Noetherian lattice modules and semi-local completions

by

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§ 1. Introduction. The general a-adic completion of a Noetherian lattice module was developed and studied in [1], and some specific results for Noetherian lattice modules over local Noether lattices were obtained. Some of those results were generalized in [2] to Noetherian lattice modules over semi-local Noether lattices. In this paper we are concerned with completions of Noetherian lattice modules over semi-local Noether lattices.

In § 2 the basic concepts are given. Some preliminary results are developed in § 3 which are required later in the paper. Let $(L, p_1, \ldots, p_r)$ be a semi-local Noether lattice, let M be a Noetherian L-module, let $m = p_1 \wedge \ldots \wedge p_r$, and let M^* be the m-adic completion of M. In § 4 it is shown that the L-module $[AM^*, BM^*]$ with the m-adic metric is the m-adic completion of the Noetherian L-module $[A, B]$, where A and B are elements of M such that $A \leqslant B$ (Theorem 4.2). In § 5 we establish that the extension map $A \to AM^*$ of $M \to MM^*$ is a lattice isomorphism (Theorem 5.3). Thus, the Noetherian L-module M is lattice isomorphic to a sublattice of its m-adic completion.

§ 2. Preliminary remarks. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let L be a multiplicative lattice and let M be a complete lattice. We shall denote elements of L by $a, b, c, \ldots,$ with the exception that the null element and unit element of L will be denoted by 0 and I, respectively. We shall denote elements of M by $A, B, C, \ldots,$ with the exception that the null element and unit element of M will be denoted by 0_M and $\mathfrak{M}$, respectively. When no confusion is possible, 0 will also be used in place of 0_M. Recall that M is an L-module ([1], Definition 2.2) in case there is a multiplication between

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elements of L and M, denoted by aA for a in L and A in M, which satisfies: (i) $(ab)A = a(bA)$; (ii) $(\bigvee_\alpha a_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha,\beta} a_\alpha B_\beta$; (iii) $IA = A$; and (iv) $0A = 0$; for all a, a_α, b in L and for all A, B_β in M.

Let M be an L-module. For a, b in L and for A, B in M, (i) $a:b$ denotes the largest c in L such that $cb \leqslant a$; (ii) $A:B$ denotes the largest c in L such that $cB \leqslant A$. An element A in M is said to be meet principal in case $\big(b \wedge (B:A)\big)A = bA \wedge B$, for all b in L and for all B in M; A is said to be join principal in case $b \vee (B:A) = (bA \vee B):A$, for all b in L and for all B in M; and, A is said to be principal in case A is both meet and join principal. If each element of M is the join (finite or infinite) of principal elements, M is called principally generated. M is said to be Noetherian if M satisfies the ascending chain condition, is modular, and is principally generated. If L is a Noetherian L-module, L is called a Noether lattice. For other general properties and definitions concerning Noetherian lattice modules, the reader is referred to [1] and [2].

We state the following results for convenience. The reader is referred to [2] for their proofs.

Definition 2.1. Let L be a multiplicative lattice and let M be a Noetherian L-module. For a in L and A in M, let $T(a, A)$ be the collection of all sequences $\langle B_i \rangle$, $i = 1, 2, \ldots$, of elements of M satisfying

$$a^i A \geqslant B_i \geqslant B_{i+1} \geqslant aB_i\,, \tag{2.1}$$

for all integers $i \geqslant 1$. For $\langle C_i \rangle$ and $\langle B_i \rangle$ in $T(a, A)$, define

$$\langle C_i \rangle \leqslant \langle B_i \rangle \quad \text{if and only if} \quad C_i \leqslant B_i\,, \quad \text{for all integers } i \geqslant 1 \tag{2.2}$$

$$\langle C_i \rangle \vee \langle B_i \rangle = \langle C_i \vee B_i \rangle\,, \tag{2.3}$$

$$\langle C_i \rangle \wedge \langle B_i \rangle = \langle C_i \wedge B_i \rangle\,. \tag{2.4}$$

It is easily seen that $T(a, A)$ forms a complete, modular lattice under the relation $\leqslant$ with the resulting join and meet being given by (2.3) and (2.4). The resulting lattice will be denoted by $R(a, A)$.

