Galois spaces, representable spaces and strongly locally homogeneous spaces

by

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1. Introduction. Recently it has been shown that if G and K are two groups, then there exists a complete metric space X such that the full homeomorphism group of X is isomorphic to G and such that X has a complete subspace Y whose full homeomorphism group is isomorphic to K ([5], Theorem 3). This result points out rather dramatically that a topological space is not characterized by the algebraic nature of its full group of homeomorphisms. Nevertheless, within a given class $\mathcal{C}$ of topological spaces, it may be true two members of $\mathcal{C}$ are homeomorphic if and only if their corresponding full homeomorphism groups are isomorphic. Indeed, J. V. Whittaker has shown that this is true of the class $\mathcal{C}$ of all compact Euclidean manifolds with or without boundary ([9], Theorem 4). In [3], P. Fletcher and R. L. Snider introduced the class of topological Galois spaces and conjectured that two Hausdorff Galois spaces would be homeomorphic if and only if their full homeomorphism groups were isomorphic. In their own words, the authors of [3] offered only "scant evidence" that their conjecture might hold; and, in any case, the conjecture remains unresolved.

In this paper we study topological Galois spaces and two subclasses, strongly locally homogeneous Galois spaces and representable Galois spaces. It is conceivable that the conjecture of P. Fletcher and R. L. Snider might prove false for Galois spaces and yet hold in the more restrictive classes of strongly locally homogeneous Galois spaces or representable Galois spaces. On the other hand examples of two (non-homeomorphic) strongly locally homogeneous Galois spaces with isomorphic full homeomorphism groups would quash their conjecture once and for all.

We believe that the results and examples of this paper might aid in the search for counterexamples to the conjecture given in [3] or to the corresponding conjectures for representable Galois spaces and strongly locally homogeneous Galois spaces — if, indeed, such counterexamples do exist.

We show that every linear topological space is a Galois space and that every locally convex linear topological space is strongly locally homogeneous. We also show that every representable continuum is decomposable.

It is known that every strongly locally homogeneous space is representable and that every representable space without isolated points is a Galois space ([2], Theorem 4). Furthermore, the unit circle and the unit circle together with its center, with their usual subspace topologies, are two non-homeomorphic strongly locally homogeneous spaces which clearly have isomorphic full homeomorphism groups. For these reasons, we assume throughout this paper that all spaces under consideration are Hausdorff spaces without isolated points.

2. Preliminaries. Let $(X, \mathcal{T})$ be a topological space. We let $H(X)$ denote the group of all homeomorphisms from the space $(X, \mathcal{T})$ onto itself and let i denote the identity of $H(X)$. If $A \subset X$, then $A' = \{h \in H(X): h|A = i|A\}$ and if G is a subgroup of $H(X)$, then $G' = \{x \in X: g(x) = x$ for each $g \in G\}$. If $G \subset H(X)$ and $A \subset X$, then $G(A) = \{g(a): g \in G, a \in A\}$. We often write $\{x'\}$ for $\{x\}'$ and $G(x)$ for $G(\{x\})$.

DEFINITION ([4]). A topological space $(X, \mathcal{T})$ is *strongly locally homogeneous* if for every neighborhood of any point x, there exists a subneighborhood U_x such that for any $z \in U_x$ there exists a homeomorphism $g \in (X-U_x)'$ with $g(x) = z$.

DEFINITION ([2]). A topological space $(X, \mathcal{T})$ is *representable* provided that if F is a closed set and $x \in X-F$, then $F'(x)$ contains an open set about x.

DEFINITION ([3]). A topological space $(X, \mathcal{T})$ is a *Galois space* provided that for each closed set F, $F = F''$.

Every strongly locally homogeneous space is a representable space and every representable space without isolated points is a Galois space ([2], Theorem 4).

It is convenient to give alternate characterizations of Galois spaces and representability.

