Approximating continua from within

by

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Introduction. It is known that the product of continua with the fixed point property (fpp) need not have fpp and also that the cone over a continuum with fpp need not have fpp [10]. However for certain kinds of continua, the fixed point property is preserved under products and cones. In this paper we introduce the notion of approximation from within and use it to exhibit a large class of continua with the above property. This class is shown to contain among others all smooth metric dendroids and all fans. In addition we exhibit a class of tree chainable continua with fpp.

1. Approximation from within. Let X be a continuum ($\equiv$ compact, connected Hausdorff space) and let $\mathcal{U}$ be an open set in $X\times X = X^2$ containing the diagonal of X^2, $\Delta X^2 = \{(x,x)|\ x\in X\}$. A subcontinuum Y of X is called a *$\mathcal{U}$-subcontinuum of X* provided there is a continuous function f from X onto Y such that $(x, f(x))\in\mathcal{U}$ for all $x\in X$. Now let P be a topological property and suppose that for each open set $\mathcal{U}$ in X^2 containing ΔX^2 there is a $\mathcal{U}$-subcontinuum Y of X with property P. Then we shall say that X *can be approximated from within by subcontinua with property* P, or more briefly we shall say that X is *approximately* P. Clearly the property of being approximately P is a topological property.

If X is also metric then it is easy to see that X is approximately P if and only if for each positive number ε there is a continuous function f from X into X such that $f(X)$ has property P and f moves no point of X more than ε.

As an example of what we have in mind, consider the '$\sin\frac{1}{x}$ curve':

$$X = \left\{\left(x, \sin\frac{1}{x}\right)\Big|\ 0 < x \leqslant 1\right\} \cup \{(0,y)|\ -1\leqslant y\leqslant 1\}.$$

It can be seen that for each positive integer n, there is a retraction of X onto a subarc of X which moves no point of X more than $1/n$. Consequently X is an approximate arc; that is X is approximately an arc.

One readily sees that if a continuum has property P then it is approximately P. The converse however depends on the property P. For example the $\sin\frac{1}{x}$ curve shows that the converse is false for the properties of being an arc, being an absolute retract (AR) or being locally connected. Other examples show that many topological properties do not coincide with their approximations. We mention only that there is an indecomposable continuum which is approximately an arc.

We shall see that several properties do agree with their approximations.

1.1. THEOREM. *Let X be a continuum which can be approximated from within by continua with* fpp. *Then X has* fpp.

Proof. Suppose $f\colon X\to X$ is a continuous function such that $f(x)\neq x$ for all $x\in X$. Choose a finite cover of X by open sets $U_1, U_2, \ldots, U_n$ so that $f(U_i)\cap U_i=\emptyset$ for all i. Let $\mathfrak{U}=\bigcup_{i=1}^{n}U_i^2$ and note that $\mathfrak{U}$ is open in X^2 containing ΔX^2. Let Y be a $\mathfrak{U}$-subcontinuum of X with fpp and let $g\colon X\overset{\text{onto}}{\to} Y$ be a continuous function such that $(x, g(x))\in\mathfrak{U}$ for all $x\in X$. Now consider the continuous function $g\circ(f|Y)$, where $f|Y$ denotes the restriction of f to Y. Since Y has fpp, there is a point $x_0\in Y$ so that $g(f(x_0))=x_0$. Hence $(f(x_0), x_0)=(f(x_0), g(f(x_0)))\in\mathfrak{U}$, and from the symmetry of $\mathfrak{U}$ we get that $(x_0, f(x_0))\in U_i^2$ for some i. This contradicts the choice of the U_i's, so X has fpp.

