# Invariant metric properties of maps

by

**Benjamin Halpern** (Bloomington, Ind.)

Let $(X, d)$ be a precompact connected metric space with at least two distinct points and $f\colon X \to X$ a continuous 1-1 onto map. A generalized diameter on $n$-tuples is a continuous positive valued function $m$ from the set of $n$-tuples $(x_1, \ldots, x_n)$, $x_i \in S$ with $x_i \neq x_j$ if $i \neq j$, such that for each $\varepsilon > 0$ there exists a $\zeta > 0$ that $(x_1, \ldots, x_n) \leqslant \zeta$ implies $\operatorname{diam}(x_1, \ldots, x_n) = \max_{i,j} d(x_i, x_j) \leqslant \varepsilon$, and $\operatorname{diam}(x_1, \ldots, x_n) \leqslant \zeta$ implies $m(x_1, \ldots, x_n) \leqslant \varepsilon$. Examples of generalized diameters are $m(x_1, \ldots, x_n) = \operatorname{diam}(x_1, \ldots, x_n)$, $m(x_1, \ldots, x_n) = \sum_{i,j} d(x_i, x_j)$, $m(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} d(x_i, x_{i+1})$, and $m(x_1, \ldots, x_n) = \inf_{p \in X} \max_i d(p, x_i)$.

A generalized diameter is weakly symmetric if $m(x_1, x_2, \ldots, x_n) = m(x_n, x_{n-1}, \ldots, x_2, x_1)$ for all $(x_1, \ldots, x_n)$ with $x_i \in X$ and $x_i \neq x_j$ if $i \neq j$. We prove that there always exists invariant $n$-tuples for a weakly symmetric generalized diameter $m$, that is $m\big(f(x_1), \ldots, f(x_n)\big) = m(x_1, \ldots, x_n)$ for some $(x_1, \ldots, x_n)$ with $x_i \in X$ and $x_i \neq x_j$ if $i \neq j$. If $X$ is also locally connected then there are a continuum (in cardinality) of invariant $n$-tuples. If in addition, the local connectivity satisfies a certain uniformity condition (which always holds for locally connected compact metric spaces) then there is a continuum (in cardinality) of invariant $n$-tuples with arbitrarily small diameters.

**Section 1. Definitions and examples.** Let $(X, d)$ be a metric space. We will denote by $X^n$ the cartesian product of $X$ with itself $n$ times, $X^n = X \times X \times \ldots \times X$ $n$ times. Set $GD_n(X)$ ($GD_n$ if $X$ is understood) $= \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j$ for some $i \neq j\}$. Let $S_n$ represent the group of permutation of the set $\{1, 2, \ldots, n\}$. Set $\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$ for any $\sigma \in S_n$ and $(x_1, \ldots, x_n) \in X^n$. If $f$ is a function from $X$ into $X$ and $(x_1, \ldots, x_n) \in X^n$ then set $f(x_1, \ldots, x_n) = \big(f(x_1), \ldots, f(x_n)\big)$. $\boldsymbol{R}$ will denote the real numbers and $\boldsymbol{R}^+ = \{t \in \boldsymbol{R} \mid t > 0\}$.

(1.1) DEFINITION. We will say that a generalized diameter $m$ is *symmetric* provided $m\bar{x} = m\sigma\bar{x}$ for all $\bar{x} \in X^n - GD_n$ and all $\sigma \in S_n$. We will say $m$ is *weakly symmetric* provided $m(x_1, x_2, \ldots, x_{n-1}, x_n) = m(x_n, x_{n-1}, \ldots, x_2, x_1)$ for all $(x_1, \ldots, x_n) \in X^n - GD_n$.

Now suppose $f\colon X \to X$ and $m$ is a generalized diameter on $n$-tuples.

(1.2) DEFINITION. An *invariant* $n$-*tuple* for $f$ and $m$ is an $n$-tuple $\bar{x} \in X^n - GD_n$ satisfying $m\bar{x} = mf\bar{x}$.

We recall that a metric space $(X, d)$ is called precompact provided any one (and thus all) of the following equivalent conditions holds.

(a) The completion of $(X, d)$ is compact.

(b) $(X, d)$ can be isometrically embedded in a compact metric space.

(c) Every infinite sequence in $X$ has a Cauchy subsequence.

We will say that $X$ is *nontrivial* provided $X$ has at least two distinct elements. We also will need some notation from [2] and will refer to definition from [2] by their numbers.

We say that two point $x, y$ of a topological space $Y$ can be *separated in* $Y$ (notation $xS_Yy$) if there exist disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cup V = Y$. We write $xS'_Yy$ provided $xS_Yy$ does not hold.

(1.3) EXAMPLES OF GENERALIZED DIAMETERS.

(a) $m(x_1, \ldots, x_n) = \operatorname{diam}(x_1, \ldots, x_n)$.

(b) $m(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} d(x_i, x_{i+1}) + d(x_n, x_1)$ (the perimeter of the polygon with vertices $x_1, \ldots, x_n$).

(c) $m(x_1, \ldots, x_n) = \sum_{i,j} d(x_i, x_j)$.

(d) $m(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} \big(d(x_i, x_{i+1})\big)^p$ where $p > 0$.

(e) $m(x_1, x_2, x_3, x_4) = \big(d(x_1, x_2)\big)^5 + 3d(x_3, x_4) + e^{d(x_1, x_3)} - 1$.

(f) $m(x_1, \ldots, x_n) = \inf_{p \in X} \max_{1 \leqslant i \leqslant n} d(p, x_i)$ (roughly, the radius of the smallest disk containing the points $x_1, \ldots, x_n$).

(g) If $X = \mathbf{R}^k$ and $d(x, y) = \|x - y\|$ then we can set $m(x_1, \ldots, x_n) = \max_{1 \leqslant i \leqslant n} \|p - x_i\|$ where $p = \frac{1}{n} \sum_{i=1}^{n} x_i$.

## Section 2. There exists invariant $n$-tuples.

(2.1) THEOREM. *Suppose $(X, d)$ is a nontrivial connected precompact metric space and $f\colon X \to X$ is a 1-1 onto continuous map from $X$ onto $X$. If $m$ is a weakly symmetric generalized diameter on $n$-tuples then there exists an invariant $n$-tuple for $f$ and $m$. In fact, if $X$ is also locally connected and $I$ is the set of invariant $n$-tuples for $f$ and $m$ then cardinality of $I$ = cardinality of $X$ = cardinality of the continuum.*



The proof will be based on the following propositions and lemmas which are of some interest in themselves. Proposition 2.2 is a generalization of a theorem of Freudenthal and Hurewicz [1].

(2.2) PROPOSITION. *Let $(X, d)$ be a precompact metric space and $m$ a generalized diameter on $n$-tuples. If $f\colon X \to X$ is a 1-1 function then $m\bar{x} \leqslant mf\bar{x}$ for all $\bar{x} \in X^n - GD_n$ implies $m\bar{x} = mf\bar{x}$ for all $\bar{x} \in X^n - GD_n$.*

Proof. Assume $f$ is 1-1 and $m\bar{y} \leqslant mf\bar{y}$ for all $\bar{y} \in X^n - GD_n$. Suppose $\bar{x} = (x_1, \ldots, x_n) \in X^n - GD_n$ and $m\bar{x} < mf\bar{x}$. Next consider the sequence $\{f^k\bar{x}\}_{k=0,1,\ldots}$. We wish to extract from $\{f^k\bar{x}\}_{k=0,1,\ldots}$ a subsequence $\{f^{k_i}\bar{x}\}_{i=0,1,\ldots}$ with the following two properties.

1) For each $j$, $1 \leqslant j \leqslant n$, either

(a) $i \neq l$ implies $f^{k_i}(x_j) \neq f^{k_l}(x_j)$ or

(b) $f^{k_i}(x_j) = f^{k_l}(x_j)$ for all $i$ and $l$.

2) $\{f^{k_i}(x_j)\}_{i=0,1,\ldots}$ is a Cauchy sequence for each $j = 1, 2, \ldots, n$.

First we will produce a subsequence satisfying 1) and condition 3) (below).

3) $k_{i+1} - k_i = a$ constant independent of $i$.

We do this by induction on the number of $j$'s for which 1) holds. Clearly $k_i = i$ satisfies 3) and 1) for $j \leqslant 0$. Now assume the inductive hypothesis that we have a sequence $k_i$ which satisfies 3) and 1) for $j \leqslant N$. If 1) (a) holds for $j = N+1$ the sequence $k_i$ already satisfies 3) and 1) for $j \leqslant N+1$. So assume the contrary, $f^{k_i}(x_{N+1}) = f^{k_l}(x_{N+1})$ for some $i < l$. Set $k'_p = k_i + p(k_l - k_i)$. Then it is easily verified that $k'_p$ satisfies 3) and 1) for $j \leqslant N+1$ (3) is needed so that $k'_p$ is a subsequence of $k_i$). Thus by a finite induction, we obtain a subsequence $\big(f^{k_i}(\bar{x})\big)_{i=0,1,\ldots}$ satisfying 1).

Now that we have a subsequence satisfying 1) we simply envoke the hypothesis that $X$ is precompact to extract a subsequence which satisfies both 1) and 2).

