(b) *If X is deformable into A relative to x_0, then $\Omega^n A$ and $\Omega^n X \times \Omega^{n+1}(X, A)$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

THEOREM 9. (a) *The fiber structures $(\Omega^2(X, A), \partial, \Omega A)$ and $(\Omega A \times \Omega^2 X, \pi_1, \Omega A)$ are homotopy equivalent if and only if ΩA is contractible in ΩX.*

(b) *If A is contractible in X relative to x_0, then $\Omega^{n+1}(X, A)$ and $\Omega^n A \times \Omega^{n+1} X$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

THEOREM 10. (a) *The fiber structures $(\Omega X, i, \Omega(X, A))$ and $(\Omega(X, A) \times \Omega A, \pi_1, \Omega(X, A))$ are homotopy equivalent if and only if $\partial\colon \Omega(X, A) \to A$ is null-homotopic.*

(b) *If A is a retract of X, then $\Omega^n X$ and $\Omega^n(X, A) \times \Omega^n A$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

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Topologies for probabilistic metric spaces

by

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1. Introduction. The purpose of the present paper is to generalize some of the topological notions for probabilistic metric spaces introduced by Schweizer and Sklar [4], and by Thorp [5]. The fundamental tool for this task is the "profile function" a monotone non-decreasing function defined on the non-negative half of the real line and having its values in the closed unit interval. It will be shown that such a function gives rise to a generalized topology [3] on any PM-space $(S, \mathcal{F})$.

A condition sufficient to strengthen these g-topologies to topologies is established but its non-necessity is shown by several examples which are of some interest in their own right, with a view, perhaps, towards possible applications.

Finally, this approach to generalized topologies for PM-spaces is compared with that of Thorp [5] and a rather mild condition for the equivalence of these two is demonstrated.

The only concepts required for an understanding of these results are those of PM-space, triangular norm (t-norm) and Menger space. These may be found in Schweizer and Sklar [4], among others.

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2.

DEFINITION 2.1. A function φ is a *profile function* if $\mathrm{Dom}\,\varphi = [0, \infty)$, φ is non-decreasing and $0 \leqslant \varphi(x) \leqslant 1$ for all x in $[0, \infty)$.

DEFINITION 2.2. Let $(S, \mathcal{F})$ be a PM-space, let φ be a profile function, let $p \in S$, let $A \subseteq S$, and let $\varepsilon, \lambda > 0$ be given. Then:

a. The set $N_p(\varphi;\ \varepsilon, \lambda) = \{q \in S\colon\ F_{pq}(\varepsilon) > \varphi(\varepsilon) - \lambda\}$ is called the $(\varphi;\ \varepsilon, \lambda)$*-neighborhood of* p;

b. p is a φ*-accumulation point of* A if $(N_p(\varphi;\ \varepsilon, \lambda) - \{p\}) \cap A \neq \emptyset$ for every $\varepsilon, \lambda > 0$;

c. A is φ*-closed* if $\varphi(A) \subseteq A$, where $\varphi(A)$ is the set of φ-accumulation points of A.

THEOREM 2.1. *Let $(S, \mathcal{F})$ be a* PM-*space, let φ be a profile function and let A_a be a φ-closed set for each a in some index set Λ.* Then

$$A = \bigcap \{A_a: a \in \Lambda\} \quad \textit{is } \varphi\textit{-closed.}$$

Proof. Observe first that, if $A \subseteq B$, then $\varphi(A) \subseteq \varphi(B)$. For, if $p \in \varphi(A)$, then for every $\varepsilon, \lambda > 0$, there exists $q \in A$ such that $F_{pq}(\varepsilon) > \varphi(\varepsilon) - \lambda$. But $q \in A$ implies that $q \in B$ and, hence, $p \in \varphi(B)$. Next, $A \subseteq A_a$, for all $a \in \Lambda$, implies that $\varphi(A) \subseteq \varphi(A_a)$, for all a, so that $\varphi(A) \subseteq \bigcap_a \varphi(A_a) \subseteq \bigcap_a A_a = A$.