The above result can be applied to many continua to show that they have fpp. For example, the $\sin\frac{1}{x}$ curve has fpp by 1.1. As another example consider the '$\sin\frac{1}{x}$ disk':

$$X=\left\{(x,y)\mid 0<x\leqslant 1,\ -2\leqslant y\leqslant \sin\frac{1}{x}\right\}\cup\{(0,y)\mid -2\leqslant y\leqslant 1\}.$$

As with the $\sin\frac{1}{x}$ curve there are retractions of X onto subdisks of X which are close to the identity map on X. So X is an approximate disk and hence by 1.1 has fpp. More generally, we have:

1.2. COROLLARY. *Each continuum which can be approximated from within by* AR's *has* fpp.

Proof. This follows immediately from 1.1 since AR's have fpp.

We use the term λ-dendroid to denote a continuum (not necessarily metric) which is hereditarily unicoherent and hereditarily decomposable. A *dendroid* is an arcwise connected λ-dendroid and a *tree* is a locally connected dendroid. Many λ-dendroids are known to have fpp. Thus in [1], Borsuk generalized the well-known fact that metric trees have fpp to metric dendroids. In [12], Ward removed the metric condition on Borsuk's result and showed that dendroids have even the strong fpp (a continuum X has the strong fpp provided each continuous function f from X into the space of closed subsets of X with the finite topology, there is a point $x\in X$ such that $x\in f(x)$). Some fixed point theorems for λ-dendroids have been obtained by Ward [14] and Charatonik [2], however, the question of whether every λ-dendroid has fpp remains open even in the metric case.

The next corollary follows immediately from 1.1 and Ward's results [12].

1.3. COROLLARY. *Each λ-dendroid which can be approximated from within by dendroids has* fpp.

We remark that 1.1 can also be used to show that certain λ-dendroids which are not approximate dendroids have fpp. For example, the λ-dendroid constructed by Charatonik in [3] as an example of a λ-dendroid which admits no nontrivial monotone mapping into a dendroid can be shown to have fpp using 1.1 and the fact that a wedge of two continua with fpp has fpp. (A continuum X is called a wedge of two subcontinua X_1 and X_2 if $X=X_2\cup X_2$ and $X_1\cap X_2$ is a point.)

A *chainable* (*tree-chainable*) continuum is a continuum for which any open cover has a finite open refinement whose nerve is an arc (tree). Metric chainable continua are known to have fpp, but it is unknown for tree-chainable continua. Recently Cook [5] has shown that metric λ-dendroids are tree-chainable.

1.4. THEOREM. *Let X be an approximately tree-chainable continuum. Then X is tree-chainable.*

Proof. Let $\mathfrak{W}$ be an open cover of X and let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be open covers of X such that $\mathfrak{V}_2$ is star refinement of $\mathfrak{V}_1$ and $\mathfrak{V}_1$ is a star-refinement of $\mathfrak{W}$; that is, for each $V\in\mathfrak{V}_1$, the star of V, $\bigcup\{V'\mid V'\in\mathfrak{V}_1$ and $V\cap V'\neq\emptyset\}$ is contained in some member of $\mathfrak{W}$. We can do this since X is fully normal. Let $\mathfrak{U}=\bigcup\{V^2\mid V\in\mathfrak{V}_2\}$ and note that $\mathfrak{U}$ is an open set in X^2 and contains ΔX^2. Let Y be a tree-chainable $\mathfrak{U}$-subcontinuum of X and let $g\colon X\overset{\text{onto}}{\to} Y$ be a continuous function such that $(x, g(x))\in\mathfrak{U}$ for each $x\in X$. Let $\mathfrak{V}'$ be the open cover of Y obtained by intersecting the elements of $\mathfrak{V}_2$ with Y, and consider the open cover $\mathfrak{V}''$ of X whose members are of the form $g^{-1}(V)$ for $V\in\mathfrak{V}'$. To see that $\mathfrak{V}''$ refines $\mathfrak{W}$, first fix $y\in Y$. Now for each $x\in g^{-1}(y)$, we have $(x,y)\in\mathfrak{U}$ so x and y both lie in some member of $\mathfrak{V}_2$. But this implies that $g^{-1}(y)\cup\{y\}$ is contained in some member of $\mathfrak{V}_1$. Consequently for each $V\in\mathfrak{V}'$, $g^{-1}(V)\cup V=\bigcup\{g^{-1}(y)\cup\{y\}\mid y\in V\}$ is contained in some member of $\mathfrak{W}$, and hence $\mathfrak{V}''$ refines $\mathfrak{W}$. Now since Y is tree chainable there is a finite open cover $\mathcal{O}=\{O_1, \ldots, O_n\}$ of Y which refines $\mathfrak{V}'$ and whose nerve is a tree. Let $\mathcal{O}'$ be the finite open cover of X whose elements are of the form $\mathcal{O}'_i=g^{-1}(O_i)$