Theorem 2.2. *Let L be a multiplicative lattice, let M be a Noetherian L-module, let a be an element of M, and let $\langle B_i \rangle$, $i = 1, 2, \ldots$, be an element $R(a, A)$. Then there exists a natural number n such that $B_{m+i} = a^i B_m$, for all integers $m \geqslant n$ and for all integers $i \geqslant 0$* ([2], Theorem 3.2).

A Noether lattice is called semi-local if it has only finitely many maximal elements. If L is a semi-local Noether lattice with maximal elements $p_1, p_2, \ldots, p_k$, we will say that $(L, p_1, p_2, \ldots, p_k)$ is a semi-local Noether lattice. For the rest of this section, $(L, p_1, p_2, \ldots, p_k)$ is a semi-local Noether lattice, M is a Noetherian L-module, $m = p_1 \wedge \ldots \wedge p_k$, M^* is the m-adic completion of M, and L^* is the m-adic completion of L (see [2], Corollary 3.4).

Lemma 2.3. *Let $\langle A_i \rangle$, $i = 1, 2, \ldots$, be a sequence of elements of M satisfying $A_{i+1} \leqslant A_i \vee m^i \mathfrak{M}$, for all integers $i \geqslant 1$. Then the sequence $\langle A_i \rangle$ is Cauchy* ([2], Lemma 5.4).

Proposition 2.4. *Let B, C be elements of M^*. Let $\langle B_i \rangle$ and $\langle C_i \rangle$ be the completely regular representatives of B and C, respectively. Then the sequence $\langle B_i \wedge C_i \rangle$ is a representative of $B \wedge C$* ([2], Proposition 5.5).

Proposition 2.5. *M^* is modular* ([2], Proposition 5.6).

Proposition 2.6. *Let A, B be elements of M^*. Let $\langle A_i \rangle$ and $\langle B_i \rangle$ be the completely regular representatives of A and B, respectively. Then the sequence $\langle A_i : B_i \rangle$ is a representative of $A:B$* ([2], Proposition 5.7).

Theorem 2.7. *Let $\langle A_i \rangle$ be a Cauchy sequence of principal elements of M. Then the equivalence class determined by $\langle A_i \rangle$ is a principal element in M^* (considered as an L^*-module)* ([2], Theorem 5.8).

Theorem 2.8. *L^* is a Noether lattice and M^* is a Noetherian L^*-module* ([2], Theorem 5.9).

Theorem 2.9. *Let m^* be the greatest lower bound of the maximal elements of L^*. Then,* ([2], Theorem 6.2)

(2.5) *L^* is a semi-local Noether lattice with maximal elements $p_1 L^*, \ldots, p_n L^*$;*

$$m = m^* \cap L; \tag{2.6}$$

and,

$$(p_1 \wedge \ldots \wedge p_n) L^* = mL^* = m^* = p_1 L^* \wedge \ldots \wedge p_n L^*. \tag{2.7}$$

§ 3. Preliminary results. Throughout this section we will have $(L, p_1, p_2, \ldots, p_k)$ is a semi-local Noether lattice, M is a Noetherian L-module, $m = p_1 \wedge \ldots \wedge p_k$, M^* is the m-adic completion of M, and L^* is the m-adic completion of L.

We will need the following generalization of an unpublished result due to E. W. Johnson.

Theorem 3.1. *M is a complete L-module with respect to the m-adic metric, if and only if, given any decreasing sequence $\langle B_i \rangle$, $i = 1, 2, \ldots$, of elements of M and positive integer n, $B_i \leqslant (\bigwedge_j B_j) \vee m^n \mathfrak{M}$, for all sufficiently large integers i.*

Proof. Assume M is a complete L-module with respect to the m-adic metric on M. Let $\langle B_i \rangle$, $i = 1, 2, \ldots$, be a decreasing sequence of elements of M. By Lemma 2.3, $\langle B_i \rangle$ is a Cauchy sequence. Thus, by our assumption, there exists an element C in M such that $B_i \to C$ as $i \to +\infty$ (in the m-adic metric).

