Non-uniqueness of homotopy factorizations into irreducible polyhedra

by

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We say (the based homotopy type of) a polyhedron X is *irreducible* if the only homotopy factorizations $X \simeq Y \times W$ are the trivial ones $Y \simeq X$, $W \simeq *$ and $W \simeq X$, $Y \simeq *$. In [3], P. Hilton and J. Roitberg illustrated the failure of the cancellation law for products in the homotopy category of compact polyhedra with an example of two 3-sphere bundles over the n-sphere of distinct homotopy types whose products with the 3-sphere have the same homotopy type. Their example suffices to establish the title of this paper. To further illustrate the non-uniqueness of homotopy factorizations of polyhedra into irreducible polyhedra we construct here the extreme situation of four irreducible compact polyhedra A, B, S, and P of distinct homotopy types for which $A \times P \simeq B \times S$.

In our example S is the 3-sphere and P is real projective 3-space. The 3-sphere S is irreducible since a consideration of the homology groups of possible factors shows that a simply connected Moore polyhedron $M(G, n)$ is reducible iff the group G admits a decomposition $G \approx H \oplus K$ with $H \otimes K = 0$ and $\mathrm{Tor}(H, K) = 0$. Projective space P is irreducible because a non-trivial factorization $P \simeq Y \times W$ (say, with W the simply connected factor) would determine a factorization of the universal coverings $\widetilde{P} \simeq \widetilde{Y} \times W$, and, as $\widetilde{P} = S$ is irreducible, would allow one to conclude $\widetilde{Y} \simeq *$, $W \simeq S$, and $P \simeq K(\mathbf{Z}_2, 1) \times S$, which contradicts the vanishing of the higher homology groups of P.

For the construction of A and B, let (B^{n+1}, S^n) be the $n+1$-ball and its bounding n-sphere with $n > 4$. Given α: $S^n \to S$, and β: $S^n \to P$ we form the maps

$$g_\alpha = \alpha \times 1_S \circ m_S\colon\ S^n \times S \to S \times S \to S$$

and

$$g_\beta = \beta \times 1_P \circ m_P\colon\ S^n \times P \to P \times P \to P,$$

where m_S: $S\times S\to S$ and m_P: $P\times P\to P$ are induced by quaternionic multiplication and the double covering p: $S\to P$, and we construct the adjunction spaces

$$A = S\bigcup_{g_\alpha} B^{n+1}\times S \quad \text{and} \quad B = P\bigcup_{g_\beta} B^{n+1}\times P.$$

PROPOSITION 1. *The polyhedra A and B are reducible if and only if the maps α: $S^n\to S$ and β: $S^n\to P$ are inessential.*

Proof. A consideration of the homology groups of A and B shows that the only possible non-trivial factorizations are $A\simeq S^{n+1}\times S$ and $B\simeq S^{n+1}\times P$. But no such maps can preserve the nth homotopy groups if α and β are essential. For the converse, one can prove directly that $A\simeq S^{n+1}\times S$ and $B\simeq S^{n+1}\times P$ if α and β are inessential.

The construction we make for the homotopy equivalence $A\times P\simeq B\times S$ takes advantage of the large supply of homotopy equivalences h: $S\times P\to\to P\times S$. We can associate with a map h: $S\times P\to P\times S$ the 2×2 matrix $\emptyset(h) = \big(d(h_{IJ})\big)$ of degrees of the four maps h_{IJ}: $I\to J$ $(I, J = S, P)$ obtained from h by restriction and projection. Since the degree function d: $[S, P]\to \boldsymbol{Z}$ has image $2\boldsymbol{Z}$, the matrix $\emptyset(h)$ has off-diagonal entry $d(h_{SP})$ even. Conversely, given such a matrix (n_{IJ}) we can select maps $\overline{n}_{IJ}$: $I\to J$ $(I, J = S, P)$ with degrees $d(\overline{n}_{IJ}) = n_{IJ}$ and we can construct a map

$$\{(\overline{n}_{IJ})\}\colon\ S\times P\to P\times S$$

with $\emptyset\big(\{(\overline{n}_{IJ})\}\big) = (n_{IJ})$ by requiring that the projection on the Jth factor be

$$\overline{n}_{SJ}\times\overline{n}_{PJ}\circ m_J\colon\ S\times P\to J\times J\to J \qquad (J = S, P)\,.$$

The value of this correspondence is that a map h: $S\times P\to P\times S$ is a homotopy equivalence iff the matrix $\emptyset(h)$ is invertible ([4]). So we can construct a homotopy equivalence for each pair of relatively prime integers n_{SS} and n_{SP} with the later even.

PROPOSITION 2. *If an invertible matrix (n_{IJ}) as above satisfies the conditions*

(i) $n_{SS}(n_{SS}-1)\equiv 0 \mod 24$, $n_{SP}(n_{SP}-2)\equiv 0 \mod 96$

and

(ii) $0\simeq n_{SS}\alpha$: $S^n\to S$, $\beta\simeq 1/2n_{SP}\alpha\circ p$: $S^n\to S\to P$,

then the homotopy equivalence $\{(\overline{n}_{IJ})\}$: $S\times P\to P\times S$ extends to a homotopy equivalence $A\times P\to B\times S$.

Proof. Now we have

$$A\times P = S\times P \bigcup_{g_\alpha\times 1_P} B^{n+1}\times S\times P$$

and

$$B\times S = P\times S \bigcup_{g_\beta\times 1_S} B^{n+1}\times P\times S\,.$$

Therefore it suffices to show that

$$g_\alpha\times 1_P\circ\{(\overline{n}_{IJ})\}\simeq 1_{S^n}\times\{(\overline{n}_{IJ})\}\circ g_\beta\times 1_S\,,$$

for then, as in the simplier setting where the adjunction spaces are mapping cones ([2, p. 40]), the homotopy equivalences $\{(\overline{n}_{IJ})\}$ and $1_{S^n}\times\times\{(\overline{n}_{IJ})\}$ determine a homotopy equivalence $A\times P\to B\times S$ which extends $\{(\overline{n}_{IJ})\}$.

It is known that $\overline{n}$: $S, m_S\to S$, m_S is a homomorphism if and only if $n(n-1)\equiv 0 \mod 24$ ([1, Theorem A]), hence the same holds for $\overline{2n}\simeq\overline{n}\circ p$: $S, m_S\to S$, $m_S\to P, m_P$, since p: $S\to P$ is a homomorphism. These facts make it easy to show that in the presence of conditions (i) and (ii) the homotopy relation above holds, which completes the proof.

If α: $S^n\to S$ has order a prime p (e.g., a generator of the p-primary component $\boldsymbol{Z}_p$ of $\pi_{2p}(S)$) with $p\equiv 1 \mod 24$, then we can choose the relatively prime integers $n_{SS} = p$, $n_{SP} = 48$ and obtain an invertible matrix (n_{IJ}) which satisfies the hypotheses of Proposition 2 for $\beta = 24\alpha\circ p$: $S^n\to S\to P$. In this way we obtain an example of four irreducible compact polyhedra A, B, S, and P of distinct homotopy types with $A\times P\simeq B\times S$.

We close by remarking that the example shows that even the basic irreducible polyhedron S fails to be "prime" in that there is a product it divides involving only factors that it fails to divide. Perhaps there are no "prime" polyhedra.

References

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