Monotone decompositions of continua irreducible about a finite subset

by

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This paper is concerned with monotone decompositions of continua which are irreducible about a finite subset. The term "continuum" is taken to mean a compact connected metric space. For definitions not given here, the reader is referred to [5].

The present work deals only with hereditarily decomposable continua and is essentially an extension of the work of Miller [4]. I am grateful to the referee for bringing to my attention the work of Kuratowski in this area ([2], [3]) which does not assume hereditary decomposability. These results, along with the work of Thomas [7], would significantly strengthen the present theorems. However, they seem to invite a different approach as well as suggesting further areas of investigation and, consequently, are left for future investigation.

Miller [4] has shown that a hereditarily decomposable irreducible continuum M contains two, and only two, mutually exclusive continua E_1 and E_2, called the E-continua of M, such that M is irreducible between two of its points if and only if one of them belongs to E_1 and the other to E_2.

In the same paper it is shown that a hereditarily decomposable irreducible continuum M has a monotone upper semi-continuous decomposition G into an arc, the ends of which are the E-continua of M, such that no member of G has an interior with respect to M. In the present discussion, this result will be referred to as Miller's decomposition theorem.

This paper generalizes the above results. In particular, Section 1 develops the concept of the ends of a hereditarily decomposable continuum which is irreducible about a finite subset while section 2 is devoted to proving a generalization (Theorem 2.4) of Miller's decomposition theorem.

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The third section, which is concerned with the application of Theorem 2.4 to K-like continua, where K is a tree, was motivated in part by a theorem of Bing [1] which is similar to Miller's decomposition theorem. In Section 3 it is shown that a generalization of this theorem of Bing's is a corollary to Theorem 2.4. Examples are given to show that if M is a K-like continuum with the decomposition G determined by Theorem 2.4, K and G are not necessarily as closely related as one might hope.

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1. n-ended continua. The following sequence of theorems, while primarily intended as a basis for proving Theorem 2.4, can be compared to Lemma A of [4].

LEMMA 1.1. *Suppose M is a hereditarily decomposable continuum which is irreducible about the subset $W = \{A_1, \ldots, A_n\}$ but is not irreducible about a subset containing fewer than n points. For $i = 1, 2, \ldots, n$, let E_i be the set of all points P of M such that M is irreducible about $(W-A_i)+P$. If M_i is a subcontinuum of M that is irreducible about $W-A_i$, then $\overline{M-M_i}$ is a continuum that is irreducible from E_i to M_i. Moreover, E_i is an E-continuum of $\overline{M-M_i}$.*

Proof. M is irreducible about $M_i+\{A_i\}$, so by Theorem 48 in Chapter I of [5], $M-M_i$ is connected. Thus $I = \overline{M-M_i}$ is a continuum, which is hereditarily decomposable since M is hereditarily decomposable.

It is a straightforward matter to show that I is irreducible from E_i to M_i and that the E-continuum of I containing E_i contains no points other than those in E_i.

THEOREM 1.2. *Let M be an hereditarily decomposable continuum which is irreducible about n, but no fewer, of its points. Then there are n mutually disjoint subcontinua $E_1, E_2, \ldots, E_n$ of M, none of which has an interior with respect to M, such that M is irreducible about the finite point set K if and only if K intersects each of $E_1, E_2, \ldots, E_n$. Moreover, the n subcontinua are unique.*

Proof. Let $A_1, A_2, \ldots, A_n$ be n points such that M is irreducible about $W = \{A_1, A_2, \ldots, A_n\}$. For each $i = 1, \ldots, n$, let E_i be defined as in Lemma 1.1. It is clear that A_i is in E_j if and only if $i = j$.

Having defined the point sets $E_1, E_2, \ldots, E_n$, let us now verify the following statements about them:

(i) For each i, E_i is a subcontinuum of M which has no interior with respect to M.

(ii) If $j \neq i$, then E_i and E_j have no point in common.

(iii) If K is a subset of M which intersects each of the E_i, then M is irreducible about K.

(iv) If M is irreducible about the finite subset K, then K intersects each of the E_i.

