Since both x and y belong to $g^{-1}[M]$ and $g^{-1}[M] \subset g^{-1}[W]$, there exist disjoint open 3-cells U_x and U_y containing x and y, respectively, and contained in $g^{-1}[W]$.

By Lemma 6, there is an arc θ in $(\operatorname{Int} A) \cap g^{-1}[W]$ from a point of U_x to a point of U_y. By Lemma 7, there is an arc φ in $(\operatorname{Int} B) \cap g^{-1}[W]$ from a point of U_x to a point of U_y. Let γ be a simple closed curve formed by joining θ and φ by an arc in U_x and by an arc in U_y. Then γ is a loop in $g^{-1}[W]$ and it is clear that $\gamma \not\sim 0$ in $A^* \cup B^*$. However, W has the property that each loop in $g^{-1}[W]$ is homotopic to 0 in $A^* \cup B^*$. This is a contradiction, and thus Z has no brick decomposition.

The following summarizes our results.

THEOREM. *There exist compact metric spaces X and Y such that* (1) *X, Y, and $X \cap Y$ have brick decompositions, but* (2) *$X \cup Y$ has no brick decomposition.*

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Determinateness in the low projective hierarchy *

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Introduction. This paper contains results on $\boldsymbol{\Sigma}_1^1$ determinateness, determinateness for certain fragments of the Boolean algebra generated by the $\boldsymbol{\Sigma}_1^1$ sets, and $\boldsymbol{\Delta}_2^1$ determinateness. We use L for the class of constructible sets. In the first section, we use L_α for the set of all sets of level $< \alpha$ in the constructible hierarchy, and $L_\alpha(x)$ for the set of all sets of level $< \alpha$ in the constructible hierarchy starting from x. In Sections 2, 3, 4, however, we found it convenient to use, respectively, $L(\alpha)$ and $L^x(\alpha)$, and to use L^x for the class of sets constructible from x. (No confusion will arise as to which of the possible notions of relative constructibility is used.) By K determinateness, where $K \subset (N^N \times N^N)$, we mean that $(\forall A \in K)$ (A is determined). By "A is determined" we mean that the game G_A has a winning strategy for either player I or player II, where G is played as follows: players I, II play alternately, starting with I. Each move is an integer. The result of the game is an element x of $N^N \times N^N$, and I is deemed the winner if $x \in A$; II is deemed the winner otherwise.

In Section 1 we consider Σ_1^1 determinateness (a relativized version, Theorem 1′, is stated in Section 4). Previously, there were two main results about Σ_1^1 determinateness. The first is the result of D. Martin [4] that $\boldsymbol{\Sigma}_1^1$ determinateness follows from (*) of Section 2, which in turn follows from measurable cardinals. The second is that $\Sigma_1^1(\boldsymbol{\Sigma}_1^1)$ determinateness implies that every uncountable Σ_1^1 ($\boldsymbol{\Sigma}_1^1$) subset of N^N contains a perfect subset (Davis [2], Theorems 4.1, 4.2). The former result suggested that it would be worthwhile to find a consistency proof for Σ_1^1 ($\boldsymbol{\Sigma}_1^1$) determinataness using the consistency of some currently formulated axioms about sets. The latter result showed that, by Solovay [10], Σ_1^1 ($\boldsymbol{\Sigma}_1^1$) determinateness implies that $N^N \cap L$ is countable ($N^N \cap L(x)$ is countable for all $x \subset \omega$), and so Σ_1^1 determinateness cannot be proved from the currently formulated axioms about sets, since they are all compatible with $(\forall x)(x \in L)$. But it was still possible that Σ_1^1 determinateness was

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equivalent to $L \cap N^N$ is countable (or $\boldsymbol{\Sigma}_1^1$ determinateness equivalent to $L(x) \cap N^N$ countable for all $x \subset \omega$).

By proving that Σ_1^1 determinateness fails in all forcing extensions of L, we refute the possibility, and show that no consistency proof of Σ_1^1 determinateness can be given using current methods (i.e., current axioms about sets and current methods of proof). Thus Σ_1^1 determinateness provides an example of a sentence known not to hold in any forcing extension of L but not known to imply (*) for $x = \emptyset$.

Now if $N^N \cap L^x$ is countable for all $x \subset \omega$, then Ω must be inaccessible in L^x for all $x \subset \omega$. Hence Σ_1^1 determinateness implies that Ω is inaccessible in L^x for all $x \subset \omega$. This raises the question of how large Ω must appear to be in L^x, $x \subset \omega$, if $\boldsymbol{\Sigma}_1^1$ determinateness holds. We show in Section 2 that using a slight (?) extension of $\boldsymbol{\Sigma}_1^1$ determinateness, written $\boldsymbol{\Sigma}_1^1$ & $\boldsymbol{\Pi}_1^1$ determinateness, we can conclude that Ω is quite large in every L^x; e.g., that it is the Ωth inaccessible in every L^x, $x \subset \omega$. The best previous results along the lines of proving that Ω appears large in every L^x, $x \subset \omega$ from determinateness were the one cited above for $\boldsymbol{\Sigma}_1^1$ determinateness and a result of Solovay (personal communication) that $\boldsymbol{\Pi}_2^1$ determinateness implies (*).

In the second part of Section 2 we extend the method of proof in Martin [4] to prove that certain Boolean combinations of $\boldsymbol{\Sigma}_1^1$ sets are determinate, using (*). Just as in [4], we make use of an auxiliary open game on ordinals.

In Section 3 we prove that $\boldsymbol{\Delta}_2^1$ determinateness implies (*). The best previous result was due to Solovay (personal communication), that $\boldsymbol{\Pi}_2^1$ determinateness implies (*). This result of Section 3 is due, independently, to D. A. Martin.

§ 1. Our first goal is to give the definition of semi-generic well-ordering. We use the treatment of forcing in Shoenfield [7].

DEFINITION 1.1. For each countable admissible ordinal α let C_α be the notion of forcing whose conditions are the functions from some finite initial segment of ω into α, and where $p \leqslant q \equiv q \subset p$. A semi-generic enumeration of α is a function $g\colon \omega \to \alpha$ which intersects every dense subset of C_α which lies in L_{α^++1}, where α^+ is the least admissible ordinal greater than α.

DEFINITION 1.2. Let α be countable and admissible. Then $x \subset \omega$ is called a *semi-generic well-ordering of type* α if there is a semi-generic enumeration, g, of α, such that $x = \{2^n 3^m\colon g(n) \in g(m)\}$.

LEMMA 1.1. *Let α be a countable and admissible, and let g be any semi-generic enumeration of α. Then $L_{\alpha^+}(g)$ is an admissible set, where α^+ is the least admissible greater than α.*

Proof. See Barwise [1], Appendix B. Set $b = \alpha$, $\tau = \alpha^+$. In [1], forcing is presented using the ramified language, and of course we have the equivalence of that notion of genericity with our semi-genericity.

We let $\alpha^{<\omega}$ be the set of conditions in C_α.

LEMMA 1.2. *Let α be countable and admissible. Let g be any semi-generic enumeration of α, and let $y \subset \omega$ have $y \in L_{\alpha^+}(g)$. Then there is a function $f\colon \alpha^{<\omega} \times \omega \to \{0, 1\}$ such that $f \in L_{\alpha^+}$ & $(\forall n)\big(n \in y \equiv (\exists p \in \alpha^{<\omega})(p \subset g$ & $f(p, n) = 1)\big)$.*

Proof. Choose a term t in the forcing language which defines y in $L_{\alpha^+}(g)$. Set $f(p, n) = 1$ if p forces $\overline{p} \in t$; 0 if p does not force $\overline{n} \in t$.