for $i = 1, \ldots, n$. Then the nerve of $\mathcal{O}'$ is also a tree and $\mathcal{O}'$ refines $\mathcal{W}$. We conclude that X is tree-chainable.

From 1.4 and Cook's result we immediately obtain

1.5. COROLLARY. *Each continuum which can be approximated from within by metric λ-dendroids is tree-chainable.*

The fact that dendroids have fpp together with 1.4 combine to tell us that many tree-chainable continua have fpp.

1.6. COROLLARY. *Each approximate metric dendroid is a tree-chainable continuum with* fpp.

As mentioned previously, Ward has shown that dendroids have the strong fpp. In 1.8 below we extend this result to approximate dendroids.

1.7. THEOREM. *Let X be a continuum with the approximate strong* fpp. *Then X has the strong* fpp.

Proof. Suppose, to the contrary, that there is a continuous function $F\colon X \to K(X)$, the space of closed subsets of X under the finite topology, such that for each $x \in X$, $x \notin F(x)$. Fix $x \in X$. Since $\{x\}$ and $F(x)$ are disjoint closed sets there are open sets S and T such that $x \in S$, $F(x) \subset T$ and $\bar{S} \cap \bar{T} = \emptyset$.

Let $\{T\}'$ denote the collection of all closed subsets of X contained in T. Then $\{T\}'$ is an open subset of $K(X)$ and $F(x) \in \{T\}'$. Hence there is a set V open in X such that $x \in V$ and $F[V] \subset \{T\}'$. Let $U_x = S \cap V$, and note that $x \in U_x$. Also if $y \in U_x$, then $F(y) \in \{T\}'$; that is, $F(y) \cap U_x = \emptyset$. Therefore for each $x \in X$, there is an open set U_x such that $U_x \cap \{F(y)\colon y \in U_x\} = \emptyset$. By the compactness of X, we can find a finite subset N of X such that $\{U_x\}_{x\in N}$ covers X. Let $\mathcal{U} = \bigcup_{x\in N} U_x^2$. Then $\mathcal{U}$ is open in X^2 and contains ΔX^2. Let $g\colon X \to X$ be a continuous function such that $(x, g(x)) \in \mathcal{U}$ for each $x \in X$ and $g(X)$ has the strong fpp. It follows from [11], p. 165 that the function $H\colon g(X) \to K(g(X))$ defined by $H(x) = g(F(x))$ is continuous. Hence there is an $x_0 \in g(X)$ such that $x_0 \in H(x_0)$, and thus $g(y) = x_0$ for some $y \in F(x_0)$. But $(y, x_0) = (y, g(y)) \in \mathcal{U}$, so that $(y, g(y)) \in U_x^2$ for some $x \in N$. This contradicts the fact that $U_x \cap \bigcup\{F(z)\colon z \in U_x\} = \emptyset$. We conclude that X has the strong fpp.

1.8. COROLLARY. *Each approximate dendroid has the strong* fpp.

The last three results in this section give additional information about when a property P coincides with its approximation. In particular 1.9 will prove useful in the next section.

1.9. THEOREM. *Let* P *be a topological property, then approximately* P *coincides with its approximation.*

Proof. Let X be a continuum which can be approximated from within by continua which are approximately P.