(v) If $F_1, F_2, \ldots, F_n$ are subcontinua of M such that M is irreducible about the finite subset K if and only if K intersects each of the F_i, then $\{F_1, \ldots, F_n\} = \{E_1, \ldots, E_n\}$.

Proof of (i). By Theorem 46 of Chapter 1 of [5], there is a subcontinuum M_i of M which is irreducible about $W-A_i$. M_i does not intersect E_i.

By Lemma 1.1, $I = \overline{M-M_1}$ is an irreducible continuum and E_i is an E-continuum of I. This implies that E_i is a subcontinuum of I which has no interior with respect to I. It follows that E_i is a subcontinuum of M with no interior with respect to M.

Proof of (ii). Suppose there is a point P which belongs to both E_i and E_j, where $i \neq j$. M is not irreducible about $[W-(A_i+A_j)]+P$, so there is a proper subcontinuum M' of M containing $[W-(A_i+A_j)]+P$. Then $M'+E_i$ contains $(W-A_j)+P$, about which M is irreducible. Thus $M'+E_i = M$, so $M-E_i \subset M'$. But this implies that M' is M, contradicting the supposition that M' be a proper subcontinuum of M.

Proof of (iii). Let K be a subset of M which intersects each of the E_i. Suppose M is not irreducible about K. Then there is a proper subcontinuum M' of M containing K. There is a j such that A_j is not in M'. K contains a point B_j of E_j.

Since M' intersects each of the E_i, $M'' = M'+(\bigcup_{i\neq j} E_i)$ is connected, and thus is a subcontinuum of M. M'' cannot be M because it does not contain A_j. But M'' does contain $(W-A_j)+B_j$, about which M is irreducible.

Thus M is irreducible about K.

Proof of (iv). Denote the finite point set K, about which M is irreducible, as $\{P_1, P_2, \ldots, P_t\}$. Suppose there is an i such that K does not intersect E_i. For each $j = 1, 2, \ldots, t$, M is not irreducible about $(W-A_i)+P_j$, so there is a proper subcontinuum N_j which is irreducible about this set. N_j does not intersect E_i. Then $N = \bigcup_{j=1}^{t} N_j$ is also a proper subcontinuum of M, proper because it does not intersect E_i. But N contains K, and M is irreducible about K.

Proof of (v). Suppose $F_1, F_2, \ldots, F_n$ are subcontinua of M such that M is irreducible about the finite subset K of M if and only if K intersects each of the F_i. Since M is not irreducible about a subset containing $n-1$ points, no two of the F_i can have a point in common. M is irreducible about W, so each member of W belongs to exactly one of the F_i. Let us assume that the F_i are indexed so that, for each i, A_i belongs to F_i.

It is then a routine matter to show that $F_i = E_i$ for each i. Thus

$$\{F_1, \ldots, F_n\} = \{E_1, \ldots, E_n\}.$$

This completes the proof of Theorem 1.2.

If M is a hereditarily decomposable continuum which is irreducible about n, but no fewer, of its points, then the subcontinua $E_1, E_2, \ldots, E_n$ of Theorem 1.2 will be called the *ends* of M. It will be said that *M has n ends*, or equivalently, that *M is an n-ended continuum.*

In the following theorems it will be assumed that M is a hereditarily decomposable n-ended continuum whose ends are $E_1, E_2, \ldots, E_n$ and that $W = \{A_1, A_2, \ldots, A_n\}$, where, for each i, A_i is a point of E_i (making W a set about which M is irreducible).

The statement that the point set K is strongly connected means that if P and Q are two points of K, then K contains a continuum which contains both P and Q. (See [5]).

THEOREM 1.3. *For each $i = 1, 2, \ldots, n$, $M - E_i$ is strongly connected.*

Proof. Let P and Q be two points of $M - E_i$. Since P and Q are not in E_i there are proper subcontinua M_P and M_Q containing $(W - A_i) + P$ and $(W - A_i) + Q$, respectively, neither of which intersects E_i. Then $M' = M_P + M_Q$ is a subcontinuum of M containing P and Q which is a subset of $M - E_i$.

