then

$$f \in \bigcap_{t\in\omega} B_{\varphi_t}^{k_t}.$$

So $\mathcal{A}$ is not point-finite, and the proof of the Lemma is complete.

As consequences of this Lemma we have the following two theorems:

THEOREM 1. *The space S is not totally metacompact.*

In fact, $\mathcal{B}' = \mathcal{B} \cup \{\{x\}|\ x \in S\backslash D\}$ is an open base of S and $\mathcal{B}'$ contains no point-finite covering of S, because, by the Lemma, $\mathcal{B}$ contains no point-finite covering of D.

THEOREM 2. *The space D (which is homeomorphic to the space of the rationals) is not absolutely paracompact.*

This is clear, because D is a closed subspace of the paracompact space S and $\mathcal{B}$ is a "wrong" outer base of D in S.

Remarks. (1) S is a paracompact space with the property that S^d (the set of all limit points of S) is totally paracompact, but S itself is not totally paracompact.

(2) Every C-scattered paracompact space is absolutely paracompact [6]. Every paracompact locally compact space and every $F_\sigma \cap G_\delta$-absolute metrizable space is C-scattered paracompact [6]. Hence, there are many absolutely paracompact spaces.

(3) Since there is a homeomorphism h from ω^ω onto the space of all irrationals in the unit interval I (cf. [4], p. 143), $h(S)$ is a closed subspace of the space $I_{h(D)}$ (cf. [1], p. 216). Since $I_{h(D)}\backslash h(S)$ is a discrete open set, $I_{h(D)}$ is not totally metacompact.

E. Michael (cf. [3] or [1], p. 218) proved that $\omega^\omega \times I_{h(D)}$ is not normal. Here, as we have seen above, neither ω^ω (cf. [2]) nor $I_{h(D)}$ is totally metacompact.

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A metrization theorem for developable spaces*

by

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1. Preliminaries. The usual approach to the problem of metrizability of developable spaces has been through the use of the rich global properties enjoyed by the developable spaces, and the additional assumption of normality. In [2] the application of a significant local property, the first countability of a developable space, was made in giving another (non-normal) dimension to this problem. The additional global property used in this application was sequential mesocompactness. A family of subsets $\{F_a\colon a \in A\}$ of a space X is said to be cs-*finite* if for each convergent sequence, $\{P_i\}$ in X, $F_a \cap \{P_i\colon i \in N\} \neq \emptyset$ for at most finitely many $a \in A$. Accordingly, a Hausdorff space X is called *sequentially mesocompact* provided: every open covering of X has a cs-finite open refinement.

In this paper, the use of both the local and global properties of developable spaces is made to yield a metrization theorem, Theorem 2.1, for developable spaces which improves both of the following theorems.

THEOREM 1.1 ([1], Theorem 10). *A developable space is metrizable if and only if it is collectionwise normal.*

THEOREM 1.2 ([2], Theorem 4.2). *A developable space is metrizable if and only if it is sequentially mesocompact.*

A non-normal simultaneous generalization of sequentially mesocompact spaces and collectionwise normal spaces is introduced.

DEFINITION. A Hausdorff space X is said to have *property* (ω) if for each discrete collection of closed sets $\{F_a\colon a \in A\}$ in X, there exists a cs-finite collection of open sets $\{G_a\colon a \in A\}$ such that $F_a \subset G_a$, for each $a \in A$ and $G_a \cap F_\beta = \emptyset$, if $a \neq \beta$.

Let $\{F_a\colon a \in A\}$ be any discrete collection of closed sets in a space X. Suppose X is sequentially mesocompact, and consider the open covering $\mathfrak{U} = \{X - \bigcup_{\gamma\neq a} F_\gamma\colon a \in A\}$ of X. Let $\mathcal{G} = \{G_\beta\colon \beta \in B\}$ be a cs-finite open

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refinement of $\mathcal{U}$. For each $\alpha \in A$, let $H_\alpha = \{G_\beta\colon G_\beta \cap F_\alpha \neq \emptyset\}$. Then $\{H_\alpha\colon \alpha \in A\}$ is a cs-finite open collection such that $F_\alpha \subset H_\alpha$, for each $\alpha \in A$ and $H_\alpha \cap F_\beta = \emptyset$, if $\alpha \neq \beta$. Hence, every sequentially mesocompact space has property (ω). Suppose X is collectionwise normal. Dowker [3] has shown that there exists a discrete collection of open sets $\mathcal{V} = \{V_\alpha\colon \alpha \in A\}$ such that $F_\alpha \subset V_\alpha$ for each $\alpha \in A$ and $\mathrm{Cl}(V_\alpha) \cap F_\gamma = \emptyset$, if $\alpha \neq \gamma$. Since every discrete collection is cs-finite, $\mathcal{V}$ is cs-finite. Thus every collectionwise normal space has property (ω).