DEFINITION 1.3. Let A be a transitive set. Then a relational structure (B, R) is called an *ε-extension of A* if

$$A \subset B \;\&\; (\forall x \in A)(\forall y)\big(R(y, x) \to y \in A\big) \;\&\; (\forall x \in A)(\forall y \in A)\big(R(x, y) \equiv x \epsilon y\big).$$

LEMMA 1.3. *Let α be countable and admissible, and let (ω, R) be isomorphic to some ε-extension of L_{α^+}. Let $f\colon \alpha^{<\omega} \times \omega \to \{0, 1\}$ have $f \in L_{\alpha^+}$. Then $f \in L_{\alpha+\omega}(R)$.*

Combining Lemmas 1.2 and 1.3 we obtain

LEMMA 1.4. *There is a $\beta < \alpha^+$ such that whenever g is a semi-generic enumeration of α and (ω, R) is isomorphic to some ε-extension of L_{α^+} and $y \subset \omega$ has $y \in L_{\alpha^+}(g)$, then $y \in L_\beta(g, R)$.*

LEMMA 1.5. *Let α be admissible, g a semi-generic enumeration of α. Then for all $\beta < \alpha^+$ there is a $y \subset \omega$ with $y \in L_{\alpha^+}(g) - L_\beta(g)$.*

Proof. If not, then $L_{\alpha^+}(g)$ must satisfy the existence of a first uncountable ordinal, which must be $> \alpha$, at the same time satisfied to be admissible, which is a contradiction.

LEMMA 1.6. *Let α be admissible, g a semi-generic enumeration of α, (ω, R) isomorphic to some ε-extension of L_{α^+}. Then $R \notin L_{\alpha^+}(g)$.*

Proof. Combine Lemma 1.4 with Lemma 1.5.

LEMMA 1.7. *Let α be admissible, y a semi-generic well-ordering of type α, (ω, R) isomorphic to some ε-extension of L_{α^+}. Then R is not hyperarithmetic in y.*

Proof. From Lemma 1.6 together with the observation that the subsets of ω present in an admissible set are always closed under relative hyperarithmeticity.

The following is due to Ville (unpublished).

LEMMA 1.8. *There is a sentence φ such that for all admissible α, any ε-extension of $\alpha+1$ satisfying φ must be isomorphic to an ε-extension of L_{α^+}, and furthermore, φ holds in $(L_{\alpha^+}, \varepsilon)$.*

We now immediately obtain

LEMMA 1.9. *Let α be admissible, y a semi-generic well-ordering of type α, (ω, R) isomorphic to some ε-extension of $\alpha+1$ satisfying φ. Then R is not hyperarithmetic in y.*

LEMMA 1.10. *There is a Σ_1^1 subset Y of $N^N \times N^N$ such that no winning strategy is hyperarithmetic in any semi-generic well-ordering.*

Proof. Take $Y = \{(f, g)$: either g does not code a well-ordering of type some admissible ordinal α or else g does code a well-ordering of type α, α admissible and f codes an (ω, R) isomorphic to some ε-extension of $\alpha+1$ satisfying $\varphi\}$. Then Y can easily be seen to be Σ_1^1. If there is a winning strategy for II for Y then the set of plays for II is a Σ_1^1 set of coded well-orderings, and hence must be bounded in order type by, say, α. But then I could thwart this winning strategy for II by playing any (ω, R) isomorphic to L_{β^+}, where β is the least admissible ordinal greater than α. Hence there is no winning strategy for II for Y. If there is a winning strategy for I for Y that is hyperarithmetic in some semi-generic well-ordering of type α, y, then II can thwart this winning strategy for I by playing a code for y. Then the winning strategy directs I to play something hyperarithmetic in y, and so I loses by Lemma 1.9.

DEFINITION 1.4. A notion of forcing is a partially ordered set C with a maximum element. See Shoenfield [7]. A regular class in a class Y satisfying ZFC which contains all the ordinals and which is transitive.

DEFINITION 1.5. Let C be a notion of forcing. A dense subset of C is an $x \subset C$ such that for all $y \in C$ there is an $a \in x$ such that $a \leqslant_C y$.

DEFINITION 1.6. Let C be a notion of forcing, Y be a regular class. Then x is *C-generic over Y* just in case $x \subset C$ and $(\forall y)(\forall z)\big((y \in x \;\&\; y \leqslant_C z) \to \to z \in x\big)$ and for all dense subsets $y \subset C$ with $y \in Y$ we have $x \cap y \neq \emptyset$. We let $Y(x)$ be the least regular class containing $Y \cup \{x\}$. If we write "x is C-generic", we mean "x is C-generic over L".

The following is a basic lemma about forcing. See [7].

LEMMA 1.11. *Let P be a sentence in the language of set theory, and let Y be a regular class, $C \in Y$ a notion of forcing. Suppose that there is a C-generic $x \subset C$ such that $Y(x) \models P$. Then there is an $a \in x$ such that for all C-generic $y \subset C$ with $a \in y$, we have $Y(x) \models P$.*

LEMMA 1.12. *The property of being a relation R on ω such that $(\omega, R) \approx (L_{\alpha^+}, \varepsilon)$ for some admissible α, is Π_1^1.*

From this we may obtain

LEMMA 1.13. *The property of being a semi-generic well-ordering is Σ_2^1.*

LEMMA 1.14. *The property of being hyperarithmetic in some semi-generic well-ordering is Σ_2^1.*

Now by the absoluteness theorem [6] we have

LEMMA 1.15. *There is a sentence P such that for all regular classes Y, we have $Y \models P$ if and only if for all $x \subset \omega$ with $x \in Y$, x is hyp in some semi-generic well-ordering.*

We fix this P.

DEFINITION 1.7. For $x \in L$ let $\mathrm{CA}(x)$ be the least cardinal in L, α, such that $x \in L_\alpha$.

The following is a basic lemma about forcing. See [7].

LEMMA 1.16. *Let $C \in L$ be a notion of forcing and let $x \subset C$ be C-generic over L. Then for all $y \subset \omega$ with $y \in L(x)$ we have that $y \in L_{CA(C)}(x)$.*

The following is a recursion-theoretic refinement of the usual theorem which says that generic sets exist.

LEMMA 1.17. *Let $C \in L$, $a \in C$, C a notion of forcing, and let y be any well-ordering on ω of type $\mathrm{CA}(C)$. Then there is a generic $x \subset C$ with $a \in x$ such that $x \in L_{CA(C)+CA(C)}(y)$.*

From Lemma 1.16 we obtain

LEMMA 1.18. *Let $C \in L$, $a \in C$, C a notion of forcing, and let y be any well-ordering on ω of type $\mathrm{CA}(C)$. Then there is a generic $x \subset C$ with $a \in x$, such that every $z \subset \omega$ with $z \in L(x)$ is hyperarithmetic in y.*

Choosing y to be any semi-generic well-ordering of type $\mathrm{CA}(C)$, we obtain the following from Lemma 1.15.