Hence, by ([1], Remark 3.6), for each integer $n \geqslant 1$, $C \vee m^n \mathfrak{M}$

$= B_i \vee m^n\mathfrak{M}$, for all sufficiently large integers i. Therefore, for each positive integer n and each positive integer k, we have

$$C \vee m^n\mathfrak{M} = B_i \vee m^n\mathfrak{M} \leqslant B_k \vee m^n\mathfrak{M},$$

for all sufficiently large integers i. Consequently, by ([2], Corollary 3.4), it follows that

$$C = \bigwedge_n (C \vee m^n\mathfrak{M}) \leqslant \bigwedge_n (B_k \vee m^n\mathfrak{M}) = B_k,$$

for each positive integer k. Thus $C \leqslant \bigwedge_k B_k$. Then, for any positive integer n,

$$B_i \leqslant B_i \vee m^n\mathfrak{M} = C \vee m^n\mathfrak{M} \leqslant (\bigwedge_k B_k) \vee m^n\mathfrak{M},$$

for sufficiently large integers i.

To show the opposite implication, assume that given any decreasing sequence $\langle B_i \rangle$, $i = 1, 2, \ldots$, of elements of M and any positive integer n, $B_i \leqslant (\bigwedge_j B_j) \vee m^n\mathfrak{M}$, for all sufficiently large integers i. We wish to show that M is a complete L-module with respect to the m-adic metric. Let $\langle C_i \rangle$, $i = 1, 2, \ldots$, be a Cauchy sequence of elements of M. Let $\langle C_i' \rangle$, $i = 1, 2, \ldots$, be a regular subsequence of $\langle C_i \rangle$ ([1], Lemma 4.11). Set $D_i = C_i' \vee m^i\mathfrak{M}$, for $i = 1, 2, \ldots$ Then $\langle D_i \rangle$ is a completely regular Cauchy sequence ([1], Lemma 4.12). Since

$$\operatorname*{limit}_{i\to\infty} C_i = \operatorname*{limit}_{i\to\infty} C_i' = \operatorname*{limit}_{i\to\infty} D_i$$

if any one of these limits exist, it is sufficient to show that $\operatorname*{limit}_{i\to\infty} D_i$ exists. We shall show that $\operatorname*{limit}_{i\to\infty} D_i = \bigwedge_j D_j$. Let $\varepsilon > 0$. Let n be the least natural number k such that $2^k < \varepsilon$. Since D_i is a completely regular Cauchy sequence, it is decreasing ([1], Remark 4.8). Consequently, by assumption,

$$D_i \leqslant (\bigwedge_j D_j) \vee m^n\mathfrak{M},$$

for all sufficiently large integers i. It follows that

$$D_i \vee m^n\mathfrak{M} = (\bigwedge_j D_j) \vee m^n\mathfrak{M},$$

for all sufficiently large integers i. This implies that $d_m(D_i, \bigwedge_j D_j) \leqslant 2^{-n}$, for all sufficiently large integers i. ([1], Remark 3.6). Thus, $\operatorname*{limit}_{i\to\infty} D_i = \bigwedge_j D_j$, in the m-adic metric, q.e.d.

If A, B are elements of M with $A \leqslant B$, then the set $\{D \text{ in } K \mid A \leqslant D \leqslant B\}$ is a sublattice of M, and will be denoted by $[A, B]$.

Remark 3.2. Let A, B be elements of M with $A \leqslant B$, and let d be an element of L such that $dC \leqslant A$, for all C in $[A, B]$. For b, c in $[d, I]$, define $b \circ c = bc \vee d$. For C in $[A, B]$, and b in $[d, I]$, define $b \circ C = bc \vee A$. These definitions of multiplication make $[d, I]$ into a multiplicative lattice, and $[A, B]$ into a Noetherian $[d, I]$-module (see [2], Remarks 2.3 and 2.4).

Proposition 3.3. *Let A and B be elements of M such that $A \leqslant B$. Then the extension map $C \to CM^*$ of the Noetherian L-module $[A, B]$ with the m-adic metric to the L-module $[AM^*, BM^*]$ with the m-adic metric is an isometry.*

Proof. Let C and D be elements of $[A, B]$. A routine calculation shows that

$$CM^* \vee m^n \circ (BM^*) = DM^* \vee m^n \circ (BM^*),$$

if and only if,

$$C \vee m^n \circ B = D \vee m^n \circ B,$$

for each nonnegative integer n. It follows from this that the map is an isometry, q.e.d.

