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Noetherian lattice modules and semi-local completions

by

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§ 1. Introduction. The general ω-adic completion of a Noetherian lattice module was developed and studied in [1], and some specific results for Noetherian lattice modules over local Noether lattices were obtained. Some of those results were generalized in [2] to Noetherian lattice modules over semi-local Noether lattices. In this paper we are concerned with completions of Noetherian lattice modules over semi-local Noether lattices.

In § 2 the basic concepts are given. Some preliminary results are developed in § 3 which are required later in the paper. Let \( (L, p_1, \ldots, p_r) \) be a semi-local Noether lattice, let \( M \) be a Noetherian \( L \)-module, let \( m = p_{i_1} \ldots p_{i_r} \), and let \( M^* \) be the \( m \)-adic completion of \( M \). In § 4 it is shown that the \( L \)-module \([AM^*, BM^*]\) with the \( m \)-adic metric is the \( m \)-adic completion of the Noetherian \( L \)-module \([A, B]\), where \( A \) and \( B \) are elements of \( M \) such that \( A \leq B \) (Theorem 4.2). In § 5 we establish that the extension map \( A \to AM^* \) of \( M \to MM^* \) is a lattice isomorphism (Theorem 5.3). Thus, the Noetherian \( L \)-module \( M \) is lattice isomorphic to a sublattice of its \( m \)-adic completion.

§ 2. Preliminary remarks. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let \( L \) be a multiplicative lattice and let \( M \) be a complete lattice. We shall denote elements of \( L \) by \( a, b, c, \ldots \), with the exception that the null element and unit element of \( L \) will be denoted by \( 0 \) and \( I \), respectively. We shall denote elements of \( M \) by \( A, B, C, \ldots \), with the exception that the null element and unit element of \( M \) will be denoted by \( 0_M \) and \( 1_M \), respectively. When no confusion is possible, \( 0 \) will also be used in place of \( 0_M \). Recall that \( M \) is an \( L \)-module ([1], Definition 2.2) in case there is a multiplication between

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elements of \( L \) and \( M \), denoted by \( aA \) for \( a \in L \) and \( A \in M \), which satisfies:

(i) \((ab)A = a(bA)\);
(ii) \(\bigvee aA \bigvee B = \bigvee aA, B\);
(iii) \(IA = A\); and
(iv) \(0A = 0\) for all \( a, A, B, L, R \) in \( M \).

Let \( A \) be an \( L \)-module. For \( a, b \) in \( L \) and for \( A, B \) in \( M \), (i) \( a \cdot b \) denotes the largest \( c \) in \( L \) such that \( ab \leq c \); (ii) \( A \cdot B \) denotes the largest \( c \) in \( L \) such that \( c \leq AB \). An element \( A \) in \( M \) is said to be meet principal in case \( (b \cap (B : A))A = \bigwedge (b \cap B)A \) for all \( b \) in \( L \) and for all \( B \) in \( M \); \( A \) is said to be join principal in case \( b \cap B = \bigwedge (b \cap B)A \) for all \( b \) in \( L \) and for all \( B \) in \( M \); and, \( A \) is said to be principal in case \( A \) is both meet and join principal. If each element of \( M \) is the join (finite or infinite) of principal elements, \( M \) is called principally generated. \( M \) is said to be Noetherian if \( M \) satisfies the ascending chain condition, is modular, and is principally generated. If \( J \) is a Noetherian \( L \)-module, \( J \) is called a Noetherian lattice.

We state the following results for convenience. The reader is referred to [1] and [2] for their proofs.

**Definition 2.1.** Let \( L \) be a multiplicative lattice and let \( M \) be a Noetherian \( L \)-module. For \( a \in L \) and \( A \in M \), let \( T(a, A) \) be the collection of all sequences \( \langle B_i \rangle \), \( i = 1, 2, \ldots \), of elements of \( M \) satisfying

\[
aA \leq B_1 \leq B_{i+1} \leq aA, \tag{2.1}
\]

for all \( i \geq 1 \). For \( \langle A \rangle \) and \( \langle B \rangle \) in \( T(a, A) \), define

\[
\langle A \rangle \leq \langle B \rangle \quad \text{if and only if} \quad A_i \leq B_i, \quad \text{for all integers} \quad i \geq 1. \tag{2.2}
\]

\[
\langle A \rangle \vee \langle B \rangle = \langle C \rangle, \tag{2.3}
\]

\[
\langle A \rangle \wedge \langle B \rangle = \langle C \rangle, \tag{2.4}
\]

It is easily seen that \( T(a, A) \) forms a complete, modular lattice under the relation \( \leq \) with the resulting join and meet being given by (2.3) and (2.4). The resulting lattice will be denoted by \( R(a, A) \).