Let $\mathcal{U}$ be an open set containing ΔX^2. It follows from [9], p. 197, that there is an open set $\mathcal{V} \supset \Delta X^2$ so that $\mathcal{V} \circ \mathcal{V} \subset \mathcal{U}$, where $\mathcal{V} \circ \mathcal{V} = \{(x, y) \in X^2 |$ for some $z \in X$, (x, z) and $(z, y) \in \mathcal{V}\}$. Now let Y be a $\mathcal{V}$-subcontinuum of X which is approximately P and let f be a continuous function from X onto Y so that $(x, f(x)) \in \mathcal{V}$ for each $x \in X$. Now $\Delta Y^2 \subset \mathcal{V} \cap Y^2$, which is open in Y^2, hence there is a $\mathcal{V} \cap Y^2$-subcontinuum Z of Y with property P. Let g be a continuous function from Y onto Z so that $(y, g(y)) \in \mathcal{V} \cap Y^2$ for each $y \in Y$. Then the function fg is a continuous function from X onto Z. For each $x \in X$ $(x, f(x)) \in \mathcal{V}$ and $(f(x), gf(x)) \in \mathcal{V} \cap Y^2 \subset \mathcal{V}$ so $(x, gf(x)) \in \mathcal{V} \circ \mathcal{V} \subset \mathcal{U}$. Consequently Z is a $\mathcal{U}$-subcontinuum of X with property P, and the theorem is proved.

1.10 THEOREM. *Let X be a continuum which is approximately* P. *Then X can be embedded in a product of continua each of which has property* P.

Proof. Let $\mathcal{A}$ denote a collection of continua with property P such that for each open $\mathcal{U}$ in X^2 containing ΔX^2 some member of $\mathcal{A}$ is a $\mathcal{U}$-subcontinuum of X. Then X can be embedded in the product of the collection $\mathcal{A}$ by the embedding lemma in [9], p. 116.

A property P is called (finitely) productive if whenever $\{X_\alpha\}_{\alpha\in A}$ is a (finite) collection of spaces having property P, then $\prod X_\alpha$ has property P. P is called hereditary provided it is inherited by subspaces.

1.11. COROLLARY. *If* P *is a productive and hereditary property, then* P *and approximately* P *coincide.*

2. Products and cones. Throughout this section P will denote an arbitrary topological property. First we establish a condition for approximately P to be preserved under products.

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of continua and let X denote the product continuum $\prod_{\alpha\in A} X_\alpha$. Define $h\colon X^2 \to \prod_{\alpha\in A} X_\alpha^2$ by

$$\big(h(x, y)\big)_\alpha = (x_\alpha, y_\alpha)\,,$$

where $\big(h(x, y)\big)_\alpha$ is the αth coordinate of $h(x, y)$ and x_α, y_α are the αth coordinates of x and y respectively.

2.1. LEMMA. *The function h is a homeomorphism of X^2 onto $\prod_{\alpha\in A} X_\alpha^2$ taking ΔX^2 onto $\prod_{\alpha\in A} \Delta X_\alpha^2$. Further, if $\mathcal{U}$ is an open set in X^2 containing ΔX^2, then there is a finite set $F \subset A$ and open sets U_α in X_α^2 containing ΔX_α^2 for each $\alpha \in F$ such that $h^{-1}\big(\prod_{\alpha\in F} U_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha^2\big) \subset \mathcal{U}$.*

Proof. The proof of the first assertion is straightforward. To see the second assertion, let $h(\mathcal{U}) = \mathcal{W}$. Then $\mathcal{W}$ is open in $\prod_{\alpha\in A} X_\alpha^2$ and contains $\prod_{\alpha\in A} \Delta X_\alpha^2$. Let $\mathfrak{B}$ be the collection of all sets of the form

$$\prod_{\alpha\in F} \overline{U}_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha^2,$$