It should also be noted that $\varphi(\emptyset) = \emptyset$.

Definition 2.1 is not as restrictive as it appears. It is not hard to show that for any one-place function φ on $[0, \infty)$ there is a function $\hat{\varphi}$, satisfying the conditions of Definition 2.1, and such that $\varphi(A) = \hat{\varphi}(A)$, for every subset A of the PM-space.

The property of φ-closed sets established in Theorem 2.1 has been used [1, 3] as the defining property of what has come to be called a "generalized topology", distinguished from "topology" by the additional property, that, in a topology a finite union of closed sets is also a closed set. Often, the definition is given in neighborhood terminology as follows.

DEFINITION 2.3. If, with each point p of a set S is associated a family of sets $\mathcal{N}_p = \{N_p\}$, called neighborhoods of p, having the property that each member of the family contains p, then the collection $\{\mathcal{N}_p: p \in S\}$ is called a *generalized topology* (g-topology) for S.

Thus, since $F_{pp}(\varepsilon) = 1 \geqslant \varphi(\varepsilon) - \lambda$, for any $\varepsilon, \lambda > 0$, implies $p \in N_p(\varphi; \varepsilon, \lambda)$, we may, given a PM-space $(S, \mathcal{F})$ and a profile function φ, refer to the *φ-g-topology* on S.

A more detailed analysis of the topological structures generated by profile functions is made possible by determining which of the so-called "neighborhood axioms" are satisfied. For convenience, these axioms are listed here:

N_1: *For each neighborhood, V_p, of p, $p \in V_p$.*

N_2: *For each point p and each pair of neighborhoods U_p and V_p of p, there is a neighborhood W_p of p such that $W_p \subseteq U_p \cap V_p$.*

N_3: *For each point p and each neighborhood V_p of p there is a neighborhood U_p of p such that for each $q \in U_p$ there exists a neighborhood V_q of q such that $V_q \subseteq V_p$.*

N_4: *For each point p and each point q in a neighborhood V_p of p there exists a neighborhood V_q of q such that $V_q \subseteq V_p$.*

N_5: *If $p \neq q$, there exist neighborhoods V_p of p and V_q of q such that $p \in V_p$, $q \bar{\epsilon} V_p$, $q \in V_q$, $p \bar{\epsilon} V_q$.*

N_6: *If $p \neq q$, p and q have disjoint neighborhoods V_p and V_q such that $p \in V_p$ and $q \in V_q$.*

THEOREM 2.2. *Let $(S, \mathcal{F})$ be a* PM-*space, let φ be a profile function and let*

$$\mathcal{N}_p = \{N_p(\varphi;\ \varepsilon, \lambda):\ \varepsilon > 0, \lambda > 0, p \in S\}.$$

Then, for the family $\mathcal{N}_p$,

(i) N_1 *is always satisfied.*

(ii) N_2 *is satisfied if $F_{pq} - \varphi$ is non-decreasing for every pair p, q of distinct points in S.*

(iii) N_5 *is satisfied if for every pair of distinct points p, q in S, there is a number $x_0 > 0$ such that $\varphi(x_0) > F_{pq}(x_0)$.*

(iv) N_6 *is satisfied if $(S, \mathcal{F})$ is a Menger space under a t-norm, T, having the property that, for every pair of distinct points p, q in S, $\sup_{x < \varphi(0+)} T(x, x) > F_{pq}(0+)$.*

Proof. (i) For every $\varepsilon, \lambda > 0$, we have $F_{pp}(\varepsilon) = 1 > 1 - \lambda \geqslant \varphi(\varepsilon) - \lambda$, since $\varphi(\varepsilon) \leqslant 1$. Hence always $p \in N_p(\varphi;\ \varepsilon, \lambda)$.