**Theorem 2.2.** Let \( L \) be a multiplicative lattice, let \( M \) be a Noetherian \( L \)-module, let \( a \) be an element of \( L \), and let \( \langle B_i \rangle \), \( i = 1, 2, \ldots \), be an element of \( R(a, A) \). Then there exists a natural number \( n \) such that \( a^{n+1} = a^n \), for all integers \( n \geq n \) and for all integers \( i \geq 0 \) \([2], \text{Theorem 3.2}\).

A Noetherian lattice is called semi-local if it has only finitely many maximal elements. If \( L \) is a semi-local Noetherian lattice with maximal elements \( p_1, p_2, \ldots, p_n \), we will say that \( \langle p_1, p_2, \ldots, p_n \rangle \) is a semi-local Noetherian lattice. For the rest of this section, let \( Z \) be a semi-local Noetherian lattice, \( M \) be a Noetherian \( L \)-module, \( m = p_1 \wedge \cdots \wedge p_n \), \( M' \) is the \( m \)-adic completion of \( M \), and \( L' \) is the \( m \)-adic completion of \( L \) (see \([2], \text{Corollary 3.4}\).

**Lemma 2.3.** Let \( \langle A_i \rangle \), \( i = 1, 2, \ldots \), be a sequence of elements of \( M \) satisfying \( A_{i+1} \leq A_i \vee m^i M \), for all integers \( i \geq 1 \). Then the sequence \( \langle A_i \rangle \) is Cauchy \([2], \text{Lemma 5.4}\).

**Proposition 2.4.** Let \( B, C \) be elements of \( M' \). Let \( \langle B_i \rangle \) and \( \langle C_i \rangle \) be the completely regular representatives of \( B \) and \( C \), respectively. Then the sequence \( \langle B_i \cap C_i \rangle \) is a representative of \( B \cap C \) \([2], \text{Proposition 5.5}\).

**Proposition 2.5.** \( M' \) is modular \([2], \text{Proposition 5.6}\).

**Proposition 2.6.** Let \( A, B \) be elements of \( M' \). Let \( \langle A_i \rangle \) and \( \langle B_i \rangle \) be the completely regular representatives of \( A \) and \( B \), respectively. Then the sequence \( \langle A_i \cup B_i \rangle \) is a representative of \( A \cup B \) \([2], \text{Proposition 5.7}\).

**Theorem 2.7.** Let \( \langle A \rangle \) be a Cauchy sequence of principal elements of \( M \). Then the equivalence class determined by \( \langle A \rangle \) is a principal element in \( M' \) (considered as an \( L' \)-module) \([2], \text{Theorem 5.8}\).

**Theorem 2.8.** \( L' \) is a Noetherian lattice and \( M' \) is a Noetherian \( L' \)-module \([2], \text{Theorem 5.9}\).

**Theorem 2.9.** Let \( m^* \) be the greatest lower bound of the maximal elements of \( L' \). Then, \([2], \text{Theorem 6.2}\):

\[
L' \text{ is a semi-local Noetherian lattice with maximal elements}
\]

\[
p_1 L', \ldots, p_n L', \quad m^* = m^* \cap L, \tag{2.6}
\]

and

\[
(p_1, \ldots, p_n) L^* = m L^* = m^* = p_1 L^* \cap \cdots \cap p_n L^*. \tag{2.7}
\]

§ 3. Preliminary results. Throughout this section we will have \( L, p_1, p_2, \ldots, p_n \) is a semi-local Noetherian lattice, \( M \) is a Noetherian \( L' \)-module, \( m = p_1, \ldots, p_n, M' \) is the \( m \)-adic completion of \( M \), and \( L' \) is the \( m \)-adic completion of \( L \).

We will need the following generalization of an unpublished result due to E. W. Johnson.

**Theorem 3.1.** \( M \) is a complete \( L \)-module with respect to the \( m \)-adic metric, if and only if, given any decreasing sequence \( \langle B_i \rangle \), \( i = 1, 2, \ldots \), of elements of \( M \) and positive integer \( n \), \( B_i \leq \langle \bigwedge B \rangle \cup m^n M, \) for all sufficiently large integers \( i \).