2. Main theorem.

THEOREM 2.1. *A developable space X is metrizable if and only if X is regular and has property* (ω).

Proof. The necessity is clear, because every metric space is sequentially mesocompact. To prove the sufficiency, we show that a regular developable space with property (ω) is paracompact. Let $\{G_i\colon i \in N\}$ be a development of X such that G_{i+1} refines G_i. Let $\mathcal{H} = \{H_\alpha\colon \alpha \in A\}$ be any open covering of X. (The following construction of the discrete collections is taken from Bing's proof of his Theorem 9 in [1].) Consider $\mathcal{H}$ to be well-ordered by $<$. For each $i \in N$ and for each $\alpha \in A$, let F_α^i be the set of all $p \in X$ such that (a) H_α is the first element of $\mathcal{H}$ such that $p \in H_\alpha$ and (b) $\mathrm{st}(p, G_i) \subset H_\alpha$. Let $\mathcal{F}_i = \{F_\alpha^i\colon \alpha \in A\}$ for each $i \in N$. Then for each $i \in N$, (1) $\mathcal{F}_i$ is a discrete collection of closed sets in X, and (2) $F_\alpha^i \subset H_\alpha$ and $F_\alpha^i \subset F_\alpha^{i+1}$ for $\alpha \in A$. Also, $\bigcup\{\mathcal{F}_i\colon i \in N\}$ covers X. Since X has property (ω), for each $i \in N$ there exists a cs-finite open collection $\mathcal{G}_i = \{G_\alpha^i\colon \alpha \in A\}$ such that $F_\alpha^i \subset G_\alpha^i$, for each $\alpha \in A$ and $G_\alpha^i \cap F_\gamma^i = \emptyset$, if $\alpha \neq \gamma$. Let $D_\alpha^i = G_\alpha^i \cap H_\alpha$ for each $i \in N$ and $\alpha \in A$. Then, for each $i \in N$, $\{D_\alpha^i\colon \alpha \in A\}$ is open refinement of $\mathcal{H}$. Since cs-finite families are hereditarily cs-finite (i.e., the collection consisting of one subset from each set is cs-finite), $\mathcal{D}_i = \{D_\alpha^i\colon \alpha \in A\}$ is a cs-finite open refinement of $\mathcal{H}$ which covers $\bigcup\{F_\alpha^i\colon \alpha \in A\}$. Since $\bigcup\{\mathcal{F}_i\colon i \in N\}$ covers X, $\bigcup\{\mathcal{D}_i\colon i \in N\}$ covers X. Since every developable space is first countable, $\mathcal{D}_i$ is locally finite for each $i \in N$, by [2], Lemma 3.9. Hence, $\bigcup\{\mathcal{D}_i\colon i \in N\}$ is a σ-locally finite open refinement of $\mathcal{H}$ which covers X. By Michael's characterization of paracompactness ([7], Theorem 1), X is paracompact. Accordingly, X is metrizable by Bing's Theorem ([1], Theorem 10).

3. Collectionwise normality and paracompactness of spaces with property (ω).

In [2] the paracompact property was characterized in the class of sequentially mesocompact spaces by means of various local properties (i.e., first countable, Fréchet, sequential spaces [4]). The natural question of whether or not collectionwise normality can be described, in a similar manner, for spaces with property (ω) is answered here. To do this, we will need other characterizations of collectionwise normal spaces.

A family $\{F_\alpha\colon \alpha \in A\}$ with a property Q is said to be hereditarily Q, if F'_α is any subset of F_α, for each $\alpha \in A$ then $\{F'_\alpha\colon \alpha \in A\}$ has property Q. Local finiteness, cs-finiteness and point-finiteness are some of the hereditary properties. The closure preserving property is not hereditary. The notion of hereditarily closure preserving families was first brought to my attention by Sconyers [10] in his study of $\mathfrak{M}$-paracompact spaces. I am indebted to the referee, who has informed me that Lašnev [5] introduced this notion earlier, calling these families "hereditarily conservative" in his paper characterizing the closed continuous images of metric spaces. We use this notion to obtain the following theorem.