Set $\varepsilon = mf\bar{x} - m\bar{x}$. Because $m$ is continuous at $\bar{x}$, there is $\delta > 0$ such that $d(y_i, x_i) \leqslant \delta$ for $i = 1, \ldots, n$ implies $|m(y_1, \ldots, y_n) - m(x_1, \ldots, x_n)| \leqslant \varepsilon/2$. Next, from Definition 1.1 we can find a $\delta' > 0$ such that $\bar{z} \in X^n - GD_n$ and $m\bar{z} \leqslant \delta'$ implies $\operatorname{diam}\bar{z} \leqslant \delta$. Finally, using Definition 1.1 again find a $\delta'' > 0$ such that $\bar{z} \in X^n - GD_n$ and $\operatorname{diam}\bar{z} \leqslant \delta''$ implies $m\bar{z} \leqslant \delta'$.

Since $\{f^{k_i}(\bar{x}_j)\}_{i=0,1,\ldots}$ is a Cauchy sequence for each $j = 1, \ldots, n$ we can find a $q$ such that $\operatorname{diam}\bar{y}^j \leqslant \delta''$ for $j = 1, \ldots, n$ where $\bar{y}^j = \big(f^{k_{q+1}}(x_j), f^{k_{q+2}}(x_j), \ldots, f^{k_{q+n}}(x_j)\big)$. Let $J_{1a}(J_{1b})$ be the set of $j$'s for which condition 1a (1b) above holds. For $j \in J_{1a}$ we have $m\bar{y}^j \leqslant \delta'$. Since

$$\bar{y}^j = f^{k_{q+1}}\bar{z}^j \quad \text{where} \quad \bar{z}^j = \big(x_j, f^{k_{q+2}-k_{q+1}}(x_j), f^{k_{q+3}-k_{q+1}}(x_j), \ldots, f^{k_{q+n}-k_{q+1}}(x_j)\big)$$

we can conclude that

(1) $$m\bar{z}^j \leqslant \delta' \quad \text{for} \quad j \in J_{1a}.$$

Hence

(2) $$\operatorname{diam} \bar{z}^j \leqslant \delta \quad \text{for} \quad j \in J_{1a}.$$

Now for $j \in J_{1b}$ we have $\operatorname{diam} \bar{y}^j = 0$ and since $f$ is 1-1 it follows that $\operatorname{diam} \bar{z}^j = 0$ for $j \in J_{1b}$. Thus $\operatorname{diam} \bar{z}^j \leqslant \delta$ for all $j = 1, \ldots, n$. Consequently $d(x_j, f^{k_{q+2}-k_{q+1}}(x_j)) \leqslant \delta$ for $j = 1, \ldots, n$. Hence $|m\bar{x} - mf^{k_{q+2}-k_{q+1}}\bar{x}| \leqslant \varepsilon/2$. But on the other hand $mf\bar{x} - m\bar{x} = \varepsilon$ and $mf^{k+1}\bar{x} = mff^k\bar{x} \geqslant mf^k\bar{x}$ for all $k$ and so $mf^{k_{q+2}-k_{q+1}}\bar{x} \geqslant m\bar{x} + \varepsilon$. We have thus reach a contradiction. Therefore no such $\bar{x}$ exists and consequently $m\bar{y} = mf\bar{y}$ for all $\bar{y} \in X^n - GD_n$ as we wished to prove. Q.E.D.

(2.3) Corollary. *Let $(X, d)$ be a precompact connected metric space and $m$ a generalized diameter on $n$-tuplets. If $f\colon X \to X$ is* 1-1 *and continuous, $\bar{x} \in X^n - GD_n = Z$ and $m\bar{x} < mf\bar{x}$ then there is a $\bar{y} \in Z$ such that $m\bar{y} > mf\bar{y}$ and for some $N$, $\bar{y}$ cannot be separated from $f^N\bar{x}$ in $Z$.*

Proof. Assume all the hypotheses and that $m\bar{y} \leqslant mf\bar{y}$ for all $\bar{y} \in Z$ such that for some $N$ (depending on $\bar{y}$), $\bar{y}$ cannot be separated from $f^N\bar{x}$ in $Z$. We may now proceed exactly as in Proposition 2.2 up to but not including the line marked (1). We will bypass (1) and establish (2) by another method. In order to obtain (2) under our present hypothesis we observe that by Lemma 3.9 of [2] there exists a $\sigma_j \in S_n$ such that $\sigma_j \bar{z}^j S'_Z \bar{x}$ for each $j \in J_{1a}$. Now since $f$ is continuous we have

(3) $$f^r \sigma_j \bar{z}^j S'_Z f^r \bar{x} \quad \text{for all} \quad r \geqslant 0.$$

It follows then by induction that for $r = k_{q+1}$

$$m\sigma_j \bar{y}^j = m\sigma_j f^r \bar{z}^j = mf^r \sigma_j \bar{z}^j \geqslant mf^{r-1}\sigma_j \bar{z}^j \geqslant m\sigma_j \bar{z}^j.$$

But $\operatorname{diam} \sigma_j \bar{y}^j = \operatorname{diam} \bar{y}^j \leqslant \delta$ and so $m\sigma_j \bar{y}^j \leqslant \delta'$. Thus $m\sigma_j \bar{z}^j \leqslant \delta'$ which implies $\operatorname{diam} \bar{z}^j = \operatorname{diam} \sigma_j \bar{z}^j \leqslant \delta$ for $j \in J_{1a}$. Thus we have established line (2) and we can again proceed exactly as in the proof of Proposition 2.2. Q.E.D.

Remark. The continuity of $f$ was used only to establish (3). Thus according to Proposition 2.9 below, we could replace continuity of $f$ by assuming $f^{-1}$ exists and is 1-1, onto and continuous.

A straight forward connectedness argument gives

(2.4) Lemma. *Suppose $X$ is a locally connected Hausdorff space and $F$ is a finite subset of $X$. If $x, y \in O \equiv X - F$ and $x$ cannot be separated from $y$ in $O$ then there is a closed subset $C$ of $X$ contained in $O$, $C \subset O$, such that $x, y \in C$ and $x$ cannot be separated from $y$ in $C$.*



(2.5) Notation. Given $\bar{x} = (x_1, \ldots, x_n) \in Y^n$ and $p \in Y$ set

1) $\bar{x}_i = x_i$ for $i = 1, \ldots, n$,

2) $|\bar{x}| = \{x_1, \ldots, x_n\}$,

3) ${}_{p,i}\bar{x} = (x_1, \ldots, x_{i-1}, p, x_{i+1}, \ldots, x_n)$,

4) ${}^j\bar{x} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$,

5) $W(i; \bar{x}) = \{x_1\} \times \{x_2\} \times \ldots \times \{x_{i-1}\} \times Y \times \{x_{i+1}\} \times \ldots \times \{x_n\} \cap \cap (Y^n - GD_n(Y))$.

(2.6) Lemma. *Suppose $Y$ is a locally connected Hausdorff space $n \geqslant 2$ and $g\colon Y^n - GD_n(Y) \to \mathbf{R}$ is continuous. If $g(\bar{x}) < 0$ and $g(\bar{y}) > 0$ for some $\bar{x}, \bar{y} \in Y^n - GD_n(Y) = Z$ such that $\bar{x}T\bar{y}$ (see proof for definition of $T$) then cardinality $\{\bar{z} \in Z \mid g(\bar{z}) = 0\} \geqslant$ cardinality $O$ where $O$ is some nonempty open subset of $Y$.*

Proof. The relation $\bar{x}T\bar{y}$ means that for some $j$, ${}^j\bar{x} = {}^j\bar{y}$ and $\bar{x}S'_{W(j,\bar{x})}\bar{y}$. Suppose $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$. Clearly the relation $\bar{x}S'_{W(j,\bar{x})}\bar{y}$ is equivalent to $x_j S'_{Y-|{}^j\bar{x}|} y_j$. Since $n \geqslant 2$ we can find a $k$ such that $1 \leqslant k \leqslant n$ and $k \neq j$. By Lemma 2.4 pick a closed set $C$ such that $x_j, y_j \in C \subset Y - |{}^j\bar{x}|$ and $x_j S'_C y_j$. Using the continuity of $g$ pick a neighborhood $U$ of $x_k = y_k$ such that $p \in U$ implies $g({}_{p,k}\bar{x}) < 0$ and $g({}_{p,k}\bar{y}) > 0$. We now claim that $O = U \cap (Y - C)$ satisfies the requirements of the lemma. Clearly $O$ is nonempty ($x_k \in O$) and open. Next, we will assign to each $p \in O$ a distinct $\bar{z} = \bar{z}(p) \in Z$ such that $g(\bar{z}(p)) = 0$. Let $p \in O$. By the construction of $O$ we clearly have $x_j S'_{Y-|{}^j({}_{p,k}\bar{x})|} y_j$ and consequently ${}_{p,k}\bar{x} S'_{W(j,{}_{p,k}\bar{x})} {}_{p,k}\bar{y}$. Since $g$ is continuous there must exist a $\bar{z} = \bar{z}(p) \in W(j, {}_{p,k}\bar{x})$ such that $g(\bar{z}(p)) = 0$. It follows directly from the definitions that if $p \neq q$ then $W(j, {}_{p,k}\bar{x}) \cap W(j, {}_{q,k}\bar{x}) = \emptyset$. Hence $p, q \in O$ and $p \neq q$ implies $\bar{z}(p) \neq \bar{z}(q)$. Therefore cardinality $\{\bar{z} \in Z \mid g(\bar{z}) = 0\} \geqslant$ cardinality $O$ as we wished to prove. Q.E.D.