Proof. Assume \( M \) is a complete \( L \)-module with respect to the \( m \)-adic metric on \( M \). Let \( \langle B_i \rangle \), \( i = 1, 2, \ldots \), be a decreasing sequence of elements of \( M \). By Lemma 2.3, \( \langle B_i \rangle \) is a Cauchy sequence. Thus, by our assumption, there exists an element \( C \) in \( M \) such that \( B_i \to C \) as \( i \to \infty \) (in the \( m \)-adic metric).

Hence, by \([1], \text{Remark 3.6}\), for each integer \( n \geq 1 \), \( C \cap m^i M \)}
Remark 3.2. Let $A, B$ be elements of $M$ with $A \leq B$, and let $d$ be an element of $L$ such that $dC \leq A$, for all $C$ in $[A, B]$. For $b, c$ in $[d, I]$, define $b \cdot c = bc \lor d$. For $C$ in $[A, B]$, and $b$ in $[d, I]$, define $b \cdot C = bc \lor A$.

These definitions of multiplication make $[d, I]$ into a multiplicative lattice, and $[A, B]$ into a Noetherian $[d, I]$-module (see [2], Remarks 2.3 and 2.4).

Proposition 3.3. Let $A$ and $B$ be elements of $M$ such that $A \leq B$. Then the extension map $C \mapsto CM^* = \langle BM^* \rangle$ of the Noetherian $L$-module $[A, B]$ with the $m$-adic metric to the $L$-module $[A, M^*, BM^*]$ with the $m$-adic metric is an isometry.

Proof. Let $C$ and $D$ be elements of $[A, B]$. A routine calculation shows that

$$CM^* \vee m^n = (BM^*)^* = DM^* \vee m^n \cdot (BM^*)^*,$$

if and only if,

$$C \vee m^n \cdot D = B \vee m^n \cdot D,$$

for each nonnegative integer $n$. It follows from this that the map is an isometry, q.e.d.

We will now develop some properties that will be used in later sections of this paper.

Lemma 3.4. Let $A$ be an element of $M^*$. Let $\langle D_i \rangle$, $i = 1, 2, \ldots$, be a regular Cauchy sequence for $A$. Then $\lim_{i \to \infty} d_\infty(D_i, A) = 0$.

Proof. Let $\varepsilon$ be a positive real number. Let $k$ be the least natural number $k$ such that $2^{-k} < \varepsilon$. Since $\langle D_i \rangle$ is a Cauchy sequence of elements of $M$ with the $m$-adic metric, there exists a natural number $N$ such that $d_\infty(D_i, D_j) < 2^{-k}$, for all integers $i, j \geq N$. Thus, $A \vee m^n M^* = A \vee m^N M^*$, for all integers $i, j \geq N$. Consequently, for each integer $i \geq N$, we have $A \vee m^n M^* = A \vee m^N M^*$, for all integers $j \geq N$. Now, fix $i \geq N$, and let $j$ vary. Since the constant sequence $\langle A \rangle \vee m^N M^*$, $j = 1, 2, \ldots$, is a representative of $A \vee m^N M^*$, and the sequence $\langle A \rangle \vee m^N M^*$, $j = 1, 2, \ldots$, is a representative of $A \vee (m^N M^*)^*$, by (3.5), it follows that

$$A \vee m^N M^* \vee (m^N M^*)^* = A \vee (m^N M^*)^* = A \vee m^N M^*.$$

Hence, for each integer $i \geq N$, we obtain

$$A \vee m^N M^* \vee m^N M^* = A \vee m^N M^*.$$

It follows that $d_\infty(A \vee m^N M^*, A) < 2^{-k} < \varepsilon$, for all integers $i \geq N$, q.e.d.

The following result shows that the metrics $d_\infty$ and $d_\infty$ are equivalent on $M^*$. 

\[ \text{if } A, B \text{ are elements of } M \text{ with } A \leq B, \text{ then the set } \langle D \text{ in } N \rangle \text{ is a sublattice of } M, \text{ and will be denoted by } [A, B]. \]
PROPOSITION 3.5. Let $A$, $B$ be elements of $M^*$. Then $d_{\alpha}(A, B) = d_{\alpha}(A, B)$.