where F is a finite subset of A, U_α is open in X_α^2 and contains ΔX_α^2, and $\overline{U}_\alpha$ is the closure in X_α^2 of U_α. Now it is easily shown that $\mathfrak{B}$ is a descending family of closed subsets of $\prod_{\alpha\in A} X_\alpha^2$ such that $\bigcap \mathfrak{B} = \prod_{\alpha\in A} \Delta X_\alpha^2 \subset \mathfrak{W}$. Consequently, from the compactness of $\prod_{\alpha\in A} X_\alpha^2$ we can find a member B of $\mathfrak{B}$ lying in $\mathfrak{W}$, say $B = \prod_{\alpha\in F} \overline{U}_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha^2$. Thus $h^{-1}\big(\prod_{\alpha\in F} U_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha^2\big) \subset h^{-1}(\mathfrak{W}) = \mathfrak{U}$, and the lemma is proved.

2.2. THEOREM. *If $\{X_\alpha\}_{\alpha\in A}$ is a collection of continua such that for each finite set $F \subset A$, $\prod_{\alpha\in F} X_\alpha$ is approximately* P, *then $X = \prod_{\alpha\in A} X_\alpha$ is approximately* P.

Proof. Let $\mathfrak{U}$ be an open set containing ΔX^2. By 2.1, there is a finite set $F \subset A$ and for each $\alpha \in F$ an open set U_α in X_α^2 containing ΔX_α^2 so that

$$h^{-1}\Big(\prod_{\alpha\in F} U_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha^2\Big) \subset \mathfrak{U}.$$

Fix a point x^0 in X and define

$$X^F = \{x \in X \mid x_\alpha = x_\alpha^0 \quad \text{for} \quad \alpha \in A\setminus F\}.$$

Define $f\colon X \to X^F$ by

$$\big(f(x)\big)_\alpha = \begin{cases} x_\alpha & \text{if} \quad \alpha \in F, \\ x_\alpha^0 & \text{if} \quad \alpha \in A\setminus F. \end{cases}$$

Then f is clearly a continuous retraction from X onto X^F. Also if $x \in X$, then

$$\big(h(x, f(x))\big)_\alpha = \big(x_\alpha, (f(x))_\alpha\big) = \begin{cases} (x_\alpha, x_\alpha) & \text{if} \quad \alpha \in F, \\ (x_\alpha, x_\alpha^0) & \text{if} \quad \alpha \in A\setminus F, \end{cases}$$

and so $h(x, f(x)) \in \prod_{\alpha\in F} U_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha^2$. Thus $(x, f(x)) \in \mathfrak{U}$ for each $x \in X$ and so X^F is a $\mathfrak{U}$-subcontinuum of X which is approximately P. By 1.9, it follows that X is approximately P.

The next corollary is immediate.

2.3. COROLLARY. *If* P *or approximately* P *is finitely productive, then approximately* P *is productive.*

2.4. COROLLARY. *The product of a collection of approximate* AR's *is an approximate* AR *and hence has* fpp.

Proof. Follows from 1.2, 2.2, 2.3, and the fact that the product of AR's is an AR.

In [7], J. B. Fugate has announced the following result, which will appear in print later: let X be a smooth metric dendroid [4] or a fan [4]. Then for each $\varepsilon > 0$ there is a retraction $f\colon X \to X$ such that $d\big(x, f(x)\big) < \varepsilon$ for all $x \in X$ and $f(X)$ is a dendrite. It follows immediately from this result that X can be approximated from within by AR's; that is each smooth metric dendroid and each fan is an approximate AR. Hence, using 1.9, we conclude:

2.5. COROLLARY. *Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of approximate smooth metric dendroids and approximate fans. Then $X = \prod_{\alpha\in A} X_\alpha$ has* fpp.

If X is a continuum, then the *cone* over X, $C(X)$, is defined to be the quotient space $I \times X/\{0\} \times X$.

2.6. THEOREM. *Let X be a continuum which can be approximated from within by subcontinua whose cones are approximately* P. *Then $C(X)$ is approximately* P.