LEMMA 1.19. *Assume $C \in L$, C a notion of forcing, $a \in C$, $\mathrm{CA}(C)$ is countable. Then there is a $y \subset C$, y C-generic, with $a \in y$, $L(y) \models P$.*

By Lemma 1.11 we have

LEMMA 1.20. *Assume $C \in L$, C a notion of forcing, $\mathrm{CA}(C)$ is countable. Then for all C-generic $y \subset C$ and $x \in L(y)$, we have that x is hyperarithmetic in some semi-generic well-ordering.*

By Lemma 1.10 we have

LEMMA 1.21. *There is a Σ_1^1 subset Y of $N^N \times N^N$ such that there is no winning strategy for Y present in any $L(y)$ where y is C-generic for some notion of forcing $C \in L$ with $\mathrm{CA}(C)$ countable.*

We may eliminate the hypothesis that $\mathrm{CA}(C)$ be countable by a forcing argument.

THEOREM 1. *There is a Σ_1^1 subset Y of $N^N \times N^N$ such that there is no winning strategy for Y present in any $L(y)$ where y is C-generic for some notion of forcing $C \in L$.*

Proof. It suffices to show that this theorem is satisfied to be true in any countable transitive model M of ZF. Suppose it fails in M and C provides the counterexample. Pass to any forcing extension, N, of M obtained by adding an enumeration of $\mathrm{CA}(C)$ generic over M. Then apply

Lemma 1.21 in N for C. Note that by absoluteness, anything satisfied by M to be a winning strategy for the Σ_1^1 set must be satisfied by N to be so; obviously, any $x \subset C$ satisfied by M to be C-generic must be satisfied by N to be, also.

§ 2. The first part of this section concerns $\boldsymbol{\Sigma}_1^1 \& \boldsymbol{\Pi}_1^1$ determinateness, which asserts that every pairwise intersection of a $\boldsymbol{\Sigma}_1^1$ subset of $N^N \times N^N$ with the complement of a $\boldsymbol{\Sigma}_1^1$ subset of $N^N \times N^N$ has a winning strategy.

Definition 2.1. Let $V(0) = \emptyset$, $V(\alpha+1) = \boldsymbol{P}(V(\alpha))$, $V(\lambda) = \bigcup_{\alpha<\lambda} V(\alpha)$. We say that an ordinal α is *weak* Π_1^1 indescribable just in case for all first-order $P(Y, x_1, \dots, x_k)$, $x_1, \dots, x_k \in V(\alpha)$, such that $(\forall Y \subset V(\alpha))\big((V(\alpha) \cup \{Y\}, \varepsilon) \models P(Y, x_1, \dots, x_k)\big)$, there exists $\beta < \alpha$ with $x_1, \dots, x_k \in V(\beta)$ and $(\forall Y \subset V(\beta))\big((V(\beta) \cup \{Y\}, \varepsilon) \models P(Y, x_1, \dots, x_k)\big)$. We abbreviate the latter by writing $V(\beta) \models \forall Y(P(Y, x_1, \dots, x_k))$.

Definition 2.2. Let Ω be the first uncountable ordinal, and let $Z^* = \mathrm{ZFC} - P + \exists \boldsymbol{P}(\omega)$ be ZFC without power set but with the existence of the power set of ω.

Lemma 2.1.1. *Suppose $x_1, \dots, x_k \in L_\Omega$, P first-order, $y \subset \omega$,*

$$L_\Omega \models \forall Y(P(Y, x_1, \dots, x_k)) .$$

Then there is an $\alpha < \Omega$ such that $L_\alpha(y) \models \big(L_\Omega \models \forall Y(P(Y, x_1, \dots, x_k))\big) \& Z^*$.

Proof. Note that $\Omega_2 =$ 2nd uncountable cardinal satisfies the conclusion except that $\Omega_2 < \Omega$. So instead choose a countable elementary submodel of $L_{\Omega_\omega}(y)$ and take the image of the isomorphism onto a transitive set.

Definition 2.3. If Y is a transitive set, $(Y, \varepsilon) \models Z^*$, let $\Omega(Y)$ be that ordinal in Y which is satisfied, in (Y, ε), to be the first uncountable ordinal.

Lemma 2.1.2. *Suppose $x_1, \dots, x_k \in L_\Omega$, P first-order, $y \subset \omega$, $L_\Omega \models \forall Y\big(P(Y, x_1, \dots, x_k)\big)$. Then there is an $\alpha < \Omega$ such that $L_\alpha(y) \models \big(L_\Omega \models \forall Y(P(Y, x_1, \dots, x_k))\big)$, and there is a well-ordering, z, on ω of order type $\Omega(L_\alpha(y))$ such that α is not an ordinal recursive in (y, z).*

Proof. Choose α as in Lemma 2.1.1. Then $L_\alpha(y)$ is admissible. Hence by Barwise [1], Appendix B, there is a well-ordering z on ω of type $\Omega(L_\alpha(y))$ such that $L_\alpha(y, z)$ is admissible. Then clearly α must not be an ordinal recursive in (y, z).

Lemma 2.1.3. *$\boldsymbol{\Sigma}_1^1$ determinateness implies that $(L \models \alpha$ is inaccessible$)$, where α is the first uncountable ordinal.*

Proof. See Davis [2] and Solovay [10], as described in the Introduction.

The following is well known.

Lemma 2.1.4. *We have that $L \models \big((L_\alpha = V(\alpha)) \equiv (\Omega_\alpha = \alpha)\big)$. There is a first-order $P(Y)$ such that $L \models \big(L_\alpha = V(\alpha) \equiv L_\alpha \models \forall Y(P(Y))\big)$.*

We immediately have the following using Lemmas 2.1.3 and 2.1.4.

Lemma 2.1.5. *Assume $\boldsymbol{\Sigma}_1^1$ determinateness. Suppose for every first-order P, $x_1, \dots, x_k \in L_\Omega \cap \boldsymbol{P}(\omega)$ such that $L_\Omega \models \forall Y(P(Y, x_1, \dots, x_k))$, there exists an $\alpha < \Omega$ such that $L_\alpha \models \forall Y(P(Y, x_1, \dots, x_k))$, and $x_1, \dots, x_k \in L_\alpha$. Then $(L \models \alpha$ is weak Π_1^1 indescribable$)$, where $\alpha = \Omega$.*

Theorem 2.1. *$\boldsymbol{\Sigma}_1^1 \& \boldsymbol{\Pi}_1^1$ determinateness implies that $(L \models \alpha$ is weak Π_1^1 indescribable and inaccessible$)$, where $\alpha = \Omega$.*

Proof. Using Lemma 2.1.5, let $L_\Omega \models \forall Y(P(Y, x_1, \dots, x_k))$. We consider the following subset, S, of $N^N \times N^N$: $S = \{(f, g)$: f codes a well-ordering on ω of type $|f|$ with $x_1, \dots, x_k \in L_{|f|}$, and either g does not code a well-ordering on ω or g codes a well-ordering of type $|g|$ and $(\forall Y \subset L_\alpha)$ $\big(Y \in L_g \to (L_{|f|} \cup \{Y\}, \varepsilon) \models P(Y, x_1, \dots, x_k)\big)\}$. As written, it is not obvious that S is $\boldsymbol{\Sigma}_1^1 \& \boldsymbol{\Pi}_1^1$. However, it can be seen to be such by employing the Σ_1^1 predicate of g, h, written $L_{|g|}(h)$, which means that h is a coded version of the constructible hierarchy based on the ordering g, which may, however, not be a *well-ordering*. However, if g is a well-ordering and $L_{|g|}(h)$, then h will correctly code up the constructible hierarchy up through $|g|$. Thus we have $S = \{(f, g)$: f codes a well-ordering on ω of type $|f|$ with $x_1, \dots, x_k \in L_{|f|}$, and either g does not code a well-ordering on ω or $(\exists h)\big(L_{|g|}(h) \& (\forall Y \subset L_\alpha)(Y$ coded in $h \to (L_{|f|} \cup \{Y\}, \varepsilon) \models P(Y, x_1, \dots, x_k))\big)\}$. (Note that $\{(f, g)$: f codes a well-ordering on ω of type $|f|$ with $x_1, \dots, x_k \in L_{|f|}\}$ will be Σ_1^1 since there will be a countable ordinal α such that the above set is just $\{(f, g)$: f codes a well-ordering of type $\geqslant \alpha\}$.)