PROPOSITION 2.1 ([2]), Proposition 1). *A topological spaces $(X, \mathcal{T})$ is a Galois space if and only if for each closed set F and each $x \in X-F$, there exists $h \in F'$ such that $h(x) \neq x$.*

LEMMA 2.2 ([2], Corollary to Theorem 1). *A topological space $(X, \mathcal{T})$ is representable if and only if for each closed set F and each $x \in X-F$, $F'(x) \in \mathcal{T}$.*

PROPOSITION 2.3. *A topological space $(X, \mathcal{T})$ is representable if and only if given $p \in U \in \mathcal{T}$, there exists $V \in \mathcal{T}$ with $p \in V \subset U$ such that if $y \in V$, then there exists $h \in (X-U)'$ with $h(p) = y$.*

The closed disk is an example of a Galois space which is not representable. Although it would appear that representability is a weaker property than strong local homogeneity, we do not know of a representable space which is not strongly locally homogeneous. Indeed, the following theorem shows that for a large class of spaces these two concepts are equivalent.

THEOREM 2.4. *Every locally connected representable space is strongly locally homogeneous.*

Proof. Let $(X, \mathcal{T})$ be a locally connected representable space and let $x \in U \in \mathcal{T}$. There is a connected open set V such that $x \in V \subset U$. Let $y \in V$. For each $v \in V$, there exists $W_v \in \mathcal{T}$ with $v \in W_v \subset V$ such that if $w \in W_v$, there exists $h \in (X-V)'$ with $h(v) = w$. There is a simple chain $\{W_{v_i}: i = 1, \ldots, n\}$ from x to y. We may assume that $x = v_1$ and $y = v_n$. For $1 \leqslant i \leqslant n-1$, choose $x_i \in W_{v_i} \cap W_{v_{i+1}}$. For each i with $1 \leqslant i \leqslant n-1$, let $f_i \in (X-V)'$ such that $f_i(v_i) = x_i$ and let $g_i \in (X-V)'$ such that $g_i(x_i) = v_{i+1}$. Let $h = g_{n-1} \circ f_{n-1} \circ g_{n-2} \circ f_{n-2} \circ \ldots \circ g_2 \circ f_2 \circ g_1 \circ f_1$. Then $h \in (X-V)'$ and $h(x) = y$. Therefore $(X, \mathcal{T})$ is strongly locally homogeneous.

3. Linear topological spaces.

THEOREM 3.1. *Every topological vector space is a Galois space.*

Proof. Let E be a topological vector space. Let C be a nonempty closed subset of E and let $x \in E-C$. Let f be the translation on E given by $f(z) = z-x$. Since E has a local basis at 0 consisting of star-like sets ([6], Lemma 1.0), there is a star-like open set S such that $S+S \subset f(E-C)$. Without loss of generality, we may suppose that S is symmetric. Since $C \neq 0$, there is $y \in S$, $y \neq 0$, such that the ray from 0 through y, $\mathrm{Ray}[0:y]$, intersects $f(C)$. Let $\{b\} = \mathrm{Ray}[0:y] \cap \mathrm{Bd}\, S$. Then there exists a with $0 < a < 1$ such that $y = ab$. Define the function g mapping S onto S as follows. Let $g(0) = 0$. Let $z \in S-\{0\}$. If $\mathrm{Ray}\,[0:z] \subset S$, then define $g(z) = z$. If $\mathrm{Ray}\,[0:z] \not\subset S$, then let $\{z_0\} = \mathrm{Ray}[0:z] \cap \mathrm{Bd}\, S$, $\{z_1\} = \mathrm{Ray}[0:z] \cap \mathrm{Bd}\big((a+1)/2\big) S$, $\{z_2\} = \mathrm{Ray}[0:z] \cap \mathrm{Bd}\, aS$, $\{z_3\} = \mathrm{Ray}[0:z] \cap \mathrm{Bd}\, (a/2) S$ and let $\{z_4\} = \mathrm{Ray}[0:z] \cap \mathrm{Bd}(a/4) S$. If $z \in [z_1:z_0) \cup (0:z_4]$, then define $g(z) = z$. If $z \in [z_2:z_1]$ so that there is a real number t with $0 \leqslant t \leqslant 1$ such that $z = tz_2+(1-t)z_1$, then define $g(z) = tz_3+(t-t)z_1$. If $z \in [z_4:z_2]$, so that there is a real number s with $0 \leqslant s \leqslant 1$ such that $z = sz_4+(1-s)z_2$, then define $g(z) = sz_4+(1-s)z_3$. Extend g to E by letting $g(z) = z$ for each $z \in E-S$. Let k be the translation on E given by $k(z) = z+y$ and define $h = f^{-1}k^{-1}gkf$. Then $h \in H(E)$ such that $h|C = i|C$ and such that $h(x) \neq x$.