(2.7) Propostion. *Suppose $Y$ is a locally connected Hausdorff space and $g\colon Y^n - GD_n(Y) \to \mathbf{R}$ is continuous. If $g(\bar{x}) < 0$ and $g(\bar{y}) < 0$ for some $\bar{x}, \bar{y} \in Y^n - GD_n(Y) = Z$ such that $\bar{x}S'_Z\bar{y}$ then cardinality $\{\bar{z} \in Z \mid g(\bar{z}) = 0\} = $ cardinality $O$ where $O$ is some non-empty open subset of $Y$.*

Proof. By Theorem 18.2 of [2] there is a sequence $\bar{x} = \bar{x}^0, \bar{x}^1, \ldots, \bar{x}^m = \bar{y}$ such that $\bar{x}^i T \bar{x}^{i+1}$ for $i = 0, \ldots, m-1$. Since $g(\bar{x}^0) < 0$ and $g(\bar{x}^m) > 0$ it is easy to see that there must be integers $i_0 \leqslant j_0$ such that $g(\bar{x}^{i_0}) < 0$, $g(\bar{x}^k) = 0$ for $i_0 < k < j_0$ and $g(\bar{x}^{j_0}) > 0$. We claim that if $j_0 - i_0 > 1$ then either the conclusion of the lemma holds or we can reduce the length of the sequence $\bar{x}^{i_0}, \ldots, \bar{x}^{j_0}$, that is we can find a sequence $\bar{z}^0, \bar{z}^1, \ldots, \bar{z}^l$ such that $\bar{z}^i \in Z$, $\bar{z}^i T \bar{z}^{i+1}$ for $i = 0, \ldots, l-1$, $g(\bar{z}^0) < 0$, $g(\bar{z}^k) = 0$ for $k = 0, \ldots, l-1$, $g(\bar{z}^l) > 0$ and $1 \leqslant l = l - 0 < j_0 - i_0$. Assuming this claim for a moment we see that an induction argument leads to either the desired conclusion or a pair $\bar{u}, \bar{v} \in Z$ such that $\bar{u}T\bar{v}$, $g(\bar{u}) < 0$ and

$g(\bar{v}) > 0$. But in the latter case the desired conclusion follows from Lemma 2.6. Hence it is sufficient to establish the claim.

Assume $j_0 - i_0 > 1$. Since $\bar{x}^i T \bar{x}^{i+1}$ there is an integer $k_i$ such that ${}^{k_i}\bar{x}^i = {}^{k_i}\bar{x}^{i+1}$ and $\bar{x}^i S'_{W(k_1,\bar{x}^i)} \bar{x}^{i+1}$. If $k_i = k_{i+1}$ then it follows easily that $\bar{x}^i T \bar{x}^{i+2}$ and so we could drop $x^{i+1}$ making the sought after reduction. So we may now assume $k_i \neq k_{i+1}$ for $i_0 \leqslant i < j_0$. Let $a$ be the first integer $\beta$ greater than $i_0$ such that $k_\beta = k_{i_0}$ and $\beta < j_0$ if such an integer exists and otherwise set $a = j_0$. Let $V$ be a neighborhood of $\bar{z}^{i_0+1}_{k_{i_0}}$ such that for each $p \in V$

$$\bar{z}^{i_0} T_{p,k_{i_0}} \bar{z}^{i_0+1}$$

and

$$_{p,k_{i_0}}\bar{z}^i T_{p,k_{i_0}} \bar{z}^{i+1} \quad \text{for} \quad i = i_0+1, \ldots, a-1$$

and

$$_{p,k_{i_0}}\bar{z}^a T \bar{z}^{a+1}.$$

Such a neighborhood $V$ exists due to Lemma 2.4 and the local connectedness of $Y$ at $\bar{z}^1_{k_{i_0}} = \bar{z}^a_{k_a} = \bar{z}^{i_0+1}_{k_{i_0}}$. (The last equality comes from the fact the $k_{i_0}$th spot is not altered after the transition $\bar{z}^{i_0} \to \bar{z}^{i_0+1}$ until the transition $\bar{z}^a \to \bar{z}^{a+1}$ due to the choice of $a$). Now either

(a) $\qquad g({}_{p,k_{i_0}}\bar{z}^{i_0+1}) = 0 \quad$ for all $p \in V$

or

(b) $\qquad g({}_{p,k_{i_0}}\bar{z}^{i_0+1}) > 0 \quad$ for some $p \in V$

or

(c) $\qquad g({}_{p,k_{i_0}}\bar{z}^{i_0+1}) < 0 \quad$ for some $p \in V$.

Since ${}_{p,k_{i_0}}\bar{z}^{i_0+1} \neq {}_{q,k_{i_0}}\bar{z}^{i_0+1}$ for $p \neq q$ we clearly have the desired conclusion in case (a). In either case (b) or (c) we have the desired reduction. Therefore the claim and hence the lemma is established. Q.E.D.

Let $X$ be the space $X$ considered in [2]. See article 12 of [2] for the definition of $u$ used below.

(2.8) Lemma. *Suppose $f\colon X \to X$ is a continuous* 1-1 *map. If $X$ is noncircular then either*

(i) *$x < y$ iff $f(x) < f(y)$ or*

(ii) *$x < y$ iff $f(y) < f(x)$.*

*If $X$ is circular then either*

(iii) *$x < y$ iff $f(x) <' f(y)$ where $s <' t$ iff either $s < t < f(u)$ or $f(u) \leqslant s < t$ or $t < f(u) \leqslant s$, or*

(iv) *$x < y$ iff $f(y) <'' f(x)$ where $s <'' t$ iff either $s < t \leqslant f(u)$ or $f(u) < s < t$ or $t \leqslant f(u) < s$.*

Proof. Consider the case where $X$ is non-circular first. We claim that if $f(x) < f(y) < f(z)$ then either $x < y < z$ or $z < y < x$. Suppose



$f(x) < f(y) < f(z)$. By considering $\beta = \{f(x), f(y), f(z)\}$ and $R_\beta$ we see that there are disjoint open sets $U$ and $V$ such that $f(x) \in U$, $f(z) \in V$ and $U \cup V = X - \{f(y)\}$. Then $f^{-1}(U)$ and $f^{-1}(V)$ separate $x$ from $z$ in $X - \{y\}$ and so either $x < y < z$ or $z < y < x$ (see Lemma 9.5) of [2]. This establishes our claim.

Next we will show that if neither $x < y$ nor $y < x$ holds then neither $f(x) < f(y)$ nor $f(y) < f(x)$ holds. Suppose that either $f(x) < f(y)$ or $f(y) < f(x)$ holds. Without loss of generality we may assume $f(x) < f(y)$. Then in order not to contradict the connectedness of $X$ there must be a $t'$ such that $f(x) < t' < f(y)$. And again by the connectedness of $X$ and continuity of $f$ there must be a $t$ such that $f(t) = t'$: then by the first paragraph either $x < t < y$ or $y < t < x$. Thus if neither $x < y$ nor $y < x$ holds then neither $f(x) < f(y)$ nor $f(y) < f(x)$ holds as we wished to show

It follows that either $f(\mathcal{E}^-) \subset \mathcal{E}^-$ or $f(\mathcal{E}^-) \subset \mathcal{E}^+$. Similarly, either $f(\mathcal{E}^+) \subset \mathcal{E}^-$ or $f(\mathcal{E}^+) \subset \mathcal{E}^+$. Thus $f(\mathcal{E}) = f(\mathcal{E}^+ \cup \mathcal{E}^-) \subset f(\mathcal{E})$. Since $f$ is 1-1 and $\mathcal{E}$ is finite we must have $f(\mathcal{E}) = \mathcal{E}$ and consequently $f(M) = f(X - \mathcal{E}) \subset X - \mathcal{E} = M$.

Now pick $x, y \in M$ such that $x < y$. We have two cases, case 1: $f(x) < f(y)$ and case 2: $f(y) < f(x)$. Consider case 1, $f(x) < f(y)$. We claim that if $z < x$ then $f(z) < f(x)$. Since $f(x), f(y) \in M$ and $f(z)$ is distinct from $f(x)$ and $f(y)$ it follows that $f(z)$ is comparable to $f(x)$ and to $f(y)$. Hence if $f(z) < f(x)$ fails then either (a) $f(x) < f(z) < f(y)$ or (b) $f(x) < f(y) < f(z)$ holds. In case (a) we would have either (i) $x < z < y$ or (ii) $y < z < x$. But (i) gives $x < z$ which is a contradiction and (ii) gives $y < x$ which is also impossible. Case (b) leads in the same way to contradictions. Thus $f(z) < f(x)$ as claimed. It can be shown in the same case by case way that if $x < z < y$ then $f(x) < f(y) < f(y)$ and if $y < z$ then $f(y) < f(z)$ One consequence of this is that $f(\mathcal{E}^-) \subset \mathcal{E}^-$ and $f(\mathcal{E}^+) \subset \mathcal{E}^+$. A second consequence is that if $z < t$ then by considering $\{x, y, z, t\}$ three at a time in various ways one can conclude that $f(z) < f(t)$. These two consequences prove the lemma for case 1. Case 2 can be handled in a completely analogous way. This establishes the lemma for the non-circular case.

Now suppose $X$ is circular. We can reduce this case to the non-circular case by considering the map $f|X - \{u\}\colon X - \{u\} \to X - \{f(u)\}$ where $u$ is the initial element in the order $<$ (see Definition 12.1 of [2]). The orders $<$ and $<'$ have the same significance in terms of cut points with respect to $X - \{u\}$ and $X - \{f(u)\}$ respectively as $<$ does when $X$ is non-circular. Consequently, either $x < y$ iff $f(x) <' f(y)$, or $x < y$ iff $f(y) <' f(x)$ where $x, y \in X - \{u\}$. It is now easy to see that in the former case (iii) holds and in the latter case (iv) holds. This proves the lemma. Q.E.D.