We will now develop some properties that will be used in later sections of this paper.

Lemma 3.4 *Let A be an element of M^*. Let $\langle A_i \rangle$, $i = 1, 2, \ldots$, be a representative of A. Then $\operatorname*{limit}_{i\to\infty} d_m(A_iM^*, A) = 0$.*

Proof. Let ε be a positive real number. Let k be the least natural number q such that $2^{-q} < \varepsilon$. Thus $2^{-k} < \varepsilon$. Since $\langle A_i \rangle$ is a Cauchy sequence of elements of M with the m-adic metric, there exists a natural number N such that $d_m(A_i, A_j) \leqslant 2^{-k}$, for all integers $i, j \geqslant N$. Thus, $A_i \vee m^k\mathfrak{M} = A_j \vee m^k\mathfrak{M}$, for all integers $i, j \geqslant N$. Consequently, for each integer $i \geqslant N$, we have $A_i \vee m^k\mathfrak{M} = A_j \vee m^k\mathfrak{M}$, for all integers $j \geqslant N$. Now, fix $i \geqslant N$, and let j vary. Since the constant sequence $\langle A_i \vee m^k\mathfrak{M} \rangle$, $j = 1, 2, \ldots$, is a representative of $A_iM^* \vee (m^k\mathfrak{M})M^*$ and the sequence $\langle A_j \vee m^k\mathfrak{M} \rangle$, $j = 1, 2, \ldots$, is a representative of $A \vee (m^k\mathfrak{M})M^*$ ([1], (5.8)), it follows that

$$A_iM^* \vee m^k(\mathfrak{M}M^*) = A_iM^* \vee (m^k\mathfrak{M})M^* = A \vee (m^k\mathfrak{M})M^*$$
$$= A \vee m^k(\mathfrak{M}M^*).$$

Hence, for each integer $i \geqslant N$, we obtain

$$A_iM^* \vee m^k(\mathfrak{M}M^*) = A \vee m^k(\mathfrak{M}M^*).$$

It follows that $d_m(A_iM^*, A) \leqslant 2^{-k} < \varepsilon$, for all integers $i \geqslant N$, q.e.d.

The following result shows that the metrics d_m^* and d_m are equivalent on M^*.

PROPOSITION 3.5. *Let A, B be elements of M^*. Then $d^*_m(A, B) = d_m(A, B)$.*

Proof. Let the Cauchy sequences $\langle A_i\rangle, \langle B_i\rangle, i = 1, 2, \ldots$, be representatives of A and B, respectively. Since

$$\operatorname*{limit}_{i\to\infty} d_m(A_iM^*, A) = 0 = \operatorname*{limit}_{i\to\infty} d_m(B_iM^*, B)$$

by Lemma 3.4, we have that

$$d_m(A, B) = \operatorname*{limit}_{i\to\infty} d_m(A_iM^*, B_iM^*) .$$

Also, by Definition 5.5 of [1] and Proposition 3.3, we know that

$$d^*_m(A, B) = \operatorname*{limit}_{i\to\infty} d_m(A_i, B_i) = \operatorname*{limit}_{i\to\infty} d_m(A_iM^*, B_iM^*) .$$

By combining these last two results we obtain $d^*_m(A, B) = d_m(A, B)$, q.e.d.

COROLLARY 3.6. *The three metrics d^*_m, d_m, and d_{m^*} are equal on M^*.*

PROPOSITION 3.7. *Let b be an element of L and let B be an element of M^*. Then $bL^*\cdot B = bB$.*

Proof. Let $\langle B_i\rangle, i = 1, 2, \ldots$, be the completely regular representative of B. Then the sequence $\langle bB_i\rangle, i = 1, 2, \ldots$, is a representative of bB ([1], Definition 6.5). Consequently, the sequence $\langle bB_i \vee m^i\mathfrak{M}\rangle, i = 1, 2, \ldots$, is a representative of bB ([1], Corollary 4.6). Since $\langle b\vee m^i\rangle, i = 1, 2, \ldots$, is the completely regular representative of bL^* ([1], Remark 5.2), we have that $\langle (b\vee m^i)B_i\rangle, i = 1, 2, \ldots$, is a representatives of $bL^*\cdot B$ ([1], Proposition 5.14). Thus $\langle (b\vee m^i)B_i\vee m^i\mathfrak{M}\rangle, i = 1, 2, \ldots$, is a representative of $bL^*\cdot B$. Since $(b\vee m^i)B_i\vee m^i\mathfrak{M} = bB_i\vee m^i\mathfrak{M}$, for each nonnegative integer i, the result follows, q.e.d.