Now by $\boldsymbol{\Sigma}_1^1 \& \boldsymbol{\Pi}_1^1$ determinateness, either I has a winning strategy for S or II does. If II has a winning strategy, y, then choose α and z as in Lemma 2.1.2, and have I play z. Then II will play, using strategy z, some $g \leqslant_T z$. Now g must code a well-ordering, and by Lemma 2.1.2 it is clear that it must be longer than α. But then α is an ordinal recursive in z, which is a contradiction.

So I has the winning strategy. But since I is always playing well-orderings, and since $\boldsymbol{\Sigma}_1^1$ sets of (coded) well-orderings on ω are always bounded, we let α bound the order types of the well-orderings that can be played by I. Then it is clear that there is some $\beta \leqslant \alpha$ such that $L \models \big(L_\beta \models \forall Y(P(Y, x_1, \dots, x_k))\big)$, or else by the regularity of Ω, II could thwart the winning strategy for I by playing a suitably long well- ordering.

Corollary 2.1.1. *$\boldsymbol{\Sigma}_1^1 \& \boldsymbol{\Pi}_1^1$ determinateness implies that $(L \models \alpha$ is the α-th inaccessible cardinal$)$, where $\alpha = \Omega$.*

Proof. If ($L \models \alpha$ is the βth inaccessible cardinal), for some $\beta < \Omega = \alpha$, then there will be a first-order sentence $P(\beta)$ such that $\big(L \models (V_\alpha \models P(\beta)\big)$ & & $(\forall \gamma < \alpha)\big(V_\gamma \models \sim P(\beta)\big)\big)$, violating Theorem 2.1.

We may relativize Theorem 2.1 to any $x \subset \omega$.

COROLLARY 2.1.2. *$\boldsymbol{\Sigma}_1^1$ & $\boldsymbol{\Pi}_1^1$ determinateness implies that for all $x \subset \omega$ we have ($L^x \models \alpha$ is weak Π_1^1 indescribable and inaccessible), where $\alpha = \Omega$.*

We now give a proof of determinateness for a class of sets which form a fragment of the least σ-algebra (or even Boolean algebra) generated by the analytic ($\boldsymbol{\Sigma}_1^1$) subsets of N^N, using the hypothesis (*) below.

(*) *For each hereditarily countable set x, each 1-st order formula $\varphi(v_0, \ldots, v_n, v_{n+1})$, and each sequence of uncountable cardinals $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_n$ with $\alpha_i \in \alpha_j \equiv \beta_i \in \beta_j$, we have*

$$L(x) \models \varphi(\alpha_0, \ldots, \alpha_n, x) \equiv \varphi(\beta_0, \ldots, \beta_n, x) .$$

DEFINITION 2.4. For all sets y, if $f \in y^\omega$ then $\tilde{f}(n) = \big(f(0), \ldots, f(n-1)\big)$. For $s \in y^n$ we write $\ln(s) = n$, $y^0 = \{< >\}$. We have $s = \big(s(0), \ldots, s(\ln(s)-1)\big)$, and for $i < \ln(s)$ we write $\tilde{s}(i) = \big(s(0), \ldots, s(i-1)\big)$. We say that s is an initial segment of t if $\ln(s) \leqslant \ln(t)$ and $\big(\forall i < \ln(s)\big)\big(s(i) = t(i)\big)$.

DEFINITION 2.5. A *tree* is a subset, T, of $A^{<\omega}$ such that every initial segment of an element of T is an element of T where A is some set. A path through T is an $f \in A^\omega$ such that $(\forall n)\big(\tilde{f}(n) \in T\big)$. A tree is called *well-founded* just in case it has no paths.

DEFINITION 2.6. Let $h\colon (\omega^{<\omega})^3 \to \omega$. Then we write $T_h(s, t)$, $s, t \in \omega^{<\omega}$, for the tree $\big\{a \in \omega^{<\omega}\colon (\exists i)\big(h(\tilde{a}(i), \tilde{s}(i), \tilde{t}(i)) = 0\big)\big\}$. We write $T_h(f, g)$ $= \big\{a \in \omega^{<\omega}\colon (\exists i)\big(h(\tilde{a}(i), \tilde{f}(i), \tilde{g}(i)) = 0\big)\big\}$.

The following is a basic fact about co-analytic sets.

LEMMA 2.2.1. *Every Π_1^1 subset of $(N^N)^2$ is of the form $\{(f, g)\colon T_h(f, g)$ is well-founded$\}$, for some $h\colon (\omega^{<\omega})^3 \to \omega$.*

We wish to prove $(\boldsymbol{\Pi}_1^1 \,\&\, \boldsymbol{\Sigma}_1^1) \vee \boldsymbol{\Sigma}_1^1$ determinateness from (*). For this purpose, we fix functions h_1, h_2, h_3. The set we are trying to prove determinate is $S = \{(f, g)\colon T_{h_1}(f, g)$ is well-founded and $T_{h_2}(f, g)$ is not well-founded) or $T_{h_3}(f, g)$ is not well-founded$\}$.

DEFINITION 2.7. For $s, t \in \omega^{<\omega}$ we define $s < t$ if and only if either

(1) s properly extends t or

(2) neither is an initial segment of the other and s is smaller than t at the first argument at which they differ.

The following is well known.

LEMMA 2.2.2. *Note that $<$ is a linear ordering and that for trees $T \subset \omega^{<\omega}$, we have that T is well-founded if and only if $<$ restricted to T is a well-ordering.*

We now describe an auxiliary game. Player I's moves will either consist of an integer or a pair consisting of an integer together with an element of $\Omega_{\Omega+\Omega} - (\Omega_\Omega + 1) \cup \{E\}$, where E is a special symbol for "error". In the first case, the first part of his move will be that integer and there will be no second part. In the second case, the first part will be that integer and the second part that element of $\Omega_{\Omega+\Omega} - (\Omega_\Omega + 1) \cup \{E\}$. We could have said that I's moves consist of partial functions on $\{1, 2\}$ such that the domain contains 1. Keeping this in mind, we stipulate that II's moves must be partial functions from $\{1, 2, 3\}$ into $\mathrm{On} \cup \{E\}$ whose domain contains 1, such that the first part must be an integer, the second part must be an ordinal $> \Omega_{\Omega+\Omega}$, or E, and the third part must be $< \Omega_\Omega$ or E.

The auxiliary game starts with I's 0th move, then II's 0th move, then I's 1st move, then II's 1st move, etcetera. If it is I's turn to make his nth move, then the *numerical history* of the game (up to then) is $\big((a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1})\big)$, where a_i is the first part of I's ith move, b_j is the first part of II's jth move. If it is II's turn to make his nth move, then the numerical history of the game is $\big((a_0, \ldots, a_n), (b_0, \ldots, b_{n-1})\big)$.

We fix Z as any recursive function from ω onto $\omega^{<\omega}$ such that the inverse image of every element is infinite.

DEFINITION 2.8. A coded partial isomorphism, for the purposes of this paper, is a finite partial function $f\colon \omega \to \mathrm{On}$ such that whenever $a, b \in \mathrm{Dom}(f)$ we have $Z(a) < Z(b) \equiv f(a) \in f(b)$.