(2.9) Proposition. *Let $Y$ be a connected $T_1$-space and $f\colon Y\to Y$ a continuous 1-1 map. If $\bar x,\bar y\in Y^n-GD_n(Y)=Z$ then $\bar x S'_Z\bar y$ iff $f\bar x S'_Z f\bar y$.*

Proof. If $Z$ is connected the conclusion is trivial. So assume $Z$ is not connected. Then the fundamental hypothesis in [2] are satisfied and Lemma 2.8 above applies. Keeping in mind Lemmas 3.3, 9.5 and 12.7 and paragraph 3.5 all of [2] the conclusion of Proposition 2.9 is quite apparent. The details are left to the reader. Q.E.D.

(2.10) Notation. $aC_Y b$ will mean that $a$ and $b$ are in the same component of $Y$.

(2.11) Lemma. *Let $X$ be the space considered in [2] and suppose $f\colon X\to X$ is 1-1 and continuous. Assume further that $\bar x=(x_1,\ldots,x_n)\in X^n-GD_n(X)=Z$ such that $x_1<x_2<\ldots<x_n$ and $x_i\in M$ for $i=1,\ldots,n$ if $X$ is non-circular. Then if $\bar y\in Z$ and $\bar y S'_Z f^m\bar x$ for some $m$, $-\infty<m<\infty$, then either $\bar y C_Z\bar x$ or $\tau\bar y C_Z\bar x$ where $\tau\in S_n$ is given by $\tau(i)=n+1-i$ for $i=1,\ldots,n$.*

Proof. It follows easily from Lemmas 9.5, 12.7 and 3.3 and paragraph 3.5 of [2] and Lemma 2.8 above that either $\bar y S'_Z\bar x$ or $\tau\bar y S'_Z\bar x$. Since $Z$ has only a finite number of components (see Lemmas 11.2 and 12.8 of [2]) it follows that either $\bar y C_Z\bar x$ or $\tau\bar y C_Z\bar x$ as we wished to show. Q.E.D.

The following lemma is proven in the same manner as Lemma 2.11.

(2.12) Lemma. *Under the hypothesis of Lemma 2.11 if it is also assumed that $f=h^2$ or $f=(h^{-1})^2$ for some continuous 1-1 (redundant) map $h\colon X\to X$ then the final conclusion can read $\bar y C_Z\bar x$.*

The following lemma is easily established.

(2.13) Lemma. *If $O$ is a non-empty open subset of a connected non-trivial (card $Y\geqslant 2$) precompact metric space $Y$ then cardinality $O$ = cardinality of $\mathbf{R}$.*

(2.14) Proof of Theorem 2.1. First consider the case where $Z=X^n-GD_n(X)$ is connected. If $mf\bar x=m\bar x$ for all $\bar x\in Z$ then clearly the theorem follows from Lemma 2.13. So assume $mf\bar u\neq m\bar u$ for some $\bar u\in Z$. Hence $mh\bar u>m\bar u$ for some $h\in\{f,f^{-1}\}$. Proposition 2.2 now implies that $mh\bar v<m\bar v$ for some $\bar v\in Z$. Thus we have $mf\bar x<m\bar x$ and $mf\bar y>m\bar y$ for some $\bar x,\bar y$ such that $\{\bar x,\bar y\}=\{\bar u,\bar v\}$. Set $g(\bar z)=mf\bar z-m\bar z$ for all $\bar z\in Z$. Since $Z$ is connected $g(\bar z)=0$ for some $\bar z\in Z$ and this proves the first assertion of the theorem (in this case). Now suppose $X$ is locally connected. Proposition 2.7 now yields

$$\text{cardinality }\{\bar z\in Z|\ g(\bar z)=0\}=\text{cardinality }O$$

where $O$ is some non-empty open subset of $X$. The second assertion of the theorem now follows from Lemma 2.13.

Next consider the case where $Z$ is disconnected. The fundamental hypothesis of [2] now apply to our space $X$ and consequently we assume



that it is the space considered in [2]. If $mf\bar x=m\bar x$ for all $\bar x\in Z$ such that $\bar x_1<\bar x_2<\ldots<\bar x_n$ and $x_i\in M$ for $i=1,\ldots,n$ [Set $M=X$ if $X$ is circular] then clearly the theorem follows from Lemma 2.13. So assume $mf\bar u\neq m\bar u$ for some $\bar u\in Z$ such that $\bar u_1<\bar u_2<\ldots<\bar u_n$ and $\bar u_i\in M$ for $i=1,\ldots,n$. Hence $mh\bar u>m\bar u$ for some $h\in\{f,f^{-1}\}$. Now by Corollary 2.3 and the remark that follows it we know that there is a $\bar v\in Z$ such that $mh\bar v<m\bar v$ and for some $N$, $\bar v S'_Z h^N\bar u$. Next, it follows from Lemma 2.11 that either $\bar v S'_Z\bar u$ or $\tau\bar v S'_Z\bar u$ where $\tau\in S_n$ is given by $\tau(i)=n+1-i$ for $i=1,\ldots,n$. But since $m$ is weakly symmetric we have $m\tau\bar v=m\bar v$ and $mh\tau\bar v=m\tau h\bar v=mh\bar v$. Thus by replacing $v$ by $\tau v$ if necessary we can assume $\bar v S'_Z\bar u$. Hence we have $mf\bar x<m\bar x$ and $mf\bar y>m\bar y$ for some $\bar x,\bar y\in Z$ such that $\{\bar x,\bar y\}=\{\bar u,\bar v\}$. Note also that $\bar y S'_Z\bar x$. The proof of this case can now be completed exactly as in the first case. Hence, the theorem has been established in all cases. Q.E.D.

(2.15) Corollary. *Theorem 2.1 holds with the hypothesis "$m$ is weakly symmetric" replaced by "$f=l^2$ for some continuous function $l\colon X\to X$".*

Proof. Using Lemma 2.12 in place of Lemma 2.11 the proof of Theorem 2.1 goes through in essentially the same way. Q.E.D.

The first case considered in 2.14 is actually a proof of the following corollary.

(2.16) Corollary. *Theorem 2.1 holds with the hypothesis "$m$ is weakly symmetric" replaced by "$Z=X^n-GD_n(X)$ is connected".*

In light of Corollary 2.16 it is of interest to know conditions under which $Z$ must be connected. The following two lemmas and two propositions provide a large class of spaces such that $Z$ is connected.

(2.17) Lemma. *If $Y^n-GD_n(Y)$ $(n\geqslant 2)$, is connected and $Y$ is a non-trivial connected subspace of a connected $T_1$ space $X$ then $X^n-GD_n(X)$ is also connected.*

Proof. Let $\bar x,\bar y\in X^n-GD_n(X)=Z$. We will show that $\bar x S'_Z\bar y$. Since $Y$ is non-trivial and connected there exists a $\bar z\in Y^n-GD_n(Y)=Z'\subset Z$. From $Z'\subset Z$ it follows that $\bar u,\bar v\in Z'$ and $\bar u S'_{Z'}\bar v$ implies $\bar u S'_Z\bar v$. Now using the connectedness of $Z'$ and Lemma 3.9 of [2] we have $\bar x S'_Z\sigma\bar z$, $\bar y S'_Z\tau\bar z$ for some $\sigma,\tau\in S_n$ and $\sigma\bar z S'_{Z'}\tau\bar z$. Thus $\bar x S'_Z\bar y$ as claimed. Since $\bar x$ and $\bar y$ were arbitrary elements of $Z$, $Z$ must be connected. Q.E.D.

(2.18) Lemma. *The space $T=\{(x,y)\in\mathbf{R}^2|$ either $y=0$ and $-1\leqslant x\leqslant 1$, or $x=0$ and $-1\leqslant y\leqslant 0\}$ is such that $T^n-GD_n(T)$ is connected for all $n\geqslant 2$.*

Proof. We claim that $T^n-GD_n(T)$ is pathwise connected. This means that given two ordered $n$-tuples $\bar x,\bar y$ of $n$ distinct points of $X$ we can move the points of $\bar x=(x_1,\ldots,x_n)$ around, never allowing two to coincide, until they coincide with $y_1,\ldots,y_n$ in that order. Considering the way three lines of $T$ meet at $(0,0)$ the above claim is intuitively obvious. We leave the details of a analytic proof to the reader. Q.E.D.

Combining Lemmas 2.17 and 2.18 we have

(2.19) PROPOSITION. *A connected $T_1$-space $X$ is such that $X^n - GD_n(X)$ is connected provided $T$ can be embedded in $X$.*

The converse to Proposition 2.19 holds if we restrict $X$ to be either a compact polyhedron or a compact manifold and require cardinality $X \geqslant n \geqslant 3$. This is seen by noting that if $X^n - GD_n(X)$ is connected then $X$ is connected, and if $T$ could not be embedded in $X$ then $X$ is 1-dimensional. Furthermore in the case where $X$ is a polyhedron (and hence a graph) there cannot be any vertices of $X$ which are faces (ends) of more than two edges. Thus in both cases, manifold and polyhedron, $X$ is homeomorphic to either the closed interval $[0,1]$ or the circle $\{(x,y) \in \boldsymbol{R}^2 |\ x^2+y^2 = 1\}$. But in both of these eventualities $X^n - GD_n(X)$ is not connected.