(ii) Let $N_p(\varphi;\ \varepsilon_1, \lambda_1)$, and $N_p(\varphi;\ \varepsilon_2, \lambda_2)$ be two φ-neighborhoods of p and let $\varepsilon = \mathrm{Min}[\varepsilon_1, \varepsilon_2]$, $\lambda = \mathrm{Min}[\lambda_1, \lambda_2]$. For $q \in N_p(\varphi;\ \varepsilon, \lambda)$ we have $F_{pq}(\varepsilon) > \varphi(\varepsilon) - \lambda$ from which $(F_{pq} - \varphi)(\varepsilon) > -\lambda$. Now, for $i = 1, 2$ $(F_{pq} - \varphi)(\varepsilon_i) \geqslant (F_{pq} - \varphi)(\varepsilon) > -\lambda \geqslant -\lambda_i$. But this is the statement that $q \in N_p(\varphi;\ \varepsilon_i, \lambda_i)$ for $i = 1, 2$. Thus, $N_p(\varphi;\ \varepsilon, \lambda) \subseteq N_p(\varphi;\ \varepsilon_1, \lambda_1) \cap N_p(\varphi;\ \varepsilon_2, \lambda_2)$.

(iii) Since $\varphi(x_0) > F_{pq}(x_0)$, there is a λ_0 such that $\varphi(x_0) - \lambda_0 > F_{pq}(x_0)$. Thus, $N_p(\varphi;\ x_0, \lambda_0)$ is a neighborhood of p which does not contain q; and, by symmetry, $N_q(\varphi;\ x_0, \lambda_0)$ is a neighborhood of q which does not contain p.

(iv) Let $\sup_{x < \varphi(0+)} T(x, x) = a$; and note that from $T(0, 0) = 0$ and $T[\varphi(0+), \varphi(0+)] \geqslant a > F_{pq}(0+) \geqslant 0$, it follows that $\varphi(0+) > 0$. Now suppose $p \neq q$ and let $\varepsilon > 0$ be given. Choose λ, $0 < \lambda < \varphi(0+)$, so that $T\big(\varphi(0+) - \lambda, \varphi(0+) - \lambda\big) > a - \varepsilon$.

Choose δ so that $F_{pq}(\delta) < a$, and consider the neighborhoods $N_p(\varphi;\ \delta/2, \lambda)$, $N_q(\varphi;\ \delta/2, \lambda)$. If $r \in N_p \cap N_q$ then

$$F_{pq}(\delta) \geqslant T[F_{pr}(\delta/2), F_{qr}(\delta/2)] \geqslant T[\varphi(\delta/2) - \lambda, \varphi(\delta/2) - \lambda]$$
$$\geqslant T[\varphi(0+) - \lambda, \varphi(0+) - \lambda] > a - \varepsilon.$$

Hence, since δ is independent of ε, we have $F_{pq}(\delta) \geqslant a$.

But this is a contradiction, in view of which, $N_p \cap N_q = \emptyset$.

Several remarks should be made concerning these conditions. First, (iii) is always satisfied in the "strong topology" of Schweizer and Sklar (i.e., in which $\varphi(x) = 1$ for all $x > 0$). Note that (ii) is also satisfied in this case. Also in the strong topology, (iv) reduces to the already-known requirement that the Hausdorff axiom be satisfied.

Condition (ii) also turns out to be a condition which insures that φ-closed sets be closed under finite unions and is, thus, a condition that converts g-topologies into topologies. It is a rather stringent condition but one which is satisfied in at least one other special case of interest, in addition to the strong case.

3. Special cases. In this section we study the structures generated by the family of profile functions

$$\varphi(x) = (1-\lambda_0)H(x-\varepsilon_0) \tag{3.1}$$

where $\varepsilon_0 \geqslant 0$, $0 \leqslant \lambda_0 \leqslant 1$ and H is the unit step function with step at the origin.

For the choice $\varepsilon_0 = 0 = \lambda_0$, one obtains the previously mentioned "strong topology", which was the first topological structure on PM-spaces to be investigated in any depth. The choice $\varepsilon_0 \neq 0$, $0 < \lambda_0 < 1$ gives rise to the $(\varepsilon_0, \lambda_0)$-*topology*; the $(0, \lambda_0)$-topology is called the λ_0-*topology*; and the $(\varepsilon_0, 0)$-topology is called the ε_0-*topology*. It will be noticed that these structures have been referred to as topologies rather than as g-topologies. That this usage is not unfounded will be shown shortly.