We now describe, in detail, what the legal moves are for I, II.

Suppose it is I's turn to play his nth move. Let $\big((a_0, \ldots, a_{n-1}),$ $(b_0, \ldots, b_{n-1})\big)$ be the numerical history. The first part of I's nth move must be an integer. If $Z(n) \in T_{h_1}\big((a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1})\big)$, and no E has yet been played by either player, then (a) if there is an $\alpha \in \Omega_{\Omega+\Omega} -$ $- (\Omega_\Omega + 1)$ such that λk (the second part of I's kth move if $k < n$; α if $k = n$) is a coded partial isomorphism, then I is legally required to play such an α for the second part, (b) if no such α exists, then I is legally required to play E for the second part. In all other cases, I is legally required not to have a second part at the nth move.

Suppose it is II's turn to play his nth move. Let $\big((a_0, \ldots, a_n),$ $(b_0, \ldots, b_{n-1})\big)$ be the numerical history. The first part of II's nth move must be an integer. If $Z(n) \in T_{h_2}\big((a_0, \ldots, a_n), (b_0, \ldots, b_{n-1})\big)$, and no E has yet been played by either player, then (a) if there is an $\alpha > \Omega_{\Omega+\Omega}$ such that λk (the second part of II's kth move if $k < n$; α if $k = n$) is a coded partial isomorphism, then II is legally required to play such an α for the second part, (b) if no such α exists, then II is legally required to play E for the second part. In all other cases, II is legally required to have no second part at the nth move. If $Z(n) \in T_{h_3}\big((a_0, \ldots, a_n), (b_0, \ldots, b_{n-1})\big)$, and E has not been played earlier by II, then (a) if there is an $\alpha \in \Omega_\Omega$ such that λk (the third part of II's kth move if $k < n$; α if $k = n$) is a coded partial isomorphism, then II is legally required to play such an α for the third part, (b) if no such α exists, then II is legally required to play E for the third part of II's nth move.

The outcome of the auxiliary game is defined as follows: I is the winner if and only if II eventually plays an E.

Obviously we have

LEMMA 2.2.3. *The winning set in the auxiliary game is open in the usual topology of Y^ω for the appropriate set Y.*

LEMMA 2.2.4. *The auxiliary game is determinate; furthermore, it has a winning strategy definable from $(\Omega_\Omega, \Omega_{\Omega+\Omega})$ in $L(h_1, h_2, h_3)$.*

Proof. Going through the usual proof of determinateness for open games, we see that it is a matter of assigning ordinals to trees constructible in (h_1, h_2, h_3). Then one invokes absoluteness properties of $L(h_1, h_2, h_3)$.

This above lemma enables us to use hypothesis (*) effectively.

We fix J as a winning strategy for the auxiliary game, definable from $(\Omega_\Omega, \Omega_{\Omega+\Omega})$ in $L(h_1, h_2, h_3)$.

THEOREM 2.2. *If (*) holds, then $(\boldsymbol{\Pi}_1^1 \,\&\, \boldsymbol{\Sigma}_1^1) \vee \boldsymbol{\Sigma}_1^1$ determinateness holds.*

Proof. We must show that the original game on ω^ω, the one with winning set (for I) S, is determinate.

Case 1. J is a winning strategy for I.

Consider the following strategy for I in the original game. In order to determine I's nth move when II has moved $b_0, \ldots, b_{n-1}$, we consider all possible *legal* (in the auxiliary game) sequences of moves (a_0, c_0), $(b_0, d_0, e_0), \ldots, (a_{n-1}, c_{n-1}), (b_{n-1}, d_{n-1}, e_{n-1}), (a_n, c_n)$ that would ensue if I, II were playing the auxiliary game and I was using strategy J and II was under the restriction that he must only play cardinals or E or nothing for his 2nd and 3rd parts. By indiscernibility, since J is definable in $L(h_1, h_2, h_3)$ from $(\Omega_\Omega, \Omega_{\Omega+\Omega})$, we see that the numbers $a_0, \ldots, a_n$ are independent of exactly what the d_i and e_i are. So we set I's nth move to be a_n.

We must prove that either $T_{h_1}(\lambda i(a_i), \lambda i(b_i))$ is well-founded and $T_{h_2}(\lambda_i(a_i), \lambda_i(b_i))$ is not well-founded, or $T_{h_3}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded, where $a_0, a_1, \ldots$ are the moves player I will make according to the strategy just outlined for him in the original game, where player II plays $b_0, b_1, \ldots$

Assume that this is false. We then give a line of play for II in the auxiliary game which, if I uses J, will result in a loss for I, which is a contradiction.

Case 1a. $T_{h_3}(\lambda_i(a_i), \lambda i(b_i))$ and $T_{h_2}(\lambda i(a_i), \lambda_i(b_i))$ are well-founded. Let f be any isomorphism from $T_{h_3}(\lambda i(a_i), \lambda_i(b_i))$, under $<$, into cardinals $<\Omega_\Omega$. Let g be any isomorphism from $T_{h_2}(\lambda i(a_i), \lambda i(b_i))$, under $<$, into cardinals $>\Omega_{\Omega+\Omega}$. Have II play in the auxiliary game in such a way that the first parts are $b_0, b_1, \ldots$, the second parts are $g(Z(k))$, whenever II is legally required to play an ordinal for the second part of his kth move, and the third parts are $f(Z(k))$, whenever II is legally required to play an ordinal for the third part of his kth move.

It is clear than when I responds to the above line of play using J, he will play $a_0, a_1, \ldots$ for his first parts, by the definition of $a_0, a_1, \ldots$ Hence II will never play E, and so I loses.

Case 1b. $T_{h_3}(\lambda i(a_i), \lambda i(b_i))$ is well-founded, and both $T_{h_1}(\lambda i(a_i), \lambda i(b_i))$, $T_{h_2}(\lambda i(a_i), \lambda i(b_i))$ are not well-founded. In this subcase, it is important to note that if II uses only cardinals, or E, or nothing, for his second and third parts, then not only are the first part of I's moves independent of the second and third parts of II's moves; in addition, the second parts of I's moves are independent of the second parts of II's moves. That is, if I is playing according to J in the auxiliary game, then his entire moves are completely determined by the first and third parts of II's moves, as long as II is restricted to playing cardinals for his second and third parts (or E, or nothing).

Fix f as any isomorphism from $T_{h_3}(\lambda i(a_i), \lambda i(b_i))$ into cardinals $<\Omega_\Omega$. Consider the moves $(a_0, c_0), (a_1, c_1), \ldots$ for I that ensue if player I is using J and II is playing cardinals or E or nothing for his second parts, when ordinals are required, and plays $f(Z(k))$ for his third parts whenever legally required to play an ordinal for the third part of his kth move. More precisely, we consider such arbitrarily long finite initial segments of the auxiliary game. Arbitrarily long ones do exist because every finite linear ordering can be mapped isomorphically into the cardinals $>\Omega_{\Omega+\Omega}$. Since $T_{h_1}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded, clearly for some k, $c_k = E$. Fix g such that g is any isomorphism from $\{Z(a)\colon a < k\}$, under $<$, into cardinals $>\Omega_{\Omega+\Omega}$. We now describe the line of play for II which, if I follows J, will result in a loss for I. Have the first parts of II's moves be $b_0, b_1, \ldots$ For $a < k$, have the second part of II's ath move be $g(Z(a))$ if an ordinal is required. Have II play $f(Z(a))$ for the third part of his ath move, whenever an ordinal is required. Of course, have II play legally. Then the second part of I's kth move will be E, and so II will thereafter never have second parts to his moves. Obviously, II will never play E, and so I loses.