Proof. Let $\mathfrak{U}$ be an open set in $C(X)^2$ containing $\Delta C(X)^2$. Let $\mathfrak{V} = (\eta \times \eta)^{-1}(\mathfrak{U})$ where $\eta\colon I \times X \to C(X)$ is the natural map. Then $\mathfrak{V}$ is an open set in $(I \times X)^2$ containing $\Delta(I \times X)^2$. Define $h\colon (I \times X)^2 \to I^2 \times X^2$ by $h\big((t_1, x_1), (t_2, x_2)\big) = \big((t_1, t_2), (x_1, x_2)\big)$. Then by 2.1 h is a homeomorphism taking $\Delta(I \times X)^2$ onto $\Delta I^2 \times \Delta X^2$. So $h(\mathfrak{V})$ is an open set $I^2 \times X^2$ containing $\Delta I^2 \times \Delta X^2$ and hence we can pick an open at $\mathfrak{W}$ in X^2 containing ΔX^2 such that $\Delta I^2 \times \mathfrak{W} \subseteq h(\mathfrak{V})$ and hence

(1) $$h^{-1}(\Delta I^2 \times \mathfrak{W}) \subset \mathfrak{V}.$$

Now let Y be a $\mathfrak{W}$-subcontinuum of X such that $C(Y)$ is approximately P and let $g\colon X \overset{\text{onto}}{\to} Y$ be a continuous function such that $\big(x, g(x)\big) \in \mathfrak{W}$ for $x \in X$. Consider the following diagram:

$$\begin{array}{ccccc} C(X) & \xrightarrow{k^*} & C(Y) & \xrightarrow{i^*} & C(X) \\ \uparrow{\scriptstyle\eta} & & \uparrow{\scriptstyle\eta'}\ \text{natural map} & & \uparrow{\scriptstyle\eta} \\ I \times X & \xrightarrow[\mathrm{id}_I \times g]{k} & I \times Y & \xrightarrow[\text{inclusion}]{i} & I \times X \end{array}$$

It follows from the induced function theorem [6], p. 126, that the continuous functions k^* and i^* making the squares commute exists uniquely. Further i^* is 1-1. Let $Z = i^*\big(C(Y)\big)$. Then Z is approximately P since $C(Y)$ is. All that remains to be shown is that Z is a $\mathfrak{U}$-subcontinuum of $C(X)$. Let $g^* = i^*k^*$, and let $x \in C(X)$. We wish to show that $\big(x, g^*(x)\big) \in \mathfrak{U}$. Let $(t, y) \in \eta^{-1}(x)$. Then $g^*(x) = \eta i k(t, y) = \eta\big(t, g(y)\big)$. Hence to show that $\big(x, g^*(x)\big) \in \mathfrak{U}$ it suffices to show that $\big((t, y), (t, g(y))\big) \in \mathfrak{V}$.

But $h\big((t,y),(t,g(y))\big)=\big((t,t),(y,g(y))\big)\in \Delta I^2\times\mathfrak{W}$ and this together with (1) yields the desired result. The theorem now follows from 1.9.

If P is such that whenever a continuum X has P then $C(X)$ has property P, then we shall call P a *cone-invariant* property.

2.7. COROLLARY. *If* P *is a cone invariant property, then so is approximately* P.

Proof. Let X be a continuum which is approximately P. Let $\mathfrak{U}$ be an open set in $C(X)^2$ containing $\Delta C(X)^2$. By repeating essentially the same construction as in 2.6, we can construct a $\mathfrak{U}$-subcontinuum of $C(X)$ which is homeomorphic with a cone $C(Y)$ where Y is a $\mathfrak{W}$-subcontinuum ($\mathfrak{W}$ chosen appropriately) of X with property P.

We have not seen the following theorem in the literature although it is probably well-known.