THEOREM 1.4. *If K is a subcontinuum of M which intersects E_i, then either K is contained in E_i or E_i is contained in $\mathrm{Int}_M(K)$, the interior of K with respect to M.*

Proof. Assume that K intersects, but is not contained in, E_i. Let P be a point of $K \cap E_i$ and Q be a point of $K - E_i$.

Since Q is not in E_i, M is not irreducible about $(W - A_i) + Q$. Then there is a proper subcontinuum M' of M containing $(W - A_i) + Q$, and M' does not intersect E_i.

$M' + K$ is a subcontinuum of M containing $(W - A_i) + P$. Since P is in E_i, this means that $M' + K$ is M. Then if X is any point of E_i there is a region R containing X such that R does not intersect M'. Then R is a subset of K. Thus E_i is in the interior of K with respect to M.

LEMMA 1.5. *If M' is a subcontinuum of M which is irreducible about the subset $W' = \{A_{j_1}, A_{j_2}, \ldots, A_{j_t}\}$ of W, where $2 \leqslant t \leqslant n$ and M' has t ends, then $E_{j_1}, E_{j_2}, \ldots, E_{j_t}$ are the ends of M'.*

Proof. Let $F_1, F_2, \ldots, F_t$ be the ends of M', where, for each i, A_{j_i} is in F_i.

By Theorem 1.2, for each i, F_i is a subcontinuum of M' with no interior with respect to M'. It follows that F_i has no interior with respect to M. F_i intersects E_{j_i}, for A_{j_i} is in both F_i and E_{j_i}. Then, by Theorem 1.4, F_i is contained in E_{j_i}. Also by Theorem 1.4, because $t \geqslant 2$, E_{j_i} is contained in M'.

Suppose there is a point P of E_{j_i} which is not in F_i. Because P is not in F_i, there is a proper subcontinuum M'' of M' which is irreducible about $(W' - A_{j_i}) + P$. M'' does not intersect F_i; in particular, A_{j_i} is not in M''.

Now M'' is a proper subcontinuum of M which intersects E_{j_i}, and E_{j_i} is not contained in M''. Then, by Theorem 1.4, M'' is a subset of E_{j_i}. However, $W' - A_j$ (which is non-void since $t \geqslant 2$) is in M'' but does not intersect E_{j_i}. Thus, for each i, $F_i = E_{j_i}$.

If X, Y and Z are three points of the point set M, the statement that Y *weakly separates* X *from* Z *in* M means that Y is in every subcontinuum of M which contains X and Z.

LEMMA 1.6. *If P is in $M - W$ and A_j is in W, there is a point A_k of W such that P weakly separates A_j from A_k in M.*

THEOREM 1.7. *If K is a subcontinuum of M which is irreducible about a finite subset and E_i intersects K, then either K is contained in E_i or E_i is an end of K.*

Proof. There is a positive number t such that K has t ends. If t is one, K is degenerate and the theorem is trivial. Let us assume, then, that $t \geqslant 2$.

Let $V = \{B_1, B_2, \ldots, B_t\}$ be a point set about which K is irreducible. Let $F_1, F_2, \ldots, F_t$ be the ends of K, where for each j, B_j is in F_j.

Suppose that the end E_i of M intersects, but does not contain, K and is not an end of K. The ends of K have no interior with respect to M, so, by Theorem 1.4, E_i does not intersect any of the F_j. (Also, by Theorem 1.4, $E_i \subset \mathrm{Int}_M(K)$.)

Let $P \in E_i$. There is a proper subcontinuum I of K which is irreducible from B_1 to P. Let G_1 and G_2 denote the ends of I, where $B_1 \in G_1$ and $P \in G_2$.

G_2 is a proper subcontinuum of M which intersects E_i, and $\mathrm{Int}_M(G_2) = \emptyset$, so, by Theorem 1.4, $G_2 \subset E_i$. Also the proper subcontinuum I of M intersects, but is not contained in, E_i, so $E_i \subset \mathrm{Int}_M(I)$. It follows, then, that $G_2 \subset \mathrm{Int}_M(I)$, and thus $G_2 \subset \mathrm{Int}_K(I)$.