§ 4. Completions of intervals. Throughout the remainder of this paper, $(L, p_1, p_2, \ldots, p_r)$ is a semi-local Noether lattice, M is a Noetherian L-module, $m = p_1\wedge\ldots\wedge p_r$, M^* is the m-adic completion of M, and L^* is the m-adic completion of L.

In this section we shall establish the form of completions of intervals of M. This result is needed later in the paper.

THEOREM 4.1. *Let A and B be elements of M such that $A \leqslant B$. Then, the set $[A, B]M^*$ is dense in the L-module $[AM^*, BM^*]$ with the m-adic metric.*

Proof. Let C be an arbitrary element of $[AM^*, BM^*]$. Considering C as an element of M^*, let $\langle C_i\rangle, i = 1, 2, \ldots$, of elements of M be the completely regular representative of C determined by the m-adic metric on M. Since $\langle C_i\rangle$ is completely regular, it is decreasing ([1], Remark 4.8). Thus $\langle C_i\wedge B\rangle$ is decreasing, and hence is Cauchy (Lemma 2.3). Since $\langle C_i\rangle$ and $\langle B\vee m^i\mathfrak{M}\rangle$ are the completely regular representatives of C and BM^*, respectively, the sequence $\langle C_i\wedge(B\vee m^i\mathfrak{M})\rangle$ is a representative of $C\wedge BM^*$ $(=C)$ by Proposition 2.4. Since $C_i\wedge(B\vee m^i\mathfrak{M}) = (C_i\wedge B)\vee m^i\mathfrak{M}$, for all integers $i\geqslant 1$, by modularity, and since $\langle C_i\wedge B\rangle\sim\langle(C_i\wedge B)\vee m^i\mathfrak{M}\rangle$, we have that the Cauchy sequence $\langle C_i\wedge B\rangle$ is a representative of C. Thus $(C_i\wedge B)M^*\to C$ as $i\to+\infty$ with the d^*_m metric and thus with the m-adic metric (Proposition 3.5). Since $\langle C_i\rangle$, $\langle A\vee m^i\mathfrak{M}\rangle$, and $\langle B\vee m^i\mathfrak{M}\rangle$ are the completely regular representatives of C, A, and B, respectively, and since $AM^*\leqslant C\leqslant BM^*$, it follows that $A\leqslant A\vee m^i\mathfrak{M}\leqslant C_i\leqslant B\vee m^i\mathfrak{M}$, for all $i\geqslant 1$ ([1], Proposition 5.9). Consequently, $C_i\wedge B$ is in $[A, B]$, for all $i\geqslant 1$.

Consider the sequence $\langle (m^*)^i(\mathfrak{M}M^*)\wedge BM^*\rangle$, $i = 1, 2, \ldots$ Since

$$(m^*)^i(\mathfrak{M}M^*) \geqslant (m^*)^i(\mathfrak{M}M^*)\wedge BM^* \geqslant (m^*)^{i+1}(\mathfrak{M}M^*)\wedge BM^* \geqslant m^*\big((m^*)^i(\mathfrak{M}M^*)\wedge BM^*\big) ,$$

for each $i\geqslant 1$, this sequence satisfies the conditions of Theorem 2.2. (See Theorem 2.8). Thus, (Proposition 3.7) there is a natural number n such that

$$(4.1)\quad m^{n+i}(\mathfrak{M}M^*)\wedge BM^* = m^i\big(m^n(\mathfrak{M}M^*)\wedge BM^*\big) ,\quad \text{for all integers } i\geqslant 0 .$$