Proof. Let the Cauchy sequences $(A_i), (B_i), i = 1, 2, ...$, be representatives of $A$ and $B$, respectively. Since

$$\lim_{i \to \infty} d_{\alpha}(A_i M^*, A) = 0 = \lim_{i \to \infty} d_{\alpha}(B_i M^*, B)$$

by Lemma 3.4, we have that

$$d_{\alpha}(A, B) = \lim_{i \to \infty} d_{\alpha}(A_i M^*, B_i M^*)$$

Also, by Definition 5.5 of $[1]$ and Proposition 3.3, we know that

$$d_{\alpha}(A, B) = \lim_{i \to \infty} d_{\alpha}(A_i M^*, B_i M^*).$$

By combining these last two results we obtain $d_{\alpha}(A, B) = d_{\alpha}(A, B)$, q.e.d.

COROLLARY 3.6. The three metrics $d_{\alpha}$, $d_{\beta}$, and $d_{\alpha}$ are equal on $M^*$.

PROPOSITION 3.7. Let $b$ be a nonnegative integer such that $b M^* = b B$. Then $M^* = b b M^* B = b b$.

Proof. Let $(B_i), i = 1, 2, ...$, be the completely regular representative of $B$. Then the sequence $(b B_i), i = 1, 2, ...$, is a representative of $b b B$ [1], Definition 6.5). Consequently, the sequence $(b B_i M^*, i = 1, 2, ...$ is a representative of $b B_i M^*$ [1], Corollary 4.5]. Since $(b b M^*), i = 1, 2, ...$, is the completely regular representative of $b b M^*$ [1], Remark 5.2], we have that $(b b M^*) B, i = 1, 2, ...$, is a representative of $b M^* B$. Since $(b b M^*) B, i = 1, 2, ...$, is a representative of $M^* B$, for each nonnegative integer $i$, the result follows, q.e.d.

§ 4. Completions of intervals. Throughout the remainder of this paper, $(L, p_1, p_2, ..., p_r)$ is a semi-local Noether lattice, $M$ is a Noetherian $L$-module, $M = p_1 \cap \ldots \cap p_r$, $M^*$ is the $M$-adic completion of $M$, and $L^*$ is the $M$-adic completion of $L$.

In this section we shall establish the form of completions of intervals of $M$. This result is needed later in the paper.

THEOREM 4.1. Let $A$ and $B$ be elements of $M$ such that $A \subseteq B$. Then, the set $(A, B) M^*$ is dense in the $M$-module $[A M^*, B M^*]$ with the $M$-adic metric.

Proof. Let $C$ be an arbitrary element of $[A M^*, B M^*]$. Considering $C$ as an element of $M^*$, let $(C_i), i = 1, 2, ...$, elements of $M$ be the completely regular representative of $C$ determined by the $M$-adic metric on $M$. Since $(C_i)$ is completely regular, it is decreasing [11, Remark 4.8]. Thus $(C_i, B)$ is decreasing, and hence is Cauchy (Lemma 2.3). Since

$$\langle C_i \rangle \text{ and } (B \cap \nu M^*)$$

are the completely regular representatives of $C$ and $B M^*$, respectively, the sequence $(C_i \cap (B \cap \nu M^*))$ is a representative of $C \cap (B \cap \nu M^*)$ by Proposition 2.4. Consequently, $(C_i \cap (B \cap \nu M^*)) = (C_i \cap (B \cap \nu M^*))$ for all integers $i \geq 1$, by modularity, and since $(C_i \cap (B \cap \nu M^*))$ is the Cauchy sequence of $(C_i \cap (B \cap \nu M^*))$, it is Cauchy. Thus $(C_i \cap (B \cap \nu M^*)) \to C$ as $i \to \infty$ with the $d_{\alpha}$ metric and thus with the $\nu$-adic metric (Proposition 3.5). Since $(C_i \cap \nu M^*)$ is a completely regular representative of $C_i$, $A$, and $B_i$, respectively, and since $A M^* \subseteq C \subseteq B M^*$, it follows that $A \subseteq A \cap \nu M^* \subseteq C \subseteq B \cap \nu M^*$, for all $i \geq 1$ [11, Proposition 5.9]. Consequently, $C_i \cap (B \cap \nu M^*)$ is in $(A, B)$, for all $i \geq 1$.