LEMMA 3.1. *Let $(S, \mathcal{F})$ be a PM-space, let φ be a profile function of the form* (3.1) *and let $p \in S$. Then for any $\varepsilon > 0$ and $\lambda > 0$*

$$N_p(\varphi;\ \varepsilon, \lambda) = \begin{cases} S, \ \varepsilon \leqslant \varepsilon_0 \quad \text{or} \quad 1-\lambda_0 < \lambda, \\ \{q\colon F_{pq}(\varepsilon) > 1-\lambda_0-\lambda\},\ \varepsilon > \varepsilon_0 \quad \text{and} \quad 0 < \lambda \leqslant 1-\lambda_0. \end{cases} \tag{3.2}$$

Proof. A straightforward calculation, keeping in mind the non-negativity of each F_{pq}, yields the stated result.

If we let $\delta = \lambda_0 + \lambda$ in (3.2), we obtain

$$N_p(\varphi;\ \varepsilon, \delta-\lambda_0) = \begin{cases} S, \ \varepsilon \leqslant \varepsilon_0 \quad \text{or} \quad \delta > 1, \\ \{q\colon F_{pq}(\varepsilon) > 1-\delta\},\ \varepsilon > \varepsilon_0 \quad \text{and} \quad \lambda_0 < \delta \leqslant 1. \end{cases}$$

From this and from the definition of φ-accumulation point, we have at once

LEMMA 3.2. *Let $(S, \mathcal{F})$ be a PM-space, let φ be a profile function of the form* (3.1), *let $A \subseteq S$ and let $p \in S$. Then $p \in \varphi(A)$ iff for every $\varepsilon > \varepsilon_0$ and every $\delta > \lambda_0$ there exists $q \in A$ such that*

$$F_{pq}(\varepsilon) > 1-\delta.$$

LEMMA 3.3. *Let $(S, \mathcal{F})$ be a PM-space and let φ be a profile function of the form* (3.1). *Then the union of a finite number of φ-closed subsets of S is a φ-closed set.*

Proof. Let A and B be φ-closed sets. The assumption that $p\,\overline{\in}\,\varphi(A) \cup \varphi(B)$ leads to the existence of $N_p(\varphi;\ \varepsilon_1, \delta_1-\lambda_0) = N_1$ and $N_p(\varphi;\ \varepsilon_2, \delta_2-\lambda_0) = N_2$ such that $(N_1-\{p\}) \cap A = \emptyset$ and $(N_2-\{p\}) \cap B = \emptyset$. Then, $(N_3-\{p\}) \cap (A \cup B) = \emptyset$, where $N_3 = N_p(\varphi;\ \min(\varepsilon_1, \varepsilon_2), \min(\delta_1, \delta_2)-\lambda_0)$. Thus, $p\,\overline{\in}\,\varphi(A \cup B)$. Hence, $\varphi(A \cup B) \subseteq \varphi(A) \cup \varphi(B) \subseteq A \cup B$ and $A \cup B$ is φ-closed.

THEOREM 3.1. *Let $(S, \mathcal{F})$ be a PM-space and let φ be a profile function of the form* (3.1). *Then the φ-g-topology for S is a topology in the usual sense.*

Proof. Lemma 3.3 and Theorem 2.1 provide for closure of φ-closed sets under finite unions and arbitrary intersections, respectively.

At this point, a word is in order regarding the reasons for investigating these special cases. One reason, of course, is the fact that topologies rather than g-topologies are obtained from profile functions of the form (3.1). Another reason is the probabilistic interpretations in these cases. Consider, e.g., the case in which $\varepsilon_0 = 0$ and $\lambda_0 > 0$. This yields a model intuitively equivalent to a situation in which statements about distances cannot have a probability greater than $1-\lambda_0$, i.e., there is an "upper limit" on the degree of certainty with which statements about distances can be made. Similarly, restricting ε so that $\varepsilon > \varepsilon_0$ yields a model intuitively equivalent to a situation in which statements cannot be made about distances smaller than ε_0. The foregoing reflect situations in which arbitrarily great confidence regarding arbitrarily small distances is not possible.