THEOREM 3.1. *A space X is collectionwise normal if and only if X is normal and for each discrete collection of closed sets, $\mathcal{F} = \{F_\alpha\colon \alpha \in A\}$ there exists a collection of open sets, $\mathcal{G} = \{G_\beta\colon \beta \in B\}$ such that*

(i) *$\mathcal{G}$ is hereditarily closure preserving*

and

(ii) *if $B(\alpha) = \{\beta \in B\colon G_\beta \cap F_\alpha \neq \emptyset\}$ then $\{G_\beta\colon \beta \in B(\alpha)\}$ is a covering of F_α, for each $\alpha \in A$ and* if *$\beta \in B(\alpha)$ then $G_\beta \cap F_\gamma = \emptyset$ for each $\gamma \neq \alpha$.*

Proof. The necessity is clear. We will prove the sufficiency. For each $\alpha \in A$, $\mathrm{st}(F_\alpha, \mathcal{G}) \cap (\bigcup\{F_\gamma\colon \gamma \neq \alpha\}) = \emptyset$ and, by the normality, there exists an open nbhd T_α of F_α such that $F_\alpha \subset T_\alpha \subset \mathrm{Cl}(T_\alpha) \subset \mathrm{st}(F_\alpha, \mathcal{G})$. Then $\mathrm{Cl}(T_\alpha) \cap F_\gamma = \emptyset$ for each $\gamma \neq \alpha$, and $T_\alpha = \bigcup\{G_\beta \cap T_\alpha\colon \beta \in B(\alpha)\}$. Since $\mathcal{G}$ is hereditarily closure preserving, $\{T_\alpha\colon \alpha \in A\}$ is closure preserving. Hence, since $F_\alpha \cap \mathrm{Cl}(T_\gamma) = \emptyset$ for each $\gamma \neq \alpha$, $U_\alpha = T_\alpha \setminus \mathrm{Cl}(\bigcup\{T_\gamma\colon \gamma \neq \alpha\})$ is an open nbhd of F_α, and $\{U_\alpha\colon \alpha \in A\}$ is a collection of disjoint open sets. Thus X is collectionwise normal.

Since cs-finiteness is hereditary and in a Fréchet space, cs-finite families are closure preserving [2], we have, a Fréchet space is collectionwise normal if and only if it is a normal space with property (ω). This statement can be improved by a more effective application of the normality of the space.

COROLLARY 3.2. *A sequential space X is collectionwise normal if and only if X is a normal space with property* (ω).

Proof. Since the necessity is clear, we prove only the sufficiency. Let $\{F_\alpha\colon \alpha \in A\}$ be a discrete collection of closed sets in X. Since X has property (ω), there exists a cs-finite collection of open sets $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$ such that $F_\alpha \subset G_\alpha$, for each $\alpha \in A$, and $G_\alpha \cap F_\beta = \emptyset$, if $\alpha \neq \beta$. Since X is normal, for each $\alpha \in A$ there exists an open set T_α such that $F_\alpha \subset T_\alpha \subset \mathrm{Cl}(T_\alpha) \subset G_\alpha$. Thus, $\{\mathrm{Cl}(T_\alpha)\colon \alpha \in A\}$ is a cs-finite collection of closed sets such that $F_\alpha \subset T_\alpha$, for each $\alpha \in A$ and $T_\alpha \cap F_\beta = \emptyset$, if $\alpha \neq \beta$. Since cs-finite closed families are locally finite in a sequential space [2], $\{\mathrm{Cl}(T_\alpha)\colon \alpha \in A\}$ is locally finite in X. Hence, since locally finite families are hereditarily closure preserving, the collection of open sets $\{T_\alpha\colon \alpha \in A\}$

satisfies conditions (i) and (ii) of Theorem 3.1. Thus X is collectionwise normal, and this completes the proof.

In view of Michael's theorem ([6], Theorem 2), which implies that a metacompact, collectionwise normal space is paracompact, the preceding corollary can be amended in the following manner.

Corollary 3.3. *A sequential space X is paracompact if and only if X is a normal, metacompact space with property (ω).*

In Theorem 3.1 the requirement that $\mathcal{G}$ be hereditarily closure preserving can be weakened to closure preserving, if we require, for each $\beta \in B(\alpha)$, $\mathrm{Cl}(G_\beta) \cap F_\gamma = \emptyset$ for each $\gamma \neq \alpha$, and define $U_\alpha = \big(\bigcup\{G_\beta\colon \beta \in B(\alpha)\}\big) \setminus \mathrm{Cl}\big(\bigcup\{G_\beta\colon \beta \in B(\gamma),\ \gamma \neq \alpha\}\big)$. This condition is suggestive of the cushioned refinements introduced by Michael [9], and leads naturally to the next theorem.