(2.20) PROPOSITION. *If $X$ is a non-trivial connected and locally connected Hausdorff space and $X^3 - GD_3(X)$ is connected then $X^n - GD_n(X)$ is connected for all $n \geqslant 3$.*

Proof. Suppose $X^n - GD_n(X)$ is not connected for some $n \geqslant 3$. Then by Lemmas 11.2, 12.8, and 14.2 of [2] we conclude that $X^3 - GD_3(X)$ has either 3! or (3–1)! components and is thus not connected a contradiction. Thus $X^n - GD_n(X)$ is connected for all $n \geqslant 3$. Q.E.D.

The following example shows that the hypothesis of weak symmetry for $m$ cannot just be dropped from Theorem 2.1.

(2.21) COUNTER EXAMPLE. Set $X = [-1,1]$ and let $f\colon X \to X$ be given by $f(x) = -x$ for all $x \in [-1,1]$. Define $m\colon X^2 - GD_2(X) \to \boldsymbol{R}^+$ by: for $(x,y) \in X^2 - GD_2(X)$

$$m(x,y) = \begin{cases} |x-y| & \text{if} \quad x < y\,, \\ 2|x-y| & \text{if} \quad x > y\,. \end{cases}$$

Then $m(x,y) \neq m(f(x), f(y))$ for all $(x,y) \in X^2 - GD_2(X)$.

**Section 3. There exist many small invariant $n$-tuples.** In this section we will extend Theorem 2.1 to conclude under additional hypothesis on the space $X$ that there are a continuum (in cardinality) of arbitrarily small invariant $n$-tuples. The additional hypothesis on $X$ needed (to prove the theorem) is contained in the following definition.

(3.1) DEFINITION. Let $(X,d)$ be a metric space. We say that $(X,d)$ is *uniformly locally connected* provided for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $p \in X$ there is a connected set $V_p$ such that

$$\{q \in X|\ d(q,p) < \delta\} \subset V_p \subset \{q \in X|\ d(q,p) < \varepsilon\}\,.$$



(3.2) THEOREM. *Suppose $(X,d)$ is a non-trivial connected, uniformly locally connected, precompact metric space and $f\colon X \to X$ is 1-1, onto and continuous and $\varepsilon$ is a positive number. If $m$ is a weakly symmetric generalized diameter on $n$-tuples ($n \geqslant 2$) then cardinality $I^\varepsilon$ = cardinality $X$ = cardinality of the continuum where $I^\varepsilon$ is the set of all invariant $n$-tuples $\bar{z}$ for $f$ and $m$ such that $\operatorname{diam}\bar{z} < \varepsilon$.*

(3.3) PROPOSITION. *Let $(X,d)$ be a precompact metric space, $\varepsilon_1$ a positive number, $m$ a generalized diameter on $n$-tuples and $g\colon X \to X$ an onto function. Suppose $\bar{x} \in X^n - GD_n(X) = Z$ and $m\bar{x} < \varepsilon_1$ implies that either $g\bar{x} \in GD_n(X)$ or $mg\bar{x} \leqslant m\bar{x}$. Then $x \in Z$ and $m\bar{x} < \varepsilon_1$ implies that either $g\bar{x} \in GD_n(X)$ or $mg\bar{x} = m\bar{x}$.*

Proof. Assume the hypothesis and suppose $\bar{w} \in Z$, $g\bar{w} \in Z$, $m\bar{w} < \varepsilon_1$, and $mg\bar{w} < m\bar{w}$. Set $\bar{x} = (x_1, \ldots, x_n) = (g(w_1), \ldots, g(w_n)) = g(\bar{w})$ where $\bar{w} = (w_1, \ldots, w_n)$. Since $g$ is onto we can find a function $f\colon X \to X$ such that $g(f(t)) = t$ for all $t \in X$ and $f(x_i) = w_i$ for $i = 1, \ldots, n$. This $f$ must then be 1-1 and satisfy

$$\varepsilon_1 > mf\bar{x} > m\bar{x} \tag{4}$$

and

$$mf\bar{z} \geqslant \min(\varepsilon_1, m\bar{z}) \quad \text{for all } \bar{z} \in Z\,. \tag{5}$$

These facts follow directly from the assumptions on $g$.

Now the argument used in Proposition 2.2 can be applied with only a couple minor adjustments. First we change the definition of $\varepsilon$ in the proof of Proposition 2.2 to $\varepsilon \equiv \min(mf\bar{x} - m\bar{x}, \varepsilon_1 - m\bar{x})$. Secondly we now pick $q$ so that $\operatorname{diam}\bar{y}^j \leqslant \min(\delta'', \varepsilon_1)$. With these changes the argument in Proposition 2.2 goes through and the desired contradiction is reached. Thus no such $\bar{w}$ exists and the present proposition is proved. Q.E.D.

(3.4) DEFINITION. Let $(X,d)$ be a metric space and $n \geqslant 2$. Set $Z = Z_n = X^n - GD_n(X)$ and $Z_n^\varepsilon = Z^\varepsilon = \{\bar{x} \in Z|\ \operatorname{diam}\bar{x} < \varepsilon\}$ for each $\varepsilon > 0$. Now define a relation $T^\varepsilon$ on $Z^\varepsilon$ as follows: for $\bar{x}, \bar{y} \in Z^\varepsilon$ $\bar{x} T^\varepsilon \bar{y}$ iff there exists an $i$ such that ${}^i\bar{x} = {}^i\bar{y}$ and $\bar{x} S'_H \bar{y}$ where $H = W(i, \bar{x}) \cap Z^\varepsilon$. (See Notation 2.5.) We write $\bar{x} \overline{T}^\varepsilon \bar{y}$ provided there is a sequence $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N$ taken from $Z^\varepsilon$ such that $\bar{x} = z_1$, $\bar{z}_{i-1} T^\varepsilon \bar{z}_i$ for $1 < i \leqslant N$, and $\bar{z}_N = \bar{y}$.

(3.5) PROPOSITION. *Suppose $(X,d)$ is a non-trivial connected and uniformly locally connected metric space and $\varepsilon > 0$. Then there is a $\delta'$ such that for every $\bar{x}, \bar{y} \in Z_n^{\delta'}$ there is a $\sigma \in S_n$ such that $\bar{x} \overline{T}^\varepsilon \sigma\bar{y}$.*

Proof. Since $(X,d)$ is uniformly locally connected we can find a $\delta > 0$ such that for each $p \in X$ there is a connected set $V_p$ such that $\{q \in X|\ d(q,p) < \delta\} \subset V_p \subset \{q \in X|\ d(q,p) < \frac{1}{2}\varepsilon\}$. Set $\delta' = \frac{1}{2}\delta$.

Suppose $z, z' \in X$ and $d(z, z') < \delta'$. We will show that if $\bar{u}, \bar{v} \in Z_n^{\delta'}$, $\bar{u}_1 = z$ and $\bar{v}_1 = z'$ then $\bar{u}\overline{T^\varepsilon}\sigma\bar{v}$ for some $\sigma \in S_n$. Consider the connected set $V_z$ given by the first paragraph. Clearly $|\bar{u}| \subset V_z$ and $|\bar{v}| \subset V_z$ and thus $\bar{u}, \bar{v} \in V_z^n - GD_n(V_z)$. It now follows from Lemmas 3.9 and 18.2 of [2] that $\bar{u}\overline{T}_{V_z}\sigma\bar{v}$ for some $\sigma \in S_n$ where $T_{V_z} = T$ defined with respect to the space $V_z$ instead of $X$. Note the simple fact that $A \subset B$ and $xS'_A y$ imply $xS'_B y$. Now recalling the definitions of $\overline{T}_{V_z}$ and $\overline{T^\varepsilon}$ and the inclusion $V_z \subset \{q \in X \mid d(q, z) < \frac{1}{2}\varepsilon\}$ we see that $\bar{u}\overline{T}_{V_z}\sigma\bar{v}$ implies $\bar{u}\overline{T^\varepsilon}\sigma\bar{v}$. Thus $\bar{u}\overline{T^\varepsilon}\sigma\bar{v}$ as claimed.

Now suppose $\bar{x}, \bar{y} \in Z_n^{\delta'}$. Consider the set $U = \{z \in X \mid \bar{U} \in Z_n^{\delta'}$ and $\bar{u}_1 = z$ implies $\bar{u}\overline{T^\varepsilon}\sigma\bar{x}$ for some $\sigma \in S_n\}$. It follows from the preceding paragraph that $\bar{x}_1 \in U$, $U$ is open and $X - U$ is also open. Since $X$ is connected we conclude that $U = X$. Thus $\bar{x}\overline{T^\varepsilon}\sigma\bar{y}$ for some $\sigma \in S_n$ as we wished to prove. Q.E.D.