For g-topologies generated by profile functions, a straightforward proof shows that $\varphi_1 \leqslant \varphi_2$ implies that the φ_1-g-topology is coarser than the φ_2-g-topology, in the sense that every φ_1-closed set is φ_2-closed. The following example, then, serves two purposes: it shows that the converse of the preceding statement is false and it also provides an example in which the $(\varepsilon_0, \lambda_0)$-, ε_0- and λ_0-topologies are all distinct.

EXAMPLE 3.1. Let $S = \{p, q, r\}$ and let

$$F_{pq}(x) = F_{qr}(x) = \begin{cases} x, & 0 < x \leqslant 1, \\ 1, & x > 1, \end{cases} \qquad F_{pr}(x) = \begin{cases} 5/8, & 0 < x \leqslant 5/8, \\ 1, & x > 5/8. \end{cases}$$

It is easy to verify that $(S, \mathcal{F})$ is a PM-space and easy, although somewhat tedious, to verify that the $(\varepsilon_0, \lambda_0)$-topology for S is indiscrete, the ε_0-topology is discrete and the λ_0-topology for S is the one whose proper closed sets are $\{q\}$ and $\{p, r\}$.

Thus, although the λ_0-topology is strictly coarser than the ε_0-topology, their profile functions are incomparable.

Following are several theorems which indicate that somewhat stronger results are obtainable in these special cases.

Theorem 3.2. *Let* $(S, \mathcal{F}, T)$ *be a Menger space under a t-norm* T *satisfying*

$$\sup_{x<1-\lambda_0} T(x, x) = 1-\lambda_0, \tag{3.3}$$

where $0 \leqslant \lambda_0 < 1$. Then:

(i) *the* λ_0*-topology for* S *satisfies the neighborhood axiom* N_3;

(ii) *the* λ_0*-topology for* S *is pseudo-metrizable*;

(iii) *if, in the presence of* (3.3), *there exists, for each pair of distinct points* $p, q \in S$, *an* $x > 2\varepsilon_0$ *such that* $F_{pq}(x) < 1-\lambda_0$, *the* $(\varepsilon_0, \lambda_0)$*-topology or* S *satisfies the neighborhood axiom* N_6.

Proof. (i) Let $N_p(\varphi_{\lambda_0}; \varepsilon, \delta-\lambda_0)$ be a given neighborhood of p. (Note that φ_{λ_0} and $\varphi_{\varepsilon_0,\lambda_0}$ are used to denote the profile functions for the λ_0- and $(\varepsilon_0, \lambda_0)$-topologies, respectively.) Choose $\delta' > \lambda_0$ such that $T(1-\delta', 1-\delta') > 1-\delta$. Such a δ' exists in view of (3.3). Choose a so that $0 < a < \varepsilon$. Let $q \in N_p(\varphi_{\lambda_0}; a, \delta'-\lambda_0)$ and $r \in N_q(\varphi_{\lambda_0}; \varepsilon-a, \delta'-\lambda_0)$.

Then

$$F_{pr}(\varepsilon) \geqslant T[F_{pq}(a), F_{qr}(\varepsilon-a)] \geqslant T[1-\delta', 1-\delta'] > 1-\delta .$$

Thus, $r \in N_p(\varphi_{\lambda_0}; \varepsilon, \delta-\lambda_0)$ and the theorem is proved.

(ii) This result is due to E. Thorp [5; Theorem 3.13, Part (2) and Theorem 3.14]. Thorp's proof uses generalized uniformities [1]. An alternate proof, using the Alexandroff–Urysohn metrization theorem [2] can also be constructed.