Theorem 3.4. *A space X is collectionwise normal if and only if for each discrete collection of closed sets $\{F_\alpha\colon \alpha \in A\}$ there exists a collection of open sets $\mathcal{G} = \{G_\beta\colon \beta \in B\}$ such that*

(i) *$\mathcal{G}$ is cushioned in $\{U_\alpha\colon \alpha \in A\}$, where for each $\alpha \in A$, $U_\alpha = X \setminus \bigcup\{F_\gamma\colon \gamma \neq \alpha\}$,*

and

(ii) *$\mathcal{G}$ is a covering of $\bigcup\{F_\alpha\colon \alpha \in A\}$.*

Proof. Again, we prove only the sufficiency. For each $\alpha \in A$, let $B(\alpha) = \{\beta \in B\colon G_\beta \cap F_\alpha \neq \emptyset\}$, and let $B^* = \bigcup\{B(\alpha)\colon \alpha \in A\}$. Since $\mathcal{G}$ is cushioned in $\{U_\alpha\colon \alpha \in A\}$, for each $\beta \in B$ we can assign an $\alpha(\beta) \in A$ such that, for each $B' \subset B$, $\mathrm{Cl}(\bigcup\{G_\beta\colon \beta \in B'\}) \subset \bigcup\{U_{\alpha(\beta)}\colon \beta \in B'\}$. Now let $\beta \in B(\alpha)$. Assume $\alpha(\beta) \neq \alpha$. Then $\mathrm{Cl}(G_\beta) \subset U_{\alpha(\beta)}$ and $U_{\alpha(\beta)} = X \setminus \bigcup\{F_\gamma\colon \gamma \neq \alpha(\beta)\}$. Hence $\mathrm{Cl}(G_\beta) \cap F_\alpha = \emptyset$. Thus $G_\beta \cap F_\alpha = \emptyset$, and $\beta \notin B(\alpha)$ which is a contradiction to the definition of $B(\alpha)$. Hence for each $\beta \in B(\alpha)$, $\alpha(\beta) = \alpha$. Thus $\mathrm{Cl}\big(\bigcup\{G_\beta\colon \beta \in B(\alpha)\}\big) \subset U_\alpha$. To see that X is normal, let F_1 and F_2 be disjoint closed sets. Consider the collection, guaranteed by the hypothesis, with the notation developed so far. We have,

$$F_1 \subset \bigcup\{G_\beta\colon \beta \in B(1)\} \subset \mathrm{Cl}\big(\bigcup\{G_\beta\colon \beta \in B(1)\}\big) \subset U_1 = X \setminus F_2 .$$

Thus X is normal. Now, to see that X is collectionwise normal, for each $\alpha \in A$, let

$$V_\alpha = \bigcup\{G_\beta\colon \beta \in B(\alpha)\} \setminus \mathrm{Cl}\big(\bigcup\{G_\beta\colon \beta \in B^* \setminus B(\alpha)\}\big) .$$

Since

$$\mathrm{Cl}\big(\bigcup\{G_\beta\colon \beta \in B^* \setminus B(\alpha)\}\big) \subset \bigcup\{U_{\alpha(\beta)}\colon \beta \in B^* \setminus B(\alpha)\} = \bigcup\{U_\gamma\colon \gamma \neq \alpha\} = X \setminus F_\alpha ,$$

for each $\alpha \in A$, V_α is open and $F_\alpha \subset V_\alpha$. Also $V_\alpha \cap V_\gamma = \emptyset$ if $\gamma \neq \alpha$. Thus X is collectionwise normal.

Since the collectionwise normality of a metacompact space assures its paracompactness, the question arises as to whether a metacompact space with property (ω) is sequentially mesocompact. This question has not been answered, but by a direct translation, to the cs-finite terminology, Michael's proof of Theorem 2, [6], establishes the next theorem.

Theorem 3.5. *Every open covering of a metacompact space with property (ω) has a σ-cs-finite open refinement.*

From this theorem, Michael's characterization of paracompactness by means of σ-closure preserving refinements [8] and the fact that cs-finite families in a Fréchet space are closure preserving [2], we have the following corollary.

Corollary 3.6. *A Fréchet space is paracompact if and only if it is a regular metacompact space with property (ω).*

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