Let ε be a positive real number and choose k to be the least natural number q such that $2^{-q} < \varepsilon$. We showed above that $(C_i\wedge B)M^*\to C$ as $i\to+\infty$ with the m-adic metric, so there exists a natural number N such that

$$(C_i\wedge B)M^*\vee m^{n+k}(\mathfrak{M}M^*) = C\vee m^{n+k}(\mathfrak{M}M^*) ,$$

for all integers $i\geqslant N$. Thus,

$$(4.2)\quad BM^*\wedge[(C_i\wedge B)M^*\vee m^{n+k}(\mathfrak{M}M^*)] = BM^*\wedge[C\vee m^{n+k}(\mathfrak{M}M^*)] ,$$

for all integers $i\geqslant N$. By modularity in M^* (Proposition 2.5) and (4.1) we have

$$BM^*\wedge[(C_i\wedge B)M^*\vee m^{n+k}(\mathfrak{M}M^*)] = (C_i\wedge B)M^*\vee m^k[m^n(\mathfrak{M}M^*)\wedge BM^*]$$

and

$$BM^*\wedge[C\vee m^{n+k}(\mathfrak{M}M^*)] = C\vee m^k[m^n(\mathfrak{M}M^*)\wedge BM^*] .$$

It follows that

$$(4.3)\quad (C_i\wedge B)M^*\vee m^k[m^n(\mathfrak{M}M^*)\wedge BM^*] = C\vee m^k[m^n(\mathfrak{M}M^*)\wedge BM^*] ,$$

for all integers $i\geqslant N$, by (4.2). Now, let i be an integer such that $i\geqslant N$. Then

$$\begin{aligned}(C_i\wedge B)M^*\vee m^k\circ(BM^*) &= (C_i\wedge B)M^*\vee m^k[m^n(\mathfrak{M}M^*)\wedge BM^*]\vee m^k(BM^*)\\ &= C\vee m^k[m^n(\mathfrak{M}M^*)\wedge BM^*]\vee m^k(BM^*)\\ &= C\vee m^k(BM^*)\end{aligned}$$

by (4.3). Thus, we obtain

$$(C_i \wedge B)M^* \vee m^k \circ (BM^*) = C \vee m^k \circ (BM^*) .$$

for all integers $i \geqslant N$. Hence, for $i \geqslant N$, the m-adic distance from $C_i \wedge B$ to C is less than or equal to 2^{-k} ([1], Remark 3.6), q.e.d.

THEOREM 4.2. *Let A and B be elements of M such that $A \leqslant B$. Then the L-module $[AM^*, BM^*]$ is complete with respect to the m-adic metric on $[AM^*, BM^*]$.*

Proof. We will make use of Theorem 3.1 to prove this result. Let $\langle C_i \rangle$, $i = 1, 2, \ldots$, be an arbitrary decreasing sequence in the L-module $[AM^*, BM^*]$, and let j be a positive integer. We wish to show that

$$C_i \leqslant (\bigwedge_q C_q) \vee (m^*)^j \circ (BM^*) \tag{4.4}$$

for all sufficiently large integers i.

For this, consider the sequence $\langle (m^*)^i(\mathfrak{M}M^*) \wedge BM^* \rangle$, $i = 1, 2, \ldots$ Since

$$(m^*)^i(\mathfrak{M}M^*) \geqslant (m^*)^i(\mathfrak{M}M^*) \wedge BM^* \geqslant (m^*)^{i+1}(\mathfrak{M}M^*) \wedge BM^*$$
$$\geqslant m^*\big((m^*)^i(\mathfrak{M}M^*) \wedge BM^*\big) ,$$

for each positive integer i, the sequence $\langle (m^*)^i(\mathfrak{M}M^*) \wedge BM^* \rangle$, $i = 1, 2, \ldots$, satisfies the conditions of Theorem 2.2 (recall that M^* is a Noetherian L^*-module by Theorem 2.8). Consequently, there exists a natural number n such that

$$(m^*)^{k+i}(\mathfrak{M}M^*) \wedge BM^* = (m^*)^i\big((m^*)^k(\mathfrak{M}M^*) \wedge BM^*\big) , \tag{4.5}$$

for all integers $k \geqslant n$, and for all integers $i \geqslant 0$. Since the sequence $\langle C_i \rangle$ is decreasing, and since M^* is a complete L-module with respect to the m-adic metric on M^*, by Theorem 3.1 there exists a natural number N such that

$$C_i \leqslant (\bigwedge_q C_q) \vee (m^*)^{j+n}(\mathfrak{M}M^*) , \tag{4.6}$$