(iii) In outline, the proof of this part consists of choosing, for $p \neq q$ in S, an $x > 2\varepsilon_0$ such that $F_{pq}(x) = a < 1-\lambda_0$ and a δ such that $\lambda_0 < \delta < 1$ and for which $T(1-\delta, 1-\delta) = a$. Then $N_p(\varphi_{\varepsilon_0,\lambda_0}; x/2, \delta-\lambda_0)$ and $N_q(\varphi_{\varepsilon_0,\lambda_0}; x/2, \delta-\lambda_0)$ are disjoint neighborhoods of p and q respectively.

Theorem 3.3. *If* $(S, \mathcal{F})$ *is a* PM*-space such that, for each pair of distinct points* $p, q \in S$, *there exists* $x_0 > \varepsilon_0$, *such that* $F_{pq}(x_0) < 1-\lambda_0$, *then the* $(\varepsilon_0, \lambda_0)$*-topology for* S *satisfies the neighborhood axiom* N_5.

Proof. This follows at once from Theorem 2.2, (iii).

Setting $\varepsilon_0 = 0$ in Theorem 3.3, gives the condition that the λ_0-topology for S satisfies N_5 which, in turn, together with Theorem 3.2, (ii), implies that the λ_0-topology is metrizable.

Theorem 3.4. *Let* $(S, \mathcal{F}, T)$ *be a Menger space under a t-norm,* T, *satisfying*

$$\lim_{x\to 1-\lambda_0} T(a, x) = a, \quad a \in [0, 1-\lambda_0] .$$

Then the λ_0*-topology for* S *satisfies the neighborhood axiom* N_4.

Proof. Let $p \in S$, let $\varepsilon > 0$, let $\delta > \lambda_0$ and let $q \in N_p(\varphi_{\lambda_0}; \varepsilon, \delta-\lambda_0)$ so that $F_{pq}(\varepsilon) > 1-\delta$. By the left-continuity of F_{pq}, there exists $\varepsilon' < \varepsilon$ and δ', $\lambda_0 < \delta' < \delta$, such that

$$F_{pq}(\varepsilon') > 1-\delta' > 1-\delta .$$

Let ε'' satisfy $0 < \varepsilon'' < \varepsilon-\varepsilon'$ and choose δ'' such that $\lambda_0 < \delta'' < \delta'$ and

$$T(1-\delta', 1-\delta'') > 1-\delta .$$

Let $r \in N_q(\varphi_{\lambda_0}; \varepsilon'', \delta''-\lambda_0)$. Then,

$$F_{pr}(\varepsilon) \geqslant T[F_{pq}(\varepsilon'), F_{qr}(\varepsilon-\varepsilon')] \geqslant T[1-\delta', F_{qr}(\varepsilon'')] \geqslant T[1-\delta', 1-\delta''] > 1-\delta .$$

Thus, $r \in N_p(\varphi_{\lambda_0}; \varepsilon, \delta-\lambda_0)$ and $N_q(\varphi_{\lambda_0}; \varepsilon'', \delta''-\lambda_0)$ is a subset of $N_p(\varphi_{\lambda_0}; \varepsilon, \delta-\lambda_0)$, proving the theorem.

4. Comparison. In this section we compare the foregoing study of g-topologies for PM-spaces with an analysis by E. Thorp [5]. It will be seen that, although there is some divergence, a relatively simple requirement will render the two points of view virtually identical. Thorp's study begins in a seemingly more general setting:

Definition 4.1. Let $(S, \mathcal{F})$ be a PM-space and let X be a subset of the positive quadrant. For any p in S and any (u, v) in X, the X-(u, v)-*neighborhood of* p is the set:

$$N_p(u, v) = \{q \text{ in } S\colon F_{pq}(u) > 1-v\} .$$

For p in S and (u, v) in X, the sets $N_p(u, v)$ are called X*-neighborhoods of* p.

It is easily shown that these neighborhoods induce a g-topology for S, called the X-g-topology, and denoted by $\tau(X)$. Further, one can show that $X_1 \subseteq X_2$ implies that $\tau(X_1)$ is coarser than $\tau(X_2)$, in the sense that every X_1-neighborhood of a point contains an X_2-neighborhood of that point.