Case 2. J is a winning strategy for II.

Consider the following strategy for II in the original game. In order to determine II's nth move when I has moved $a_0, \ldots, a_n$, we consider all possible *legal* (in the auxiliary game) sequences of moves (a_0, c_0), $(b_0, d_0, e_0), \ldots, (a_n, c_n), (b_n, d_n, e_n)$ that would ensure if I, II were playing the auxiliary game and II was using strategy J and I was under the restriction that he must only play cardinals or E or nothing for his second parts, and must never play Ω_Ω or $\Omega_{\Omega+\Omega}$. By indiscernibility, since J is definable in $L(h_1, h_2, h_3)$ from $(\Omega_\Omega, \Omega_{\Omega+\Omega})$, we see that the numbers $b_0, \ldots, b_n$ are independent of exactly what the d_i and e_i are. So we set I's nth move to be b_n.

We must obtain a contradiction under the assumption that either $T_{h_1}(\lambda i(a_i), \lambda i(b_i))$ is well-founded and $T_{h_2}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded, or $T_{h_3}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded, where $b_0, b_1, \ldots$ are the moves player II will make according to the strategy just outlined for him in the original game, where player I plays $a_0, a_1, \ldots$

Case 2a. $T_{h_1}(\lambda i(a_i), \lambda i(b_i))$ is well-founded and $T_{h_2}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded. Let f be any isomorphism from $T_{h_2}(\lambda i(a_i), \lambda i(b_i))$, under $<$, into cardinals in $\Omega_{\Omega+\Omega} - \Omega_\Omega$. Have I play as follows in the auxiliary game. His first parts are $a_0, a_1, \ldots$ The second part of his kth move is $f(Z(k))$ if an ordinal is required. Since $T_{h_2}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded, clearly for some k either the second or third part of II's kth move, if he follows J, will be E, ands o I wins, contrary to the choice of J.

Case 2b. $T_{h_3}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded. In this subcase, it is important to note that if I uses only cardinals, or E, or nothing for his second parts, then the third parts of II's moves are independent of exactly what those cardinals are.

Consider the moves $(b_0, d_0, e_0)(b_1, d_1, e_1), \ldots$ for II that would ensue if player II is using J and I is playing cardinals or E or nothing for his second parts. More precisely, we consider such arbitrarily long finite initial segments of the auxiliary game. Since $T_{h_3}(\lambda i(a_i), \lambda i(b_i))$ is not well-founded, clearly there is a k such that either d_k or e_k is E. In either case, let f be any isomorphism from $\{Z(a)\colon a \leqslant k\}$, under $<$, into cardinals in $\Omega_{\Omega+\Omega} - (\Omega_\Omega + 1)$. Have the first parts of I's moves be $a_0, a_1, \ldots$ For $a \leqslant k$, have the second part of I's ath move be $f(Z(a))$ if an ordinal is required. Then clearly if II uses J then he will play an E on his kth move, and so will lose, contrary to the choice of J.

§ 3. In this section we derive the hypothesis (*), used in Section 2, from the determinateness of Δ_2^1 subsets of N^N (Δ_2^1 determinateness).

We make use of a lemma due to Jensen (unpublished) which he proves by means of an elaborate forcing argument. We give a proof of this lemma directly from a sharp version of an earlier result of Saks, that every countable admissible α is ω_1^f for some $f\colon \omega \to \omega$, using the theory of relative hyperarithmeticity and hyperjump.

Definition 3.1. We let $f \leqslant_T g$ mean f is recursive in g, $f =_T g$ mean $f \leqslant_T g \;\&\; g \leqslant_T f$. We let $f \leqslant_h g$ mean f is hyperarithmetic (hyp) in g, $f =_h g$ mean $f \leqslant_h g \;\&\; g \leqslant_h f$. We let $f <_T g$ mean $f \leqslant_T g \;\&\; g \leqslant_T f$, $f <_h g$ mean $f \leqslant_h g \;\&\; {\sim} g \leqslant_h f$. We let $\mathrm{TJ}(f)$ be the (characteristic function of the) *Turing jump of* f. We let $\mathrm{HJ}(f)$ be the (characteristic function of the) *hyperjump of* f. We let $\mathrm{TJ}^0(f) = \mathrm{HJ}^0(f) = f$, $\mathrm{TJ}^{k+1}(f) = \mathrm{TJ}^{k+1}(f) = \mathrm{TJ}(\mathrm{TJ}^k(f))$, $\mathrm{HJ}^{k+1}(f) = \mathrm{HJ}(\mathrm{HJ}^k(f))$. We let $A(f) = \{\alpha\colon \alpha$ is admissible in $f\}$. Let ω_1^f be the least element of $A(f)$, ω_{k+1}^f be the least element of $A(f)$ greater than ω_k^f. Let $A = \{\alpha\colon \alpha$ is admissible$\}$.

Lemma 3.1. *Suppose $\alpha < \Omega$, $h\colon \omega \to \omega$, $\alpha \in A(h)$. Then there are uncountably many g with $h \leqslant_T g$ such that $A(h) \cap \alpha = A(g) \cap \alpha$.*

Proof. If f is a function from ω into ω generic over $L^h(\alpha)$, then $A((f, h)) \cap \alpha = A(h) \cap \alpha$. Furthermore, there are uncountably many such f.

Lemma 3.2. *Suppose $\beta \in \alpha \in A(h) \cap \Omega$. Then there is an f such that $A(f) \cap \alpha = (\alpha - (\beta+1)) \cap A(h)$, and $h \leqslant_T f$.*

Proof. Choose f^* to be a well-ordering of type β generic over $L^h(\alpha)$, and set $f = (f^*, h)$.

The following is a result of Sacks.

Lemma 3.3. *Suppose $\beta \in \alpha \in A(h) \cap \Omega$. Then there is a g such that $A(g) \cap \alpha = (\alpha - \beta) \cap A(h)$ and $h \leqslant_T g$.*

Proof. Fix f as in Lemma 3.2. Following the proof of $(\exists g)(\omega_1^g = \beta)$, which can be relativized to $(\exists g)(\omega_1^g = \beta \;\&\; h \leqslant_T g)$ in Friedman and Jensen [3], we form a model of an auxiliary theory T, and extract (g, h) from that model. But going through the completeness theorem for the infinitary language $\mathfrak{L}_\alpha$, $(\mathfrak{L}_\alpha(h))$, used there, we see that the model of T can be chosen hyperarithmetically in f, and so g can be chosen hyp in f with $\omega_1^g = \alpha$, and so $A(g) \cap \alpha = (\alpha - \beta) \cap A(h)$.

Definition 3.2. Let $Z(i)$ be the following assertion: let $\beta \in \alpha \in \Omega$, $h\colon \omega \to \omega$, $\beta \in A(h)$. Let $\alpha_1, \ldots, \alpha_i$ be the first i elements of $A(h)$, $\alpha_i \in \beta \in \alpha$. Then there is an f such that the first $i+1$ elements of $A(f)$ are $\alpha_1, \ldots, \alpha_i, \beta$, and $h \leqslant_T f$ and $A(f) \cap (\alpha - \beta) = A(h) \cap (\alpha - \beta)$.

Our immediate aim is to prove the result of Jensen (unpublished) that $(\forall i)(Z(i))$.