A straight forward compactness argument gives

(3.6) Proposition. *If $(X, d)$ is a locally connected compact metric space then $(X, d)$ is uniformly locally connected.*

(3.7) Lemma. *Let $(X, d)$ be a non-trivial connected, uniformly locally connected metric space, $n$ an integer greater than one, and $\varepsilon > 0$. Let $\delta > 0$ and $\{V_p\}_{p \in X}$ be the number and sets given by Definition 3.1 satisfying*

*(a) $V_p$ is a connected neighborhood of $p$ for each $p \in X$ and*

*(b) for each $p \in X$, $\{q \in X \mid d(q, p) < \delta\} \subset V_p \subset \{q \in X \mid d(q, p) < \varepsilon\}$.*

*If $V_p^n - GD_n(V_p)$ is connected for some $p \in X$ then $\bar{u}\overline{T^\varepsilon}\bar{v}$ for all $\bar{u}, \bar{v} \in Z^{\delta/2} = Z_n^{\delta/2}(X)$.*

Proof. Assume $p \in X$, and $V_p^n - GD_n(V_p)$ connected. By Lemma 2.13 we can find a $\bar{w} \in Z^{\delta/2}$ such that $\bar{w}_1 = p$. Let $\bar{u}, \bar{v} \in Z^{\delta/2}$. By Proposition 3.5 we have $\bar{u}\overline{T^\varepsilon}\sigma\bar{w}$ and $\bar{v}\overline{T^\varepsilon}\tau\bar{w}$ for some $\sigma, \tau \in S_n$. Clearly $\sigma\bar{w}, \tau\bar{w} \in V_p^n - GD_n(V_p)$. Now by Theorem 18.2 of [2] we have $\sigma\bar{w}\overline{T}_{V_p}\tau\bar{w}$ and hence $\sigma\bar{w}\overline{T^\varepsilon}\tau\bar{w}$ (see proof of Proposition 3.5). Using the symmetry and transitivity of $\overline{T^\varepsilon}$ we have $\bar{u}\overline{T^\varepsilon}\bar{v}$ as we wished to show. Q.E.D.

We will need to know what happens when all the $V_p^n - GD_n(V_p)$ are disconnected. Lemma 3.10 gives the answer and Lemma 3.9 provides the essential tool in investigating this question.

(3.8) Convention. If $<$ is a linear order on a set $S$ then we will say that $-\infty < x < \infty$ holds for all $x \in S$.

(3.9) Lemma. *Suppose $U$ and $V$ are non-empty open connected subsets of a Hausdorff space $X$ and $<_U, <_V$ are linear orders on $U$ and $V$ respectively inducing their respective topologies (i.e., their relative topologies induced from $X$). (Recall that the topology induced by a linear order $<$ has as a subbase all sets of the form $\{x \mid x < p\}$ and $\{x \mid x > p\}$.) Assume $U \cap V \neq \emptyset$.*



Case 1. *$V - U \neq \emptyset$ and $U - V \neq \emptyset$. Then for some $p, q \in U \cup \{-\infty, \infty\}$, $r, s \in V \cup \{-\infty, \infty\}$, $<_1 \in \{<_U, {}_U{>}\}$ and $<_2 \in \{<_V, {}_V{>}\}$ we have $p <_1 q$, $r <_2 s$, and if*

$$A_1 = \{x \in U \mid x <_1 p\}, \qquad A_2 = \{x \in U \mid q <_1 x\},$$
$$B_1 = \{y \in V \mid y <_2 r\}, \qquad B_2 = \{y \in V \mid s <_2 y\}$$

*then $A_1 = B_2$ and $A_2 = B_1$, $U \cap V = A_1 \cup A_2 = B_1 \cup B_2$ and $<_1 = <_2$ on $A_1$ and $<_1 = <_2$ on $A_2$.*

Case 2. *$V \subset U$ or $U \subset V$; without loss of generality assume $V \subset U$ in this case. Then $V = \{x \in U \mid p <_U x <_U q\}$ for some $p, q \in U \cup \{-\infty, \infty\}$ and either $<_V = <_U$ or $<_V = {}_U{>}$ on $V$.*

Proof. We recall some easily verified facts about connected linearly ordered spaces. First, if $p, q \in U$ and $p <_U q$ then there is an $r \in U$ such that $p <_U r <_U q$. Secondly, if $S$ is a non-empty subset of $U$ bounded from above (below) in $U$ then the least upper bound (greatest lower bound) of $S$, lub $S$ (glb $S$), exists.

Consider Case 1. Assume $V \cap U \neq \emptyset$, $V - U \neq \emptyset$ and $U - V \neq \emptyset$. Pick an $a \in U - V$ and fix it throughout this proof. Also set $< = <_U$. Consider the set $U' = \{x \in U \mid a < x\}$. We claim that $U' \cap V = \{x \in U \mid c < x\}$ for some $c \in U \cup \{\infty\}$. There are two cases.

Case 1A. There is no $b \in U - V$ such that $\{x \in U \mid a < x < b\} \cap V \neq \emptyset$.

Case 1B. There is a $b \in U - V$ such that $\{x \in U \mid a < x < b\} \cap V \neq \emptyset$.

Consider Case 1A. If $U' \cap V = \emptyset$ then $U' \cap V = \{x \in U \mid \infty < x\}$ and the claim holds. So we may assume $U' \cap V \neq \emptyset$. $U' \cap V$ is bounded from below in $U$ by the point $a$ and consequently $c = \text{glb}\, U' \cap V$ exists. Since $c$ is a lower bound for $U' \cap V$ we immediately have $\{x \in U \mid a < x < c\} \subset U' - V$. Next we will show that $\{x \in U \mid c < x\} \subset V$. Suppose the contrary, that is assume $b \in U - V$ and $c < b$ for some $b$. Since $c = \text{glb}\, U' \cap V$ there must be an $e \in U' \cap V$ such that $c < e < b$. But then $e \in \{x \in U \mid a < x < b\} \cap V$ which contradicts the fundamental assumption of Case 1A. Thus $\{x \in U \mid c < x\} \subset V$. In order to establish $U' \cap V = \{x \in U \mid c < x\}$ in this case, Case 1A, we only have left to show that $c \notin U' \cap V$. Suppose $c \in U' \cap V$. Since $U' \cap V$ is an open subset of $U$ and $U$ has the order topology we can find an $f \in U' \cap V$ such that $f < c$ (unless $c$ is an initial point of $U$; but in this case is ruled out because it would imply $U - V = \emptyset$ a contradiction). But $f \in U' \cap V$ and $f < c$ contradicts $c = \text{glb}\, U' \cap V$. Thus $c \notin U' \cap V$ and $U' \cap V = \{x \in U \mid c < x\}$ as claimed.

Consider Case 1B. There is a $b \in U - V$ such that $\{x \in U \mid a < x < b\} \cap V \neq \emptyset$. Pick an $e \in \{x \in U \mid a < x < b\} \cap V$. Consider the sets $S_1 = \{x \in U \mid \{y \in U \mid x < y < e\} \subset V\}$ and $S_2 = \{x \in U \mid \{y \in U \mid e < y < x\} \subset V\}$. Both $S_1$ and $S_2$ are non-empty ($e \in S_1 \cap S_2$), $a$ is a lower bound for $S_1$ and $b$

is an upper bound for $S_2$. Therefore $p = \mathrm{glb}\, S_1$ and $q = \mathrm{lub}\, S_2$ exists. Using the fact that $V$ is open it is easily seen as above that $p, q \notin V$ and $A = \{x \in U \mid p < x < q\} \subset V$. The set $A$ is open and connected and thus $A = \{x \in V \mid r <_V x <_V s\}$ for some $r, s \in V \cup \{-\infty, \infty\}$. (See Lemma 10.1 of [2].) The possibilities where $\{r, s\} \subset \{-\infty, \infty\}$ are ruled out because $e \in A \neq \emptyset$ and $V - U \neq \emptyset$. For definiteness we will assume $s \in V$. Now by Lemma 2.8 (or more precisely by the proof of Lemma 2.8) we have $<_U = <_V$ or $<_U = {}_V>$ on $A$. For definiteness assume $<_U = <_V$ on $A$. Now $(A, <_U|A)$ is a directed set and the inclusion map $A \subset X$ is then a net in $U$ converging to $q$. But this net also converges to $s$ since $<_U|A = <_V|A$. Since $X$ is Hausdorff we must have $q = s \in V$. But this contradicts $q \notin V$. Consequently Case 1B is vacuous.

This establishes the claim that $U' \cap V = \{x \in U \mid c < x\}$ for some $c \in U \cup \{\infty\}$. Similarly we must have $U'' \cap V = \{x \in U \mid x < d\}$ for some $d \in U \cup \{-\infty\}$ where $U'' = \{x \in U \mid x < a\}$. Remembering that $a \in U - V$ we see that $U \cap V = \{x \in U \mid x < d\} \cup \{x \in U \mid c < x\}$ with $d \leqslant a \leqslant c$. Set $A_1 = \{x \in U \mid x <_U d\}$ and $A_2 = \{x \in U \mid c <_U x\}$. Similarly, there is an $r, s \in V \cup \{-\infty, \infty\}$ such that $r \leqslant s$ and $U \cap V = B_1 \cup B_2$ where $B_1 = \{x \in V \mid x <_V r\}$ and $B_2 = \{x \in V \mid s <_V x\}$. Since $A_1$ and $A_2$ are connected (see Lemma 10.1 of [2]) disjoint open subsets of $U \cap V$ they are obviously the components of $U \cap V$ provided they are non-empty. Thus $\{A_1, A_2\} - \{\emptyset\}$ and $\{B_1, B_2\} - \{\emptyset\}$ are each the set of components of $U \cap V$ and are thus equal. By adjusting the definitions of $A_1$, $A_2$, $B_1$ and $B_2$ if necessary we may assume $A_1 = B_2 \neq \emptyset$. It follows that $A_2 = B_1$. We must show that $<_1 = <_2$ on $A_1$ and $<_1 = <_2$ on $A_2$.

