = \langle A, \omega, +, \cdot\rangle$. We characterize the prime models of T in

THEOREM 1. *$\mathfrak{P}$ is a prime model of T in the sense of Vaught if and only if $\mathfrak{P}$ is isomorphic to $\mathfrak{A}$ and $\mathfrak{A}$ is a model of T.*

Proof. We use theorem 3.4 of [16] that a model is prime if and only if it is a denumerable atomic model. See [16] for an explanation of our terminology. For $n < \omega$ let $\boldsymbol{n}(v_0)$ be a purely existential formula with v_0 as its free variable and containing no function variables which defines n in $\mathfrak{F}$. If $\mathfrak{P} = \langle P, N, \oplus, \odot\rangle$ is a prime model of T, we construct an iso-

* Prepared while the author was a New Jersey Research Council Faculty Fellow.