Lemma 4.1. *Let* $(S, \mathcal{F})$ *be a* PM*-space and let* X *be a subset of the positive quadrant. Define*

$$X^* = X \cup \{(u^*, v^*)\colon u^* \geqslant u,\ v^* \geqslant v \text{ for some } (u, v) \text{ in } X\} \cup \{(u, v)\colon u > 0, v > 1\} . \tag{4.1}$$

Then $\tau(X^*) = \tau(X)$.

Proof. Since $X \subseteq X^*$, $\tau(X)$ is coarser than $\tau(X^*)$. In the other direction, let $p \in S$ and suppose that $N_p(u, v)$ is an X^*-neighborhood of p. Then $(u, v) \in X^*$ which implies that

a. $(u, v) \in X$;

b. $u \geqslant u'$, $v \geqslant v'$ for some $(u', v') \in X$; or

c. $u > 0$, $v > 1$.

In case a, $N_p(u, v)$ is itself an X-neighborhood of p. In case c, $N_p(u, v) = \{q\colon F_{pq}(u) > 1 - v\} = S$, since $1 - v < 0$ and $u > 0$. S is also an X-neighborhood of p. In case b, consider $q \in N_p(u', v')$, an X-neighborhood of p. We have

$$F_{pq}(u) \geqslant F_{pq}(u') > 1 - v' \geqslant 1 - v\,,$$

so that $q \in N_p(u, v)$.

Thus, in each case, $N_p(u, v)$ contains an X-neighborhood of p, and, accordingly, $\tau(X^*)$ is coarser than $\tau(X)$ and the lemma is proved.

The import of Lemma 4.1 is that, when considering topologies generated by subsets of the positive quadrant, we may as well restrict ourselves to sets of the form (4.1) which we shall do in the sequel. Such sets will be called *determining sets*.

LEMMA 4.2. *Let X be a determining set and let*

$$\alpha(x) = \begin{cases} 1, & x = 0\,, \\ \inf\{y\colon (x, y) \in X\}\,, & x > 0\,. \end{cases} \tag{4.2}$$

Then $\operatorname{Dom}\alpha = [0, \infty)$, $\operatorname{Ran}\alpha \subset [0, 1]$ *and α is non-increasing.*

Proof. The first two properties are obvious and the third is established by noting that a determining set contains every point of the positive quadrant which is above and/or to the right of one of its points.

If X and α, called the *boundary function* of X, are as in Lemma 4.2, then the function φ, defined by $\varphi(x) = 1 - \alpha(x)$, is a profile function, called the *associated profile function*.

The following theorem illuminates the connection between X-g-topologies and φ-g-topologies.

THEOREM 4.1. *Let X be a determining set with boundary function α and associated profile function φ. Then:*

(i) *the φ-g-topology is coarser than the X-g-topology;*

(ii) *if $\big(x, \alpha(x)\big) \bar{\epsilon} X$ for any $x > 0$, the φ-g-topology and the X-g-topology are equivalent.*

Proof. (i) For $p \in S$ and $N_p(\varphi;\ \varepsilon, \lambda)$ a $(\varphi;\ \varepsilon, \lambda)$-neighborhood of p, choose $u = \varepsilon$ and v so that

$$\alpha(\varepsilon) < v \leqslant \alpha(\varepsilon) + \lambda\,.$$

Then, $(u, v) \in X$ and $q \in N_p(u, v)$ implies $q \in N_p(\varphi;\ \varepsilon, \lambda)$, i.e., $\tau(X)$ is finer than the φ-g-topology.

(ii) Part (i) is half of the desired equivalence. In the other direction, let $N_p(u, v)$ be an X-neighborhood of $p \in S$. Since $\big(u, \alpha(u)\big) \bar{\epsilon} X$, there exists a λ satisfying

$$0 < \lambda \leqslant v - \alpha(u)\,.$$

Choosing $\varepsilon = u$ yields a point $\big(\varepsilon, \alpha(\varepsilon) + \lambda\big) \in X$. Then $N_p(\varphi;\ \varepsilon, \lambda)$ is a φ-neighborhood of p and is a subset of $N_p(u, v)$. This is the other half of the equivalence.