It is known that every normed linear space is strongly locally homogeneous ([4], Theorem 4.3). A slightly stronger result also holds.

THEOREM 3.2. *Every locally convex topological vector space is strongly locally homogeneous.*

Proof. Let E be a locally convex topological vector space. Let $x \in E$ and let U be an open set containing x. Since E is locally convex, there exists an open convex set V such that $x \in V \subset U$. Let $y \in V$. Define the function h mapping V onto V as follows. Define $h(x) = y$. Let $z \in V-\{x\}$. If $\mathrm{Ray}[x:z] \subset V$, define $h(z) = z+y-x$. If $\mathrm{Ray}[x:z] \not\subset V$, then let $\{z_0\} = \mathrm{Ray}[x:z] \cap \mathrm{Bd}V$. Then there is a real number a with $0 < a < 1$ such that $z = x+a(z_0-x)$. Define $h(z) = y+a(z_0-y)$. Extend h to E by letting $h(z) = z$ for each $z \in E-V$. Then $h \in H(E)$ such that $h|E-V = i|E-V$ and such that $h(x) = y$.

COROLLARY. 3.3. *Every manifold which is modeled on a topological vector space is a Galois space. Every manifold which is modeled on a locally convex topological vector space is strongly locally homogeneous.*

4. Representable continua. Every connected representable space is homogeneous ([2], Theorem 2), so that in particular every representable continuum is a homogeneous continuum. It follows from Theorem 3.2 and it corollary that for each integer $n \geqslant 1$, $\mathcal{E}_n$ and S_n are representable continua. It also follows from this theorem that Hilbert space is a representable continuum, and we now show that the Hilbert cube is a representable space as well.

For each positive integer i, let $I_i = [0,1]$ and let $\mathring{I}_i = (0,1)$. Define the Hilbert cube, I^ω, to be $\prod_{i=1}^{\infty} I_i$ and define $\mathring{I}^\omega = \prod_{i=1}^{\infty} \mathring{I}_i$.

LEMMA 4.1. *Let $x = (x_i) \in \mathring{I}^\omega$ and let U be an open subset of I^ω containing x. Then there exists an open subset V of I^ω such that $x \in V \subset U$ and such that if $y \in V$, then there exists $h \in (I^\omega - V)'$ with $h(x) = y$.*

Proof. There exists a positive integer n and a basic open set $V = \bigcap_{i=1}^{n} \pi_i^{-1}(V_i)$ such that $x \in V \subset U$ where for each i with $1 \leqslant i \leqslant n$ there exist a_i and b_i with $0 < a_i < b_i < 1$ such that $V_i = \{t \in I_i:\ a_i < t < b_i\}$. Let $y = (y_i) \in V$. For each i with $1 \leqslant i \leqslant n$, let $c_i,\ d_i \in I_i$ such that $0 < a_i < c_i < d_i < b_i < 1$ and such that $c_i < x_i < d_i$ and $c_i < y_i < d_i$. For each i with $1 \leqslant i \leqslant n$, let $W_i = \{t \in I_i:\ c_i < t < d_i\}$. Let $g \in H\big(\prod_{i=1}^{n} I_i\big)$ such that $g\big((x_1, \ldots, x_n)\big) = (y_1, \ldots, y_n)$ and such that $g \in \big(\prod_{i=1}^{n} I_i - \prod_{i=1}^{n} W_i\big)'$. Since I^ω is homogeneous [7], and since I^ω is homeomorphic to $\prod_{i=n+1}^{\infty} I_i$, there exists $f \in H\big(\prod_{i=n+1}^{\infty} I_i\big)$ such that $f\big((x_{n+1}, x_{n+2}, \ldots)\big) = (y_{n+1}, y_{n+2}, \ldots)$. Since every homeomorphism on I^ω is isotopic to the identity ([1], Corollary 10.5), there is an isotopy $F_t\colon \prod_{i=n+1}^{\infty} I_i \to \prod_{i=n+1}^{\infty} I_i$, $0 \leqslant t \leqslant 1$ such that $F_1 = f$ and F_0 is the identity on $\prod_{i=n+1}^{\infty} I_i$. Define $h \in H(I^\omega)$ as follows. Let $z = (z_i) \in I^\omega$. For each i with $1 \leqslant i \leqslant n$ define t_i by