There is an m such that P weakly separates B_1 from B_m in K. B_m is not in G_2, and $I - G_2$ is, by Theorem 1.3, strongly connected, so B_m is not in I. There is a proper subcontinuum K' of K which is irreducible from B_m to I, and K' does not intersect G_2 since $G_2 \subset \mathrm{Int}_K(I)$ and $K' \cap I \subset \overline{K - I}$.

Let Z be a point of $K' \cap I$. There is a subcontinuum I' of I which is irreducible from Z to B_1. Since $Z \notin G_2$, I' contains no point of G_2.

Then $I'+K'$ is a subcontinuum of K containing B_1 and B_m but not P. However, this contradicts the fact that P weakly separates B_1 from B_m in K. Therefore, either E_i contains K or E_i is an end of K.

COROLLARY 1.8. *$M-\bigcup_{i=1}^{n} E_i$ is strongly connected.*

Proof. Let P and Q be two points of $M-\bigcup_{i=1}^{n} E_i$. There is a proper subcontinuum K of M containing P and Q. K has two ends F_P and F_Q containing P and Q, respectively.

For each $i=1,\ldots,n$, neither P nor Q is in E_i, so E_i is not an end of K, and K is not contained in E_i. Then, by Theorem 1.7, K does not intersect E_i.

Thus K is contained in $M-\bigcup_{i=1}^{n} E_i$. Therefore $M-\bigcup_{i=1}^{n} E_i$ is strongly connected.

2. Monotone decompositions of n-ended continua. The decomposition theorem of Miller can be generalized to n-ended continua, where n is an integer greater than or equal to 2. If n is greater than 2, the decomposition space is not an arc, but a tree. A *tree* is a locally connected, hereditarily unicoherent continuum which is the union of a finite number of arcs. Every point of a tree is either an endpoint or a cut point.

It will be convenient to distinguish between cut points of a tree which separate it into two components and those which separate it into more than two components. It is known that, if P is a point of a tree T, the number of components of $T-P$ is the same as the Menger order of P with respect to T. [8] (For a definition of Menger order, see [5].)

Several basic properties of trees will be of use in the proof of the following theorem. The following lemmas can be proved using ideas in [8].

LEMMA 2.1. *If T is an n-ended tree, then T has at most $n-2$ points of Menger order greater than 2.*

LEMMA 2.2. *If T is a tree and H and K are disjoint closed connected subsets of T, then there is a point of order 2 which separates H from K in T.*

LEMMA 2.3. *Suppose T is a tree and P, X, Y, and Z are four points of T. If P separates each of X, Y, and Z from the other two in T, then P is a point of order greater than 2 in T.*

THEOREM 2.4. *Let M be an hereditarily decomposable continuum which has n ends, where n is an integer greater than or equal to 2. Then there is a monotone decomposition G of M such that*

(1) *no member of G has an interior with respect to M,*

(2) *G is an n-ended tree with respect to its elements, and*

(3) *the end elements of G are the ends of M.*

Proof (by induction on n). Miller's theorem establishes the theorem for $n=2$.

Assume that n is greater than or equal to 3 and that the theorem holds for all- k-ended continua, where k is less than n.

Let M be an n-ended continuum, with ends $E_1,\ldots,E_n$. For each i, let A_i be a point of E_i. Then M is irreducible about $W=\{A_1,\ldots,A_n\}$. M is not irreducible about $W'=W-A_n$, so there is a proper subcontinuum M' of M which is irreducible about W'.

M' does not intersect E_n, but it does intersect each of the other ends of M, of which there are at least two, so, by Lemma 1.5, each of $E_1,\ldots,E_{n-1}$ is an end of M'. This means that M' must be an $(n-1)$-ended continuum. M' is, of course, hereditarily decomposable.

By the induction hypotheses, there is a monotone decomposition G' of M' such that no member of G' has an interior with respect to M' and G' is an $(n-1)$-ended tree with end elements $E_1,\ldots,E_{n-1}$.

Let K be the set of all members of G' which intersect $\overline{M-M'}$. It is relatively easy to verify that K is a subcontinuum of G'.