The condition in Theorem 4.1, (ii), is the "relatively simple requirement", mentioned in the introduction to this section, which yields equivalent topological structures from these different points of view.

THEOREM 4.2. *Let X be a determining set and φ the associated profile function. If ψ is any other profile function such that $\psi(x_0) > \varphi(x_0)$ for some $x_0 \in [0, \infty)$, then there is a PM-space $(S, \mathcal{F})$ for which the X-g-topology is strictly coarser than the ψ-g-topology.*

Proof. Let $S = \{p, q\}$, where

$$F_{pq}(x) = \begin{cases} [\psi(x_0) + \varphi(x_0)]/2\,, & x \leqslant x_0\,, \\ 1, & x > x_0\,. \end{cases}$$

Here, the X-g-topology is indiscrete, whereas the ψ-g-topology is discrete.

The preceding theorem shows that the results of Theorem 4.1 are "best-possible" in the sense that, for all possible PM-spaces, the associated φ-g-topology is the finest g-topology coarser than the given X-g-topology.

Theorem 4.1 elucidates the transition from X-g-topologies to φ-g-topologies. In the other direction, we have

THEOREM 4.3. *Let $(S, \mathcal{F})$ be a PM-space and φ a profile function. Let $X = \{(x, y)\colon y > 1 - \varphi(x)\}$. Then the X-g-topology and the φ-g-topology are equivalent.*

Proof. This follows at once from Theorem 4.1, (ii). For X is a determining set, with associated profile function φ, which does not contain the graph of its boundary function, $1 - \varphi$.

This relationship between X-g- and φ-g-topologies is not one-to-one, in that more than one X-g-topology can give rise to the same φ-g-topology or, more accurately, more than one X can give rise to the same φ, whereas a given φ-g-topology determines an X-g-topology uniquely. It seems that this disadvantage is overridden, however, by the advantages of being able to deal with a convenient analytic tool, in the person of a profile function, rather than with arbitrary subsets of the positive quadrant.

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Consistency statements in formal theories*

by

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In this paper, we establish several results regarding the behavior of consistency statements in formal theories in the language of arithmetic; extensions to other "larger" languages are usually straightforward. The paper continues work begun by S. Feferman in [1], employing as its chief device self-referential statements or processes, as in [1], [4], [5], [6].

In particular, we will find that a very weak theory may be able to prove its own (*RE*) consistency (see Theorem 1.5); that any reflexive theory containing Peano arithmetic arises from adding to some theory the (*RE*) statement of its own consistency (see Theorem 1.4); that the addition of a consistency statement to a theory can substantially alter the Lindenbaum algebras of the theory, and in fact render impossible homomorphisms of these algebras which commute with a finite (and specified) number of quantifiers (see Theorem 4.1). We also explore the degrees of relative interpretability between that of a theory and the theory plus its consistency (Theorems 3.1, 3.2).

We assume that the reader is familiar with the paper of Feferman [1]; when we do not specify a convention that we use, it is to be found in that reference, which we shall call "Feferman's paper".

1. Let a theory $\mathcal{A}$ be given possessing A as an axiomatization (i.e., A is a set of Gödel numbers of formulas which axiomatize $\mathcal{A}$); let $\alpha(\mathrm{w})$ be a formula in the language of arithmetic which *designates* A in the following sense:

$$\alpha(\overline{n}) \quad \text{if and only if} \quad n \in A\,.$$

Then the construction of Feferman ([1], Def. 4.1) assigns to this designator $\alpha(\mathrm{w})$ the formula $\mathrm{Prf}_\alpha(\mathrm{y}, \mathrm{x})$ in the two free variables x, y, and this formula intuitively "says" that x is a (Gödel number of a) proof

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