We first show that $<_1 = <_2$ on $A_1$. We may assume $A_1 \neq \emptyset$ for the case $A_1 = \emptyset$ is trivial. By the proof of Lemma 2.8 we know that $<_1 = <_2$ or $<_1 = {}_2>$ on $A_1$. Suppose $<_1 = {}_2>$ on $A_1$. Then $(A_1, <_1) = (B_2, {}_2>)$ is a directed set and the inclusion map $A_1 \subset X$ is a net which clearly converges to both $p$ and $s$ (we are assuming that the definitions of $A_1$, $A_2$, $B_1$ and $B_2$ as they appear in the statement of the lemma). Thus, because $X$ is Hausdorff $p = s$. But $p \in U - V$ and $s \in V - U$ a contradiction. Thus $<_1 = <_2$ on $A_1$ as desired.

The same reasoning shows that $<_1 = <_2$ on $A_2$. This completes the proof of Case 1.

Consider now Case 2. This case follows directly from Lemma 10.1 of [2] or more precisely its proof and the proof of Lemma 2.8. Q.E.D.

Let $(X, d)$, $n$, $\varepsilon$, $\delta$, and $\{V_p\}_{p \in X}$ be as in Lemma 3.7 with the further assumption that $V_p^n - GD_n(V_p)$ is disconnected for all $p \in X$. Then by Theorem 14.4 of [2] we have a linear order $<_p$ on $V_p$ such that the topology of $V_p = <_p^l$ or $<_p^c$ (See § 14 of [2] for definitions). If the topology of $V_p = <_p^c$ then it is easy to see that the removal of one point of $V_p$ other than $p$ gives a connected neighborhood $V'_p \subset V_p$ of $p$ such that $<'_p$ (the order analogous to $<_p$ but for $<'_p$) has the property that $<'^l_p =$ the topology of $V'_p$. Thus by replacing $V_p$ by $V'_p$ if necessary we may assume that $<_p^l =$ the topology of $V_p$ for all $p \in X$. Let $\tau$ be the topology of $X$.



(3.10) Lemma. *Under the above assumptions either Case* 1 *or Case* 2 *below holds.*

Case 1. *There is a point $a \in X$ and a linear order $<$ on $X$ with $a$ as an initial point such that $<^c = \tau$. Furthermore, $X$ is circular as in* [2] *and $<$ is as in* [2].

Case 2. *There is a linear order $<$ on $X$ such that $<^l = \tau$. Furthermore, in this case $X$ is non-circular as in* [2] *and $<$ is as in* [2].

Proof. Pick a $p \in X$ and consider the collection $C$ of ordered pairs $(A, <)$ such that $A$ is an open connected subset of $X$ containing $p$, $p \in A \subset X$, and $<$ is a linear order on $A$ such that $<^l =$ the topology of $A$ (relative topology induced by $X$). We say $(A, <) \precsim (B, <')$ provided $A \subset B$ and $< = <'$ on $A$. This makes $C$ a partially ordered set and we wish to infer a maximal element from Zorn's lemma. In order to do this we need to establish that each totally ordered non-empty subset $\{(A_\alpha, <_\alpha)\}_{\alpha \in \Omega}$ of $C$ has an upper bound. We claim that $(A, <)$ is an upper bound for $\{(A_\alpha, <_\alpha)\}_{\alpha \in \Omega}$ where $A = \bigcup_{\alpha \in \Omega} A_\alpha$ and the order $<$ is defined on $A$ by requiring $< = <_\alpha$ on $A_\alpha$ for all $\alpha \in \Omega$. Since $\{(A_\alpha, <_\alpha)\}_{\alpha \in \Omega}$ is totally ordered under $\precsim$ it is easy to see that $<$ is a well defined linear order on $A$. It remains to show that $<^l =$ the topology of $A$. Let $\sigma$ be the topology of $A$. Given a point $q \in A$ we will show that a subset $U \subset A$ is a $<^l$ neighborhood of $q$ if and only if it is a $\sigma$ neighborhood of $q$. We distinguish two cases. Case A: $q$ is not an end point of $A$. Case B: $q$ is an end point of $A$.

Consider Case A. Then there exists points $r, s \in A$ such that $r < q < s$. Let $A_\alpha$, $\alpha \in \Omega$, be such that $r, q, s \in A_\alpha$.

Claim $\{x \in A \mid r < x < s\} \subset A_\alpha$. Suppose not, i.e., suppose $t \in A - A_\alpha$ and $r < t < s$ for some $t$. Let $\beta \in \Omega$ be such that $A_\alpha \subset A_\beta$ and $t \in A_\beta$. Then $A_\alpha \cap \{x \in A_\beta \mid x < t\}$ and $A_\alpha \cap \{x \in A_\beta \mid t < x\}$ are disjoint open sets whose union is $A_\alpha$ and such that they separate $r$ from $s$ in $A_\alpha$. This contradicts the connectedness of $A_\alpha$ and thus establishes the claim.

Now suppose $U$ is a $<^l$ neighborhood of $q$. Then there exists $a, b \in A$ such that $q \in \{x \in A \mid a < x < b\} \subset U$ and $r \leqslant a < b \leqslant s$. Then $\{x \in A_\alpha \mid a < x < b\} = \{x \in A \mid a < x < b\} \subset U$ which shows that $U$ is a $\tau$ neighborhood and thus a $\sigma$ neighborhood of $q$.

Now suppose $U$ is a $\sigma$ neighborhood of $q$. Then $U \cap A_\alpha$ is a neighborhood of $q$ in $A_\alpha$ and since $<_\alpha^l =$ topology of $A_\alpha$ we have $q \in \{x \in A_\alpha \mid a < x < b\} \subset U$ for some $a, b \in A_\alpha$, $r \leqslant a < q < b \leqslant s$. It now follows from the claim that $\{x \in A_a \mid a < x < b\} = \{x \in A \mid a < x < b\}$ and hence $U$ is a $<^l$ neighborhood of $q$. This completes the proof in case A. Case B is just a one-sided version of case A and can be handled very analogously. We leave these details to the reader.

Thus we have established $<^l =$ the topology of $A$, and so $(A, <) \in C$. Clearly $(A, <)$ is an upper bound for $\{(A_a, <_a)\}_{a \in \Omega}$. Therefore Zorn's lemma applies and there is a maximal element $(A, <)$ in $C$.

If $A = X$ then Case 2 holds. To see that $X$ is non-circular as in [2] and $<$ is as in [2] note that $X^2 - GD_2(X)$ is disconnected because $\{(x, y) \in X^2 - GD_2(X) \mid x < y\}$ and $\{(x, y) \in X^2 - GD_2(X) \mid y < x\}$ are open. Also $<$ determines the $R_a$'s of [2] in the same manner as in lemma 9.5 of [2] and the $R_a$'s determine the $<$ of [2] through definitions 8.7 and 8.12 of [2]. From these definitions it is not hard to see that $<$ here is the same as $<$ in [2].

We now assume $A \neq X$. If $V_a \cap A = \emptyset$ for all $a \in X - A$ then $X - A$ would be open and consequently $X$ would be disconnected, a contradiction. Thus $V_a \cap A \neq \emptyset$ for some $a \in X - A$. Lemma 3.9 with $U = A$ and $V = V_a$ analyses the present situation so clearly that it is obvious that $\{a\} = V_a - A$ and $a$ is not an extreme point of $V_a$ for otherwise $(A, <)$ would not be maximal in $C$. Set $\bar{A} = A \cup V_a = A \cup \{a\}$ and extend $<$ to $\bar{A}$ by setting $a < x$ for all $x \in A$. It is then clear that $<^c =$ the topology of $\bar{A}$ and $a$ is an initial element of $\bar{A}$.

Assume for a moment that $X = \bar{A}$.

The fact that $X$ is circular as in [2] and $<$ is as in [2] can be seen as follows. $X^3 - GD_3(X)$ is disconnected because $\{(x, y, z) \in X^3 - GD_3(X) \mid x < y < z \text{ or } y < z < x \text{ or } z < x < y\}$ and $\{(x, y, z) \in X^3 - GD_3(X) \mid y < x < z \text{ or } x < z < y \text{ or } z < y < x\}$ are open. Also $<$ determines the $R_a^1$'s of [2] in the same manner as in lemma 12.7 of [2] and the $R_a$'s determine the $<$ of [2] through definition 12.1 of [2]. From this it is not hard to see that $<$ here is the same as $<$ in [2]. All that remains to be shown is that $\bar{A} = X$.

In order to show $\bar{A} = X$ we first note that $a$ is the limit of the net $(A, <, i)$ where $i\colon A \subset X$ is the inclusion map. It follows that if $b \in X - A$ and $V_b \cap A \neq \emptyset$ then $b$ also would be the limit of $(A, <, i)$ and since $X$ is Hausdorff $b$ would equal $a$. Hence for each $b \in X - \bar{A} = X - (A \cup \{a\})$ we have $V_b \cap A = \emptyset$ and thus $X - \bar{A}$ is open. But $\bar{A} = A \cup V_a$ is also open and non-empty and so because $X$ is connected we must have $X - \bar{A} = \emptyset$ as we wished to show. This completes the proof. Q.E.D.