Consider the sequence $(m^*)^i (M^* \cap B M^*), i = 1, 2, ...$ Since

$$(m^*)^i (M^* \cap B M^*) \geq (m^*)^i (M^* \cap B M^*) \geq (m^*)^i (M^* \cap B M^*) \geq (m^*)^i (M^* \cap B M^*)$$

for each $i \geq 1$, this sequence satisfies the conditions of Theorem 2.2. (See Theorem 2.8). Thus, (Proposition 3.7) there is a natural number $n$ such that

$$(4.1) \quad m^* \cap (M^* \cap B M^*) = m^* \cap (M^* \cap B M^*),$$

for all integers $i \geq 0$. Thus,

$$(4.2) \quad B M^* \cap (C_i \cap (B \cap \nu M^*)), i = 1, 2, ...$$

for all integers $i \geq 0$. By modularity in $M^*$ (Proposition 2.5) and (4.1) we have

$$B \nu M^* \cap (C_i \cap (B \cap \nu M^*) = (C_i \cap (B \cap \nu M^*) \nu B \nu M^*$$

and

$$B \nu M^* \cap (C_i \cap (B \cap \nu M^*) = (C_i \cap (B \cap \nu M^*) \nu B \nu M^*.$$
by (4.3). Thus, we obtain
\[(A \cup B)M^* \cap m^n \cap (BM^*) = C \cup m^n \cap (BM^*)\]
for all integers \(i \geq N\). Hence, for \(i \geq N\), the \(m\)-adic distance from \(A \cup B\) to \(C\) is less than or equal to \(2^{-i}\) [see (5.3), Remark 3.6], q.e.d.

**Theorem 4.2.** Let \(A\) and \(B\) be elements of \(M\) such that \(A \leq B\). Then the \(L\)-module \([AM^*, BM^*]\) is complete with respect to the \(m\)-adic metric on \([AM^*, BM^*]\).

**Proof.** We will make use of Theorem 3.1 to prove this result. Let \(\langle Ci \rangle\), \(i = 1, 2, \ldots\), be an arbitrary decreasing sequence in the \(L\)-module \([AM^*, BM^*]\), and let \(j\) be a positive integer. We wish to show that
\[(4.4) \quad C_i \leq (\bigwedge q \bigvee (m^n)^j \cap (BM^*))\]
for all sufficiently large integers \(i\).

For this, consider the sequence \(\langle (m^n)\cap (3RM^*) \cap (BM^*) \rangle\), \(i = 1, 2, \ldots\), Since
\[(4.5) \quad (m^n)^{j+i} \cap (3RM^*) \cap (BM^*) = (m^n) \cap (3RM^*) \cap (BM^*)\]
for each positive integer \(i\), the sequence \(\langle (m^n)\cap (3RM^*) \cap (BM^*) \rangle\), \(i = 1, 2, \ldots\), satisfies the conditions of Theorem 2.2 (recall that \(M^*\) is a Noetherian \(L^*\)-module by Theorem 2.8). Consequently, there exists a natural number \(n\) such that
\[(4.6) \quad C_i \leq (\bigwedge q \bigvee (m^n)^j \cap (3RM^*))\]
for all integers \(i \geq n\), and for all integers \(i \geq 0\). Since the sequence \(\langle C_i \rangle\) is decreasing, and since \(M^*\) is a complete \(L\)-module with respect to the \(m\)-adic metric on \(M^*\), by Theorem 3.1 there exists a natural number \(N\) such that
\[(4.7) \quad C_i \leq (\bigwedge q \bigvee (m^n)^j \cap (3RM^*))\]
for all integers \(i \geq N\). It follows that
\[(4.8) \quad C_i \cap m^n = (\bigwedge q \bigvee (m^n)^j \cap (3RM^*))\]
for all integers \(i \geq N\). Hence, by (4.5) and (4.6), we have
\[(4.9) \quad C_i \cap m^n \leq (\bigwedge q \bigvee (m^n)^j \cap (3RM^*))\]
for all integers \(i \geq N\). Now, let \(i\) be an integer such that \(i \geq N\). Then,
\[(4.10) \quad C_i = C_i \cap m^n \leq (\bigwedge q \bigvee (m^n)^j \cap (3RM^*))\]
by (4.5). Thus,
\[(4.11) \quad C_i \leq (\bigwedge q \bigvee (m^n)^j \cap (BM^*))\]
for all integers \(i \geq N\), which establishes (4.4), q.e.d.