$$t_i = \begin{cases} 0, & \text{if}\quad z_i \leqslant a_i \text{ or } z_i \geqslant b_i. \\ 1, & \text{if}\quad c_i \leqslant z_i \leqslant d_i. \\ a, & \text{if}\quad \text{there is } a \text{ with } 0 \leqslant a \leqslant 1 \text{ such that } z_i = ac_i+(1-a)a_i. \\ \beta, & \text{if}\quad \text{there is } \beta \text{ with } 0 \leqslant \beta \leqslant 1 \text{ such that } z_i = \beta d_i+(1-\beta)b_i. \end{cases}$$

Let $t = \min\{t_i\ 1 \leqslant i \leqslant n\}$. Let $(w_1, \ldots, w_n) = g(z_1, \ldots, z_n)$ and let $(w_{n+1}, w_{n+2}, \ldots) = F_t(z_{n+1}, z_{n+2}, \ldots)$. Define $h(z) = (w_1, w_2, \ldots, w_n, w_{n+1}, \ldots)$. The function h thus defined is an element of $(I^\omega - V)'$ such that $h(x) = y$.

THEOREM 4.2. *The Hilbert cube is a representable continuum.*

Proof. By the previous lemma, it suffices to show that if $x \in I^\omega - \mathring{I}^\omega$ and U is an open set containing x, then there is an open set V with $x \in V \subset U$ such that if $y \in V$ there is $h \in (X-V)'$ with $h(x) = y$. Let $x \in I^\omega - \mathring{I}^\omega$ and let U be an open set containing x. Let $p \in \mathring{I}^\omega$. Then there exists $f \in H(I^\omega)$ such that $f(x) = p$. By Lemma 4.1, there is an open set W such that $p \in W \subset f(U)$ and such that if $q \in W$ then there exists $g \in (I^\omega - W)'$ with $g(p) = q$. Let $V = f^{-1}(W)$ and let $y \in V$. Then $fgf^{-1} \in (I^\omega - V)'$ such that $fgf^{-1}(x) = y$.

Despite the fact that the universal curve is strongly locally homogeneous and hence representable ([8], p. 602), there is some justification for the point of view that representable continua are "well behaved"

LEMMA 4.3. *Let $(X, \mathcal{G})$ be a representable continuum, let F be a closed subset of X and let $x \in X-F$. Then $\overline{F'(x)}$ is a subcontinuum of X with interior points.*

Proof. Let C be the component of $F'(x)$ which contains x. Then $\{h(C):\ h \in F'\}$ is the collection of all components of $F'(x)$. Since $(X, \mathcal{G})$ is a Hausdorff continuum, there exists $p \in \overline{F'(x)} - F'(x)$ such that p is a limit point of C. Suppose that $p \notin F$. Then by Lemma 2.2, $F'(p)$ is an open set about p and so there exists $h \in F'$ such that $h(x) \in C \cap F'(p)$. Then $p \in F'(x)$ — a contradiction. Thus $p \in F$. It follows that p is a limit point of each component of $F'(x)$. Let $B = \bigcup \{\overline{h(C)}:\ h \in F'\}$. Then B is connected and $F'(x) \subset B \subset \overline{F'(x)}$. Consequently $\overline{F'(x)} = \overline{B}$ which is connected. By Lemma 2.2, $F'(x)$ is open, and it is clear that $\overline{F'(x)}$ is a continuum.

THEOREM 4.4. *Every representable continuum is decomposable.*

EXAMPLE. The pseudo-arc is a homogeneous plane continuum which is not representable.

EXAMPLE. The solenoid group is a metric continuum which is a topological group but which is not representable.

L. R. Ford, Jr. has given an example of a compact metric homogeneous space which is not representable ([4], pp. 494–495). We do not know, however, of a homogeneous Galois continuum which is not representable. It would therefore be interesting to know if either of the previous two examples is a Galois space [1].