Let $I=\overline{M-M'}$. By Lemma 1.1, I is a continuum which is irreducible from E_n to M'. I also is hereditarily decomposable. Thus, by Miller's theorem, there is a monotone decomposition H of I such that no member of H has an interior with respect to I and H is an arc with respect to its elements. E_n is, by Theorem 1.7, one of the ends of I; the other end, which we will call h_K, contains $I\cap M'$.

Since, by Theorem 1.4, each end of M' is in the interior of M' with respect to M, none of the ends of M' is in K. $I\cap M'$ is contained in h_K, so $K^*\cap I$ is contained in h_K.

Let $g_K=h_K+K^*$, and let $G=(G'-K)+(H-h_K)+\{g_K\}$.

We now show that G satisfies the requirements of the theorem. In order to do this, we prove the following:

(i) The members of G are subcontinua of M.

(ii) No two members of G intersect.

(iii) No member of G has an interior with respect to M.

(iv) G is upper semi-continuous.

(v) G is an n-ended tree with respect to its elements, and the ends of G are $E_1, E_2,\ldots,E_n$.

Proof of (i). It is clear that the members of G which belong to either G' or H are subcontinua of M. The only other member of G is g_K. K is a collection of subcontinua of M which intersect h_K, so $(K+\{h_K\})^*=K^*+h_K=g_K$ is also a subcontinuum of M.

Proof of (ii). Because G' is upper semi-continuous, no two members of $G'-K$ intersect; likewise, no two members of $H-h_K$ intersect. Further-more, since $(G'-K)^*$ and $(H-h_K)^*$ are contained in $M'-I$ and $I-M'$ respectively, no member of $G'-K$ intersects any member of $H-h_K$.

The only other possibility is for some member of G other than g_K to intersect g_K. However, it is not difficult to show that the upper semi-continuity of G' and H prevents this from happening.

Thus no two members of G intersect.

Proof of (iii). Clearly, no member of $G'-K$ or of $H-h_K$ has an interior with respect to M.

It is a straightforward matter to show that K^* has no interior with respect to M. Thus, since h_K also has no interior with respect to M, it follows that g_K has no interior with respect to M.

Thus no member of G has an interior with respect to M.

Proof of (iv). The fact that G' and H are upper semi-continuous assures us that G is upper semi-continuous at all elements other than g_K.

It is a routine matter to show that, if D is a domain containing g_K, then D contains a domain D' such that every member of G which intersects D' is contained in D. Thus G is upper semi-continuous at g_K also.

Proof of (v). G' is an $(n-1)$-ended tree with respect to its elements, and the end elements of G' are $E_1, \ldots, E_{n-1}$. Clearly $G'' = (G'-k)+\{g_K\}$ is also an $(n-1)$-ended tree. Similarly, since H is an arc with respect to its elements, $H' = (H-h_K)+\{g_K\}$ is an arc, and the end elements of H' are g_K and E_n.

Then $G''+H$, which is G, is an n-ended tree with respect to its elements, and the end elements of G are $E_1, \ldots, E_n$.

This completes the proof of Theorem 2.4.

In considering the decomposition of M described in Theorem 2.4, it is apparent that the decomposition G is uniquely determined, so that the result is a specific decomposition. Note that the requirement that G satisfy properties (1) and (3) assures that G is *minimal* (cf. [8]) in the sense that if H is any decomposition of M into a tree and $g \in G$ then there is an element h of H such that $g \subset h$.

Thus as a result of Theorem 2.4 we have the following:

COROLLARY 2.5. *If M is an hereditarily decomposable continuum with n ends ($n \geqslant 2$), and G is the decomposition of M described in Theorem 2.4, then G is a minimal monotone decomposition of M into a tree.*

3. Monotone decompositions of K-like continua.

If K is a tree, the statement that the continuum M is *K-like* means that, for every positive number ε, there is an ε-K-chain covering M.

A theorem of Rosen (Theorem 3 in [4]) implies that, if K is a tree with n endpoints and the continuum M is K-like, then M is irreducible about a subset containing n points.

Thus a generalization of Bing's theorem, which was mentioned earlier, is a corollary to Theorem 2.4.