**Theorem 4.3.** Let \(A\) and \(B\) be elements of \(M\) such that \(A \leq B\). Then the \(L\)-module \([AM^*, BM^*]\) with the \(m\)-adic metric is the \(m\)-adic completion of the Noetherian \(L\)-module \([A, B]\).

**Proof.** This follows immediately from Proposition 3.3 and Theorems 4.1, 4.2, by the uniqueness of the completion (up to an isomorphism), q.e.d.

**§ 5. The extension isomorphism.** In this section we establish some results about resoludation and show that the extension map is a lattice isomorphism.

**Theorem 5.1.** Let \(A\) be an element of \(M\) and let \(B\) be a principal element of \(M\). Then \((A:B)L^* = AM^*:BM^*\).

**Proof.** Since \((A:B) \leq A\), we obtain \((A:B)L^* \cap (AM^*:BM^*) = (A:B)L^* \cap AM^*:BM^* = (A:B)L^* \cap AM^*:BM^*\). Therefore, we need only show that \((A:B)L^* \geq AM^*:BM^*\).

Let \(n\) be a nonnegative integer. Since \(L^*\) is dense in \(L^*\), there exists an element \(x\) of \(L^*\) such that \(xL^* \cap (m^n)^j = (AM^*:BM^*) \cap (m^n)^j\). Since \((AM^*:BM^*) \cap (m^n)^j \leq AM^*:BM^*\), we have
\[(4.12) \quad (xL^*)^j \cap (m^n)^j \leq (AM^*:BM^*) \cap (m^n)^j\]
for all integers \(i \geq N\). Thus, \(xL^* \cap (m^n)^j \leq (AM^*:BM^*) \cap (m^n)^j\), and hence \(xL^* \cap (m^n)^j \leq (AM^*:BM^*) \cap (m^n)^j\).

Since \(n\) was arbitrary, we have
\[(4.13) \quad (AM^*:BM^*) \cap (m^n)^j \leq (A:B)L^* \cap (m^n)^j\]
for all nonnegative integers $n$. Now, since $L^*$ is a semi-local Noether lattice (Theorem 2.9), we have

$$AM^* : BM^* = \bigwedge_n ((AM^* : BM^*) \lor (m^n))$$

$$\leq \bigwedge_n ((A : B)L^* \lor (m^n)) = (A : B)L^*$$

by (17), Corollary 3.4, q.e.d.

**Corollary 5.2.** Let $A$ be an element of $M$ and let $B$ be a principal element of $M$. Then $(A \land B)M^* = AM^* \land BM^*$.

**Proof.** Since principal elements extend to principal elements (Theorem 2.7), $BM^*$ is a principal element of $M^*$. It follows that

$$(A \land B)M^* = [(A : B)B]M^* = [(A : B)L^*[BM^*]$$

$$= (AM^* : BM^*)[BM^*] = AM^* \land BM^*$$

by the theorem and the definition of a principal element, q.e.d.

The following theorem shows that $M$ is lattice isomorphic to $MM^*$ considered as a sublattice of $M^*$.

**Theorem 5.3.** The extension map $A \rightarrow AM^*$ of $M \rightarrow MM^*$ is a lattice isomorphism.

**Proof.** Let $A$ and $B$ be elements of $M$. Recall that $(A \lor B)M^* = AM^* \lor BM^*$ by definition (11), Definition 5.4), and that the extension map is one-to-one (11), Proposition 5.3). Hence we need only show that $(A \lor B)M^* = AM^* \lor BM^*$.

Since $M$ is a Noetherian $L$-module, there are principal elements $P_1, ..., P_n$ in $M$ such that $P_i = P_i \lor P_n \lor (A \land B)$. The proof is by induction on $n$. Assume $B = P_i \lor (A \land B)$. Since $P_i$ is principal in $M$, $P_i \lor (A \land B)$ is a principal element of the Noetherian $L$-module $A \land B, [2]$. Hence,

$$(A \land B)M^* = \{A, P_i \lor (A \land B)\}M^* = \{A, P_1 \lor (A \land B)\}M^* \rightarrow [A, B, [2]^*$$

$$= [A, B, [2]^* \lor (P_1 \lor (A \land B))^*[A, B, [2]^*$$


by Corollary 5.2 and Theorem 4.3. Thus the case when $n = 1$ holds. Assume the result holds for case $n = k$ and suppose $B = P_i \lor P_k \lor (A \land B)$. The proof follows.

**References**


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