Lemma 3.4. $Z(0)$.

Proof. This is, verbatim, Lemma 3.3.

Lemma 3.5. *Suppose P is a Σ_1^1 predicate with parameter h, and there are uncountably many solutions to P, and $\mathrm{HJ}(h) \leqslant_T g$. Then $(\exists f)(P(f) \;\&\; (f, \mathrm{HJ}(h)) =_h g)$.*

Proof. Let $Q(f)$ be the Σ_1^1 predicate $P(f) \;\&\; {\sim} f \leqslant_h h$. Then by hypotheses, Q has solutions. Put Q in the form $(\exists a)(\forall n)(R(\bar{a}(n), \bar{f}(n),$

$\hat{h}(n))$, where R is a recursive predicate. We simultaneously define functions f, a such that $\mathrm{A}n\big(R(\tilde{a}(n), \tilde{f}(n), \tilde{h}(n))\big)$, in stages. Suppose we have reached stage k, and have by then defined $f(0) = a_0, \ldots, f(k-1) = a_{k-1}$, $a(0) = b_0, \ldots, a(q) = b_q$, and $(\exists f)(\exists a)\big((\forall n) R(\tilde{a}(n), \tilde{f}(n), \tilde{h}(n))\ \&\ f$ extends $(a_0, \ldots, a_{k-1})$ & a extends $(b_0, \ldots, b_q)\big)$.

Case 1. There are at least two number a_k such that $(\exists f)(\exists a)\big((\forall n) R(\tilde{a}(n), \tilde{f}(n), \tilde{h}(n))\ \&\ f$ extends $(a_0, \ldots, a_{k-1}, a_k)$ & a extends $(b_0, \ldots, b_q)\big)$. Then if $g(q+1) = 0$ then define $f(k)$ to be the least such a_k. If $g(q+1) = 1$ then define $f(k)$ to be the second least such a_k. Define $a(q+1)$ to be least with $(\exists f)(\exists a)(\forall n) R(\tilde{a}(n), \tilde{f}(n), \tilde{h}(n))\ \&\ f$ extends $(a_0, \ldots, a_{k-1}, f(k))$ & a extends $(b_0, \ldots, b_q, a(q+1))$.

Case 2. Case 1 does not apply. Then define $f(k)$ such that $(\exists f)(\exists a)\big((\forall n) R(\tilde{a}(n), \tilde{f}(n), \tilde{h}(n))\ \&\ f$ extends $(a_0, \ldots, a_{k-1}, f(k))$ & a extends $(b_0, \ldots, b_q)\big)$. Do not define $a(q+1)$.

To see that a gets defined we have to show that Case 1 applide infinitely often. Suppose not. Then there would be finite sequences $(a_0, \ldots, a_k)$, $(b_0, \ldots, b_q)$ such that there is one and only one f with $(\exists a)\big((\forall n) R(\tilde{a}(n), \tilde{f}(n), \tilde{h}(n))\ \&\ f$ extends $(a_0, \ldots, a_k)$ & a extends $(b_0, \ldots, b_q)\big)$. But then this unique f would be hyp in h, which is a contradiction. By the way f, a were defined, clearly $(f, a) \leqslant_T \mathrm{TJ}(g)$. Since Case 1 applied infinitely often, we have $g \leqslant_T \mathrm{TJ}\big(f, \mathrm{HJ}(h)\big)$. Hence $\big(f, \mathrm{HJ}(h)\big) =_h g$.

Lemma 3.6. *$Z(n) \to Z(n+1)$.*

Proof. Suppose $Z(n)$. Let $\beta \in a \in \Omega \cap A(h)$. Let $a_1, \ldots, a_{n+1}$ be the first $n+1$ elements of $A(h)$, $a_{n+1} \in \beta \in a$. Let $P(f) \equiv$ "the first $n+1$ elements of $A(f)$ are $a_1, \ldots, a_{n+1}$ and $h \leqslant_T f$". Then by Lemma 3.1, P has uncountably many solutions. Note that P is a Σ_1^1 predicate with parameter $\mathrm{HJ}^n(h)$, since $A\big(\mathrm{HJ}^n(h)\big) = A(h) - \{a_1, \ldots, a_n\}$. For we can express $P(f)$ by saying that every ordinal recursive in $\mathrm{HJ}^n(h)$ has $< n+1$ ordinals $\leqslant$ it that are admissible in h, and every well-ordering present in $L^f(a_n+1)$ is isomorphic to some linear ordering recursive in $\mathrm{HJ}^n(h)$. Now by Lemma 3.5, $(\exists f)\big(P(f)\ \&\ \big(f, \mathrm{HJ}^{n+1}(h)\big) =_h g\big)$, for any g with $\mathrm{HJ}^{n+1}(h) \leqslant_T g$. Note that $\beta \in A\big(\mathrm{HJ}^{n+1}(h)\big)$, and so by Lemma 3.3 choose g with $A(g) \cap a = (a-\beta) \cap A\big(\mathrm{HJ}^{n+1}(h)\big)$, and $\mathrm{HJ}^{n+1}(h) \leqslant_T g$. Then choose f with $P(f)\ \&\ \big(f, \mathrm{HJ}^{n+1}(h)\big) =_h g$. Note $\omega_1^f = a_1\ \&\ \ldots\ \&\ \omega_{n+1}^f = a_{n+1}$. Note $\beta \in A\big(\mathrm{HJ}^{n+1}(h)\big)$. So $\omega_{n+2}^f = \beta$. Since $f \leqslant_h g$, clearly $\omega_{n+2}^f = \beta$, and $A(f) \cap (a-\beta) = A(g) \cap (a-\beta) = A(h) \cap (a-\beta)$. So $Z(n+1)$ follows from $Z(n)$.

Lemma 3.7 (Jensen). *$(\forall n)\big(Z(n)\big)$. In particular, for every h, if $a_1, \ldots, a_n$ are countable and admissible in h, then there is an f such that $h \leqslant_T f\ \&\ \omega_1^f = a_1\ \&\ \ldots\ \&\ \omega_n^f = a_n$.*

Proof. From Lemmas 3.4 and 3.6.

We now show, using Lemma 3.7, that Δ_2^1 determinateness implies (*) of Section 2.

Definition 3.3. An ordinal $a < \Omega$ is called *ω-stable in h* if and only if $\big(L^h(a), \varepsilon\big)$ is an elementary submodel of $\big(L^h(\Omega), \varepsilon\big)$.

The following is well known from the theory of indiscernibles. (Corollary 3.2 of Silver [9].)