5. Quasi-reasonable topologies.

DEFINITION ([4] and [8]). Let X be a topological space. A topology $\mathcal{C}$ for $H(X)$ is *quasi-reasonable* provided that there is a point $x \in X$ such that, with the quotient topology induced by $\mathcal{C}$, $H(X)/x'$ is homeomorphic to X under the map $\eta\colon H(X)/x' \to X$ defined by $\eta(fx') = f(x)$. A quasi-reasonable topology $\mathcal{C}$ for $H(X)$ is *reasonable* provided that $(H(X), \circ, \mathcal{C})$ is a topological group. A topological space $(X, \mathcal{C})$ is *(quasi-) reasonable* provided that $H(X)$ possesses a (quasi)-reasonable topology.

LEMMA 5.1. *Let X be a completely regular, homogeneous, representable space, let $\mathcal{U}$ be a compatible uniformity for X and let $x \in X$. Let $H(X)$ have the topology of uniform convergence generated by $\mathcal{U}$. Then the function $\Phi_x\colon H(X) \to X$ defined by $\Phi_x(g) = g(x)$ is open at the identity.*

Proof. The proof of this theorem follows with minor modifications from the proof of the corresponding theorem for strongly locally homogeneous spaces ([4], Theorem 4.1).

THEOREM 5.2. *Every homogeneous completely regular representable space is a reasonable space.*

Proof. The proof follows from Lemma 5.1 and the proof of Theorem 3.2 of [8].

COROLLARY 5.3. *Every representable continuum is a reasonable space.*

THEOREM 5.4. *Let X and Y be homogeneous completely regular representable spaces; let $x \in X$ and $y \in Y$. Suppose that $H(X)$ and $H(Y)$ have quasi-reasonable topologies and that $\Psi\colon H(X) \to H(Y)$ is a topological isomorphism from $H(X)$ onto $H(Y)$ such that $\Psi(x') = y'$. Then X is homeomorphic to Y.*

Proof. Let Ψ_1 be a homeomorphism from $H(X)/x'$ onto X and let Ψ_2 be a homeomorphism from $H(Y)/y'$ onto Y. Let $\varrho_1\colon H(X) \to H(X)/x'$ and $\varrho_2\colon H(Y) \to H(Y)/y'$ be the natural projection maps. Let $f_1 = \varrho_2 \circ \Psi$. Let $R_1 = \{(h_1, h_2) \in H(X) \times H(X)\colon f_1(h_1) = f_1(h_2)\}$ and let $\varphi_1\colon H(X) \to H(X)/R_1$ be the natural projection map. For each $\alpha \in H(X)/R_1$, there is only one $\beta \in H(Y)/y'$ such that $\alpha = f_1^{-1}(\beta)$. For each $\alpha \in H(X)/R_1$ define $g_1(\alpha) = \beta$. Then $g_1\colon H(X)/R_1 \to H(Y)/y'$ is a one-to-one mapping onto $H(Y)/y'$. Since $g_1 \circ \varphi_1 = f_1$ which is continuous, g_1 is continuous.

[1] It is now known that the solenoid group is a Galois space (Notices American Mathematical Society **18** 71T-G46).

We now show that $H(X)/R_1 = H(X)/x'$. For $h \in H(X)$ and $k \in H(Y)$, $\Psi(hx') = \Psi(h)y'$ and $\Psi^{-1}(ky') = \Psi^{-1}(k)x'$. Let $\alpha \in H(X)/R_1$. Then there exists $k \in H(Y)$ such that $\alpha = f_1^{-1}(ky')$. It follows that $\alpha = f_1^{-1}(ky') = \Psi^{-1}(ky') = \Psi^{-1}(k)x' \in H(X)/x'$. Thus $H(X)/R_1 \subset H(X)/x'$. Similarly $H(X)/x' \subset H(X)/R_1$. Thus g_1 is a one-to-one continuous function from $H(X)/x'$ onto $H(Y)/y'$. It can be shown in a straight forward manner that g_1^{-1} is also continuous so that $\Psi_2 \circ g_1 \circ \Psi_1^{-1}$ is a homeomorphism from X onto Y.

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