for all integers $i \geqslant N$. It follows that

$$\begin{aligned} C_i = C_i \wedge BM^* &\leqslant BM^* \wedge \big((\bigwedge_q C_q) \vee (m^*)^{j+n}(\mathfrak{M}M^*)\big) \\ &= (\bigwedge_q C_q) \vee \big((m^*)^{j+n}(\mathfrak{M}M^*) \wedge BM^*\big) \\ &= (\bigwedge_q C_q) \vee (m^*)^j\big((m^*)^n(\mathfrak{M}M^*) \wedge BM^*\big) , \end{aligned} \tag{4.7}$$

for all integers $i \geqslant N$, by (4.5) and (4.6). Now, let i be an integer such that $i \geqslant N$. Then,

$$\begin{aligned} C_i = C_i \vee AM^* &\leqslant (m^*)^j(BM^*) \vee C_i \vee AM^* \\ &\leqslant (\bigwedge_q C_q) \vee (m^*)^j\big((m^*)^n(\mathfrak{M}M^*) \wedge BM^*\big) \vee (m^*)^j(BM^*) \vee AM^* \\ &= (\bigwedge_q C_q) \vee (m^*)^j(BM^*) \vee AM^* , \end{aligned}$$

by (4.7). Thus,

$$C_i \leqslant (\bigwedge_q C_q) \vee (m^*)^j \circ (BM^*) ,$$

for all integers $i \geqslant N$, which establishes (4.4), q.e.d.

THEOREM 4.3. *Let A and B be elements of M such that $A \leqslant B$. Then the L-module $[AM^*, BM^*]$ with the m-adic metric is the m-adic completion of the Noetherian L-module $[A, B]$.*

Proof. This follows immediately from Proposition 3.3 and Theorems 4.1, 4.2, by the uniqueness of the completion (up to an isomorphism), q.e.d.

§ 5. The extension isomorphism. In this section we establish some results about residuation and show that the extension map is a lattice isomorphism.

THEOREM 5.1. *Let A be an element of M and let B a principal element of M. Then $(A:B)L^* = AM^*:BM^*$.*

Proof. Since $(A:B)B \leqslant A$, we obtain $(A:B)L^* \cdot BM^* = [(A:B)B]M^* \leqslant AM^*$. Thus $(A:B)L^* \leqslant AM^*:BM^*$. Therefore, we need only show that $(A:B)L^* \geqslant AM^*:BM^*$.

Let n be a nonnegative integer. Since LL^* is dense in L^*, there exists an element x of L such that $xL^* \vee (m^*)^n = (AM^*:BM^*) \vee (m^*)^n$. Since $(AM^*:BM^*)(BM^*) \leqslant AM^*$, we have

$$\begin{aligned} (xB)M^* &\leqslant [(xL^*) \vee (m^*)^n](BM^*) = (AM^*:BM^*)(BM^*) \vee (m^*)^n(BM^*) \\ &\leqslant AM^* \vee (m^*)^n(BM^*) = (A \vee m^n B)M^* . \end{aligned}$$

Thus, $xB = (xB)M^* \cap M \leqslant (A \vee m^n B)M^* \cap M = A \vee m^n B$. Consequently, $x \leqslant (A \vee m^n B):B = (A:B) \vee m^n$, since B is a principal element of M. Hence, $xL^* \leqslant [(A:B) \vee m^n]L^* = (A:B)L^* \vee (m^*)^n$. It follows that

$$(AM^*:BM^*) \vee (m^*)^n = xL^* \vee (m^*)^n \leqslant (A:B)L^* \vee (m^*)^n .$$

Since n was arbitrary, we have

$$(AM^*:BM^*) \vee (m^*)^n \leqslant (A:B)L^* \vee (m^*)^n ,$$

for all nonnegative integers n. Now, since L^* is a semi-local Noether lattice (Theorem 2.9), we have

$$AM^*:BM^* = \bigwedge_n \big((AM^*:BM^*)\vee(m^*)^n\big)$$
$$\leqslant \bigwedge_n \big((A:B)L^*\vee(m^*)^n\big) = (A:B)L^*,$$

by ([2], Corollary 3.4), q.e.d.