2.8. THEOREM. *The cone over an* AR *is an* AR.

Proof. Let X be an AR. We can assume X lies in a Tychonoff cube I^Λ for an appropriate indexing set Λ. Consider the product space $I\times I^\Lambda$ and the set $K\subset I\times I^\Lambda$ defined by

$$K=\{(t,tx)|\ t\in I,\ x\in X\}\,.$$

Now it is easily shown that the continuous function $f\colon I\times X\to I\times I^\Lambda$ defined by $f(t,x)=(t,tx)$ induces a homeomorphism from $C(X)$ onto $f(I\times X)=K$. To show that $C(X)$ is an AR, we need only show that K is retract of $I\times I^\Lambda$. For each $n=1,2,\ldots$, let

$$K_n=K\cap\left[\frac{1}{n+1},\frac{1}{n}\right]\times I^\Lambda,\qquad X_n=K\cap\left\{\frac{1}{n}\right\}\times I^\Lambda.$$

Note that K_n and X_n are AR's, since K_n is homeomorphic with $I\times X$ and X_n is homeomorphic with X. Let $r_n\colon\left\{\frac{1}{n}\right\}\times I^\Lambda\to X_n$ be a retraction and let $k_n\colon\left[\frac{1}{n+1},\frac{1}{n}\right]\times I^\Lambda\to K_n$ be a retraction such that $k_n\Big|\left(\left\{\frac{1}{n}\right\}\times I^\Lambda\right)=r_n$ and $k_n\Big|\left(\left\{\frac{1}{n+1}\right\}\times I^\Lambda\right)=r_{n+1}$. Now the function $k\colon I\times I^\Lambda\to K$ defined by

$$k(t,y)=\begin{cases}k_n(t,y) & \text{if}\quad \dfrac{1}{n+1}\leqslant t\leqslant\dfrac{1}{n},\\ (0,0) & \text{if}\quad t=0\end{cases}$$

is a retraction, as is easily checked. This concludes the proof of the theorem.

From 2.7 and 2.8, we immediately obtain

2.9. COROLLARY. *The cone over an approximate* AR *is an approximate* AR *and hence has* fpp. *In particular, then, the cone over an approximate smooth metric dendroid or an approximate fan has* fpp.

Let $\mathfrak{D}$ be a class of continua. Let prod $\mathfrak{D}$ denote the class of all continua Y such that Y is homeomorphic with $\prod_{\alpha\in\Lambda}X_\alpha$ for some collection $\{X_\alpha\}_{\alpha\in\Lambda}$ of members of $\mathfrak{D}$, and let cone $\mathfrak{D}$ denote the class of continua Y such that $Y\in\mathfrak{D}$ or Y is homeomorphic with $C(X)$ for some $X\in\mathfrak{D}$. Inductively, let us define $\mathrm{prod}^n\mathfrak{D}=\mathrm{prod}(\mathrm{prod}^{n-1}\mathfrak{D})$ and $\mathrm{cone}^n\mathfrak{D}=\mathrm{cone}(\mathrm{cone}^{n-1}\mathfrak{D})$. Note that $\mathrm{prod}^n\mathfrak{D}=\mathrm{prod}\,\mathfrak{D}$. Now let $\langle\mathfrak{D}\rangle$ denote the class of continua Y such that for some sequence of positive integers $i_1,i_2,\ldots,i_n$,

$Y\in\mathrm{cone}^{i_n}\mathrm{prod}\,\mathrm{cone}^{i_{n-1}}\ldots\mathrm{cone}^{i_1}\mathrm{prod}\,\mathfrak{D}$. We shall call $\langle\mathfrak{D}\rangle$ the *class of continua generated by* $\mathfrak{D}$ *using products and cones*.

2.10. COROLLARY. *Let* $\mathcal{Q}$ *denote the class of approximate* AR*'s. Then* $\langle\mathcal{Q}\rangle=\mathcal{Q}$.

Proof. Follows from 2.4 and 2.9.

2.11. COROLLARY. *Let* $\mathcal{S}$ *and* $\mathcal{F}$ *denote the class of approximate smooth metric dendroids and approximate fans respectively. Then each member of* $\langle\mathcal{S}\cup\mathcal{F}\rangle$ *has* fpp.

Proof. Follows from 1.2, 2.4, and 2.9.

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