COROLLARY 3.1. *If K is a tree with m endpoints and the hereditarily decomposable continuum M is K-like, then there is an upper semi-continuous decomposition G of M such that*

(1) *no member of G has an interior with respect to M,*

(2) *G is a tree with respect to its elements, and*

(3) *the number of end elements of G is less than or equal to m.*

In the following discussion, assume that K is a tree with m endpoints, M is an hereditarily decomposable K-like continuum, n is the number of ends of M, and G is the decomposition of M described in the proof of Theorem 2.4.

The relationship between K and G is not as close as one might hope. For instance, n is not necessarily equal to m. It may even be the case that there is no n-ended tree W so that M is W-like, as the following example illustrates.

EXAMPLE 1. Let M be the set consisting of all points (x, y) of E^2 satisfying one of the following conditions:

(1) $x = 0$ and $y \in [-2, 2]$.

(2) $x \in (0, 1]$ and $y = 2\sin\frac{1}{x}$.

(3) $x \in [-1, 0)$ and $y = \sin\frac{1}{x}$.

Here, M is T-like but is not arc-like; however, M is irreducible so G is an arc.

Even if G and K have the same number of endpoints, they may fail to be homeomorphic. Consider the following example.

EXAMPLE 2. Let M be the set of all points (x, y) of E^2 that satisfy one of the following conditions:

(1) $x = 0$ and $y \in [-3, 3]$.

(2) $x \in [-1, 0)$ and either $y = 3$ or $y = 2\sin\frac{1}{x} - 1$.

(3) $x \in (0, 1]$ and either $y = -3$ or $y = 2\sin\frac{1}{x} + 1$.

In this example, M is H-like, while G is an X.

There are several questions related to those discussed in this paper that would be interesting to investigate.

If one generalizes the definition of "end of a continuum" by defining a subcontinuum E of M to be an end of M provided Lemma 1.5 is true if "E_i" is replaced by "E", a number of questions arise, such as characterizing those continua which have ends about which they are irreducible. One might then consider the possibility of obtaining interesting decompositions of such continua.

Of course, the work of Kuratowski (especially [2]), as well as that of Thomas [7] are most suggestive. A future work dealing with those ideas is planned.

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On criteria of Blumenthal for inner-product spaces

by

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1. Introduction. The problem of characterizing generalized euclidean spaces has been solved by various authors in many different ways. It is the purpose of this paper to solve the problem along the lines exhibited by Blumenthal [2]. At the same time a generalization of his criteria is obtained and a question asked by Freese in [7] is answered.

2. Four point properties. The following six classes of metric quadruples have been introduced by Wilson, Blumenthal and others.

A metric quadruple p_1, p_2, p_3, p_4 belongs to class:

C_0 if and only if p_1, p_2, p_3, p_4 are pairwise distinct;

C_1 if and only if p_1, p_2, p_3, p_4 are pairwise distinct and it contains a linear triple;

C_2 if and only if p_1, p_2, p_3, p_4 are pairwise distinct, p_3 is between p_2, p_4 and $p_2p_3 = p_3p_4$;

C_3 if and only if p_1, p_2, p_3, p_4 are pairwise distinct, p_2, p_3, p_4 are linear and $p_1p_2 = p_1p_4$;

C_4 if and only if p_1, p_2, p_3, p_4 are pairwise distinct, p_3 is between p_2, p_4 and $p_2p_3 = p_3p_4$, $p_1p_2 = p_1p_4$;

C_5 if and only if p_1, p_2, p_3, p_4 are pairwise distinct, p_3 is between p_2, p_4 while $p_2p_3 = 2p_3p_4$ and $p_1p_2 = p_1p_3$.

Definition. A metric space has the euclidean, euclidean weak, euclidean feeble, euclidean isosceles weak, euclidean isosceles feeble, euclidean external isosceles feeble four-point property provided every quadruple of its points of class $C_0, C_1, C_2, C_3, C_4, C_5$, respectively, is congruently embeddable in euclidean space.

It is known that in a complete, convex, externally convex metric space the euclidean, euclidean weak, euclidean feeble, and the euclidean external isosceles feeble four-point properties are all equivalent. See [9], [1], [2], [7]. Moreover, each of these properties implies the euclidean isosceles weak and the euclidean isosceles feeble four-point properties.