Lemma 3.8. *Suppose for each h: $\omega \to \omega$ there is a $Y \subset \Omega$, Y uncountable, such that for any formula $\varphi(v_0, \ldots, v_n, v)$ and pair of strictly increasing sequences of ordinals $a_0, \ldots, a_n, \beta_0, \ldots, \beta_n \in Y$, we have $L^h(\Omega) \models \varphi(a_0, \ldots, a_n, h) \equiv \varphi(\beta_0, \ldots, \beta_n, h)$. Then (*) holds.*

Lemma 3.9. *Suppose for each h there is a closed $Y \subset \Omega$, Y uncountable, such that for any formula $\varphi(v_0, \ldots, v_n, v)$ and any pair of strictly increasing sequences $a_0, \ldots, a_n, \beta_0, \ldots, \beta_n$ of elements of Y, we have $L^h(a_n) \models \varphi(a_0, \ldots, a_{n-1}, h)$ if and only if $L^h(\beta_n) \models \varphi(\beta_0, \ldots, \beta_{n-1}, h)$. Then (*) holds.*

Proof. For each h choose Y_h as in the hypothesis. Then consider $Y_h \cap \{a\colon a$ is ω-stable in $h\}$. Since each Y_h is closed and unbounded, clearly $Y_h \cap \{a\colon a$ is ω-stable in $h\}$ is uncountable (and closed). Now let $a_0, \ldots, a_n, \beta_0, \ldots, \beta_n$ be a pair of strictly increasing sequences of ordinals in $Y_h \cap \{a\colon a$ is ω-stable in $h\}$, and $\varphi(v_0, \ldots, v_n, v)$ a formula. Choose $a, \beta \in Y_h \cap \{a\colon a$ is ω-stable in $h\}$, $a_n < a$, $\beta_n < \beta$. Then $L^h(a) \models \varphi(a_0, \ldots, a_n, h)$, $L^h(\beta) \models \varphi(\beta_0, \ldots, \beta_n, h)$. Since a, β are ω-stable in h, we have $L^h(\Omega) \models \varphi(a_0, \ldots, a_n, h) \equiv \varphi(\beta_0, \ldots, \beta_n, h)$.

Lemma 3.10. *Suppose Δ_2^1 determinateness. Then let $Y \subset \omega^\omega$ be Δ_2^1 and have $(\forall f)\big((f \in Y\ \&\ g =_T f) \to g \in Y\big)$. Then*

$$(\exists f)\big((\forall g)(f \leqslant_T g \to g \in Y) \vee (\forall g)(f \leqslant_T g \to g \notin Y)\big).$$

Proof. This simple, yet powerful lemma is due to Martin. One considers the Δ_2^1 set $\{(f, g)\colon f \in Y\ \&\ g \leqslant_T f\}$. If the game based on this set has winning strategy h for player I then $(\forall f)(h \leqslant_T f \to f \in Y)$. If for II, then $(\forall f)(h \leqslant_T f \to f \notin Y)$.

Theorem 3. *Δ_2^1 determinateness implies (*).*

Proof. Assume Δ_2^1 determinateness. Then let h: $\omega \to \omega$, and form $B_\varphi = \{f\colon L^h(\omega_{k+1}^f) \models \varphi(\omega_1^f, \ldots, \omega_k^f, h)\}$, for each φ of $k+1$ free variables. Note that B_φ satisfies the hypothesis of Lemma 3.10. Hence let f have $(\forall g)(f \leqslant_T g \to f \in B_\varphi)$, the other case being symmetric. Let $a_1, \ldots, a_k, a_{k+1}$ be a strictly increasing sequence of ordinals in $A(f) \cap \Omega$. Then by Lemma 3.7, there is a g with $f \leqslant_T g$ such that $\omega_1^g = a_1\ \&\ \ldots\ \&\ \omega_k^g = a_k\ \&\ \omega_{k+1}^g = a_{k+1}$. Hence $g \in B_\varphi$ and $L^h(a_{k+1}) \models \varphi(a_1, \ldots, a_k, h)$.

To summarize, for each φ of $k+1$ free variable there is an f such that either for all strictly increasing $\alpha_1, \ldots, \alpha_{k+1} \in A(f) \cap \Omega$, $L^h(\alpha_{k+1}) \models \varphi(\alpha_1, \ldots, \alpha_k, h)$, or for all strictly increasing $\alpha_1, \ldots, \alpha_{k+1} \in A(f) \cap \Omega$, $L^h(\alpha_{k+1}) \models \sim\varphi(\alpha_1, \ldots, \alpha_k, h)$. Now for each formula φ, choose an f_φ with this property. Then choose g such that each $f_\varphi \leqslant_T g$. Then clearly $Y_h = \{\alpha\colon \alpha \text{ is } \infty\text{-stable in } g\} \subset A(g)$, and hence for all pairs of strictly increasing sequences $\alpha_1, \ldots, \alpha_{k+1}, \beta_1, \ldots, \beta_{k+1}$, all φ of $k+1$ free variables, we have $L^h(\alpha_{k+1}) \models \varphi(\alpha_1, \ldots, \alpha_k, h)$ if and only if $L^h(\beta_{k+1}) \models \varphi(\beta_1, \ldots, \beta_k, h)$. So we are done by Lemma 3.9, since each Y_h is closed and uncountable.

§ **4.** We mention some open questions.

We will state the open problems only for classes of subsets of $N^N \times N^N$ definable in various ways from a parameter; in every single case, there is the obvious corresponding open problem for classes defined without parameters. Thus we remark that Theorem 1 can be easily relativized to obtain

THEOREM 1'. *Let $x \subset \omega$. Then there is a subset Y of $N^N \times N^N$ which is Σ_1^1 in x such that there is no winning strategy for Y present in any $L((x, y))$, where y is C-generic over $L(x)$ for some notion of forcing $C \in L(x)$.*

1. *Does Σ_1^1 determinateness imply* (*) *of Section 2?*

Connected with 1 is the following question.

2. *Suppose that for each $x \subset \omega$ there is a set of integers, y, such that for all notions of forcing, $C \in L(x)$, and all C-generic z, we have $y \notin L((x, z))$. Then can we conclude* (*)?

As discussed in the Introduction Σ_1^1 determinateness implies that Ω is satisfied to be inaccessible in every $L(x)$, $x \subset \omega$.

3. *Does determinateness for every element in the σ-algebra generated by the Σ_1^1 sets imply that Ω is satisfied to be mahlo in every L^x, $x \subset \omega$?*

4. *Does* (*) *imply determinateness for provably Δ_2^1 sets? A provably Δ_2^1 set S is a set such that for some Σ_2^1 predicates $P(f)$, $Q(f)$, we have $L \models P(f) \equiv \sim Q(f)$, and $S = \{f\colon P(f)\}$. If not, then does* (*) *imply determinateness for the σ-algebra generated by the Σ_1^1 sets?*

Note that our proof of Theorem 3 shows that provable Δ_2^1 determinateness implies (*).

5. *Does determinateness for the σ-algebra generated by Σ_1^1 sets imply that* ZF + ∃ *measurable cardinal is consistent?*

This 5 was suggested by a result of Solovay (personal communication) that Σ_3^1 determinateness proves Con(ZF + ∃ measurable cardinal).

6. *Does Δ_2^1 determinateness fail in every forcing extension of the universe constructed from a normal ultrafilter?*

It is known that Δ_2^1 determinateness fails in the universe constructed from a normal ultrafilter, since by Martin and Solovay [5], Δ_2^1 determinateness implies that Δ_3^1 subsets of ω are not a basis for Π_2^1 subsets of ω^ω, but by Silver [8], they are satisfied to be a basis in L^μ.

Added in proof. D. A. Martin has recently obtained a positive answer to 3, a negative answer to both parts of 4, and a positive answer to 5. Since the "axiom" of determinateness (for any particular class of sets) does not remotely possess the character of an axiom of set theory as it stands, it is of interest to investigate, e.g., the "axiom" of degree determinateness, whose statement (for the class of Δ_2^1 sets) is the consequence of Lemma 3.10, and whose meaning is less obscure (and transparent to any recursion theorist). It would be of considerable intrest to reinvestigate the questions asked for determinateness, for degree determinateness. For instance, our Theorem 3 goes through, but not our Theorems 1, 2.1 and 2.2. In addition, the result of Davis, and the recent recents of Martin do not go through as they stand. In addition, one can employ other types of degrees than Turing degrees. Perhaps there is some general notion of degree perhaps lattice-theoretic for which one could study "full degree" determinateness.

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