COROLLARY 5.2. *Let A be an element of M and let B be a principal element of M. Then $(A\wedge B)M^* = AM^*\wedge BM^*$.*

Proof. Since principal elements extend to principal elements (Theorem 2.7), BM^* is a principal element of M^*. It follows that

$$(A\wedge B)M^* = \big((A:B)B\big)M^* = \big((A:B)L^*\big)(BM^*)$$
$$= (AM^*:BM^*)(BM^*) = AM^*\wedge BM^*$$

by the theorem and the definition of a principal element, q.e.d.

The following theorem shows that M is lattice isomorphic to MM^* considered as a sublattice of M^*.

THEOREM 5.3. *The extension map $A\to AM^*$ of $M\to MM^*$ is a lattice isomorphism.*

Proof. Let A and B be elements of M. Recall that $(A\vee B)M^* = AM^*\vee BM^*$ by definition ([1], Definition 5.4), and that the extension map is one-to-one ([1], Proposition 5.3). Hence we need only show that $(A\wedge B)M^* = AM^*\wedge BM^*$.

Since M is a Noetherian L-module, there are principal elements $P_1, \ldots, P_n$ in M such that $B = P_1\vee\ldots\vee P_n\vee(A\wedge B)$. The proof is by induction on n. Assume $B = P_1\vee(A\wedge B)$. Since P_1 is principal in M, $P_1\vee(A\wedge B)$ is a principal element of the Noetherian L-module $[A\wedge B, \mathfrak{M}]$. Hence,

$$(A\wedge B)M^* = \big(A\wedge(P_1\vee(A\wedge B))\big)M^* \leftrightarrow \big(A\wedge(P_1\vee(A\wedge B))\big)[A\wedge B, \mathfrak{M}]^*$$
$$= A[A\wedge B, \mathfrak{M}]^*\wedge\big(P_1\vee(A\wedge B)\big)[A\wedge B, \mathfrak{M}]^*$$
$$= A[A\wedge B, \mathfrak{M}]^*\wedge B[A\wedge B, \mathfrak{M}]^* \leftrightarrow AM^*\wedge BM^*$$

by Corollary 5.2 and Theorem 4.3. Thus the case when $n=1$ holds. Assume the result holds for case $n=k$, and suppose $B = P_1\vee\ldots\vee P_{k+1}\vee\vee(A\wedge B)$. Set $P = A\vee P_{k+1}$. Then $A\wedge B = A\wedge(B\wedge P)$. It follows that $P\wedge B = P_{k+1}\vee\big(A\wedge(P\wedge B)\big)$. Thus, applying the case $n=1$ to A and $P\wedge B$, we obtain $(A\wedge B)M^* = \big(A\wedge(P\wedge B)\big)M^* = AM^*\wedge(P\wedge B)M^*$. Also, since $B = P_1\vee\ldots\vee P_k\vee(P\wedge B)$, we obtain $(P\wedge B)M^* = PM^*\wedge BM^*$ by the induction hypothesis. By combining these results we have $(A\wedge B)M^* = AM^*\wedge(P\wedge B)M^* = AM^*\wedge(PM^*\wedge BM^*) = AM^*\wedge BM^*$. This completes the induction, q.e.d.

COROLLARY 5.4. *Let A and B be elements of M. Then $(A:B)L^* = AM^*:BM^*$.*

Proof. Since M is Noetherian, there are principal elements $P_1, \ldots, P_n$ in M such that $B = P_1\vee\ldots\vee P_n$. It follows that

$$(A:B)L^* = \big(A:(P_1\vee\ldots\vee P_n)\big)L^* = \big((A:P_1)\wedge\ldots\wedge(A:P_n)\big)L^*$$
$$= (AM^*:P_1M^*)\wedge\ldots\wedge(AM^*:P_nM^*)$$
$$= AM^*:(P_1M^*\vee\ldots\vee P_nM^*) = AM^*:BM^*$$

by Theorem 5.1 and 5.3, q.e.d.

References

[1] J. A. Johnson, *a-adic completions of Noetherian lattice modules*, Fund. Math. 66 (1970), pp. 347–373.

[2] E. W. Johnson and J. A. Johnson, *Lattice modules over semi-local Noether lattices*, Fund. Math. 68 (1970), pp. 187–201.

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