Continue to assume the hypothesis of Lemma 3.10. We would like to clear up the relation between the order $<$ of Lemma 3.10 and the orders $<_p$. Consider first Case 2 of Lemma 3.10, i.e., $<^l = \tau$. It follows from Lemma 2.8 that for some choice of $<_p$ ($<_p$ or ${}_p>$) we have $< = <_p$ on $V_p$. We will call this choice (in this case) the choice *compactible* with $<$. Now consider Case 1 of Lemma 3.10, i.e., there is a point $a \in X$ which is an initial point of $X$ under $<$ and $<^c = \tau$ and $X$ is circular as in [2]. Consider a $V_p$. If $a \notin V_p$ then it is clear from the definitions involved that the topology of $V_p$ is the same as $(<|_{V_p})^l$ where $<|_{V_p}$ is $<$ restricted to $V_p$. Thus by Lemma 2.8 there is a choice of $<_p$ such that $< = <_p$ on $V_p$. This choice of $<_p$ will be called compatible with $<$. Finally suppose $a \in V_p$. $V_p - \{p\}$ is disconnected whereas $X - \{p\}$ is connected (see lemma 12.5 of [2]) we conclude $V_p \neq X$. Then because $V_p$ is open and connected and $a \in V_p$ there must be $\alpha, \beta \in X$ such that $a < \alpha < \beta$ and $V_p = \{x \mid x < \alpha \text{ or } \beta < x\}$ (see lemma 12.5 and the proof of lemma 14.1 of [2]). It now follows from the definitions that $<_1^l$ is the topology of $V_p$ where $x <_1 y$ iff either $x < y < \alpha$ or $\beta < x < y$ or $y < \alpha < \beta < x$. Thus again by Lemma 2.8 $<_1 = <_p$ for some choice of $<_p$ and this choice will be said to be compatible with $<$.



In all cases it is easy to see that if $<_p$ and $<_q$ are both compatible with $<$ then $<_p = <_q$ on each component of $V_p \cap V_q$ (there can be at most two components). Recall that $V_p \subset \{q \in X \mid d(q, p) < \varepsilon\}$. If we insist that $4\varepsilon <$ diameter $X = \sup_{x,y \in X} d(x, y)$ then we can show that $V_p \cap V_q$ can have at most one component. Indeed, it is clear from the form that both $V_p$ and $V_q$ must have (see lemmas 10.1 and 12.5 of [2]) that if $V_p \cap V_q$ had two components then $X$ is circular and $X = V_p \cup V_q$, and consequently diameter $X \leqslant 4\varepsilon <$ diameter $X$, a contradiction. *We assume from now on that* $4\varepsilon <$ *diameter* $X$. It follows that if $V_p \cap V_q \neq \emptyset$ then $<_p = <_q$ on $V_p \cap V_q$ in all possible cases.

We may now state and prove the following lemma.

(3.11) LEMMA. *Assume the hypothesis of Lemma* 3.10 *and let* $<$ *be as in Lemma* 3.10, *either case. Suppose* $\bar{z} = (z_1, \ldots, z_n)$, $\bar{w} = (w_1, \ldots, w_n)$, $\bar{z}, \bar{w} \in Z^{\delta/2}$, $<_{z_1}$ *and* $<_{w_1}$ *are compatible with* $<$, $z_1 <_{z_1} z_2 <_{z_1} \ldots <_{z_1} z_n$ *and* $w_1 <_{w_1} w_2 <_{w_1} \ldots <_{w_1} w_n$. *Then* $\bar{z}\bar{T}^{\varepsilon}\bar{w}$.

Proof. Consider the set $Q = \{t \in X \mid$ if $\bar{t} = (t_1, \ldots, t_n) \in Z^{\delta/2}$, $t \in \{t_1, \ldots, t_n\}$, $<_{t_1}$ compatible with $<$, and $t_1 <_{t_1} t_2 <_{t_1} \ldots <_{t_1} t_n$ then $\bar{t}\bar{T}^{\varepsilon}\bar{z}.\}$ We will show that both $Q$ and $X - Q$ are open.

First note that for each $t \in X$ there exists a $\bar{t} = (t_1, \ldots, t_n) \in Z^{\delta/2}$ and a $<_{t_1}$ compatible with $<$ such that $t \in \{t_1, \ldots, t_n\}$ and $t_1 <_{t_1} t_2 <_{t_1} \ldots <_{t_1} t_n$. To see this recall that $X$ is connected and nontrivial so $V$ is also connected and nontrivial and so there must be infinitely many points arbitrarily close to $t$. The observation now follows readily.

Next consider a $t$ and $t' \in X$ such that $d(t, t') < \delta/2$. Let $\bar{t} = (t_1, \ldots, t_n)$ be as in the above paragraph and let $\bar{t}' = (t'_1, \ldots, t'_n)$ be in the same relation to $t'$. Clearly $t'_1, \ldots, t'_n \in V_{t_1}$ and as noted immediately preceding the lemma $<_{t'_1} = <_{t_1}$ on $V_{t'_1} \cap V_{t_1}$. Thus $t'_1 <_{t_1} t'_2 <_{t_1} \ldots <_{t_1} t'_n$. Now as in Proposition 2.9 $\bar{t}\bar{T}_{V_{t_1}}\bar{t}'$ and thus $\bar{t}\bar{T}^{\varepsilon}\bar{t}'$. Because $\bar{T}^{\varepsilon}$ is transitive it follows easily that both $Q$ and $X-Q$ are open. Since $z_1 \in Q \neq \emptyset$ and $X$ is connected $Q = X$ and the lemma is proved. Q.E.D.

Our next task is to formulate a result analogous to Corollary 2.3 and the Remark which follows it in the present context. Making use of Lemmas 2.8, 3.7, 3.10 and 3.11 the proof of this result is essentially the same as the proof of Corollary 2.3 and so it will be left to the reader.

(3.12) LEMMA. *Assume the hypothesis of Lemma* 3.10 *and let* $<$ *be as in Lemma* 3.10, *either case. Also assume* $X$ *is precompact and* $m$ *is a generalized diameter on* $n$*-tuples. If* $f\colon X \to X$ *is* 1-1 *and continuous* (*or alternately* $f^{-1}$ *exists and is* 1-1, *onto and continuous*), $\bar{x} = (x_1, \ldots, x_n) \in Z^{\delta/2}$, $x_1 <_{x_1} x_2 <_{x_1} \ldots <_{x_1} x_n$, *where* $<_{x_1}$ *is compatible with* $<$ *and* $m\bar{x} > mf\bar{x}$ *then there exists a* $\bar{y} \in Z^{\delta/2}$ *such that* $m\bar{y} < mf\bar{y}$ *and either* $\bar{y}\overline{T^{\varepsilon}}\bar{x}$ *or* $\bar{y}\overline{T^{\varepsilon}}\tau\bar{x}$ *where* $\tau \in S_n$ *is given by* $\tau(i) = n+1-i$ *for* $i = 1, \ldots, n$.

Next we have the analogous result to Proposition 2.7.

(3.13) LEMMA. *Assume* $X$ *is a uniformly locally connected metric space. Suppose* $g\colon X^n - GD_n(X) \to \mathbf{R}$ *is continuous,* $g(\bar{x}) < 0$ *and* $g(\bar{y}) > 0$ *for some* $\bar{x}, \bar{y} \in Z^{\delta/2}$, $\bar{x}\overline{T^{\delta/2}}\bar{y}$. *Then* $\#\{\bar{z} \in Z^{\varepsilon} |\ g(\bar{z}) = 0\} \geqslant \#0$ *for some non-empty open subset of* $X$ ($\# Q =$ *cardinality* $Q$).

Proof. It is not hard to see that the proof of Proposition 2.7 works here provided we first use $\bar{x}\overline{T^{\delta/2}}\bar{y}$ to obtain a chain $\bar{x} = \bar{x}^0, \bar{x}^1, \ldots, \bar{x}^m = \bar{y}$ with small links, $\bar{x}^i T^{\delta/2} \bar{x}^{i+1}$ and second we work through the sets $V_p$ where Lemma 2.6 can be applied directly, and there we choose the neighborhood $V$ which appears in the proof of Proposition 2.7 sufficiently small, i.e. diameter $V < \delta/2$.

We now gather the preceding lemmas together to prove Theorem 3.2.

(3.14) Proof of Theorem 3.2. Assume the hypothesis. Obtain the sets $\{V_p\}$ as in Lemma 3.7 and consider two cases. Case 1: $V_n^p - GD_n(V_p)$ is connected for some $p \in X$. Case 2: $V_p^n - GD_n(V_p)$ is disconnected for all $p \in X$.

Consider Case 1. Let $\varepsilon$ be as in the hypothesis of Theorem 3.2 and let the "$\varepsilon$" of Lemma 3.7 be called $\varepsilon'$. Let $\delta = \delta(\varepsilon)$ as in Definition 3.1 and set $\varepsilon' = \delta/2$. Finally set $\delta' = \delta(\varepsilon')$ as in Definition 3.1. We may assume $\delta' < \varepsilon' < \delta < \varepsilon$. Now, if $mf\bar{x} = m\bar{x}$ for continuum many $\bar{x} \in Z^{\delta'/2}$ then Lemma 2.13 would give the desired conclusion. So assume that this fails. We claim that $mh\bar{w} < m\bar{w}$ for some $h \in \{f, f^{-1}\}$ and $\bar{w} \in Z^{\delta'/2}$. If $mf\bar{u} < m\bar{u}$ for some $\bar{u} \in Z^{\delta'/2}$ the claim is established. So assume $mf\bar{u} \geqslant m\bar{u}$ all $\bar{u} \in Z^{\delta'/2}$. Pick any point $p \in X$. Since $X$ is connected each neighborhood of $p$ contains continuum many points. Using the continuity of $f$ at $p$ we may thus find continuum many $\bar{u} \in Z^{\delta'/2}$ such that $f\bar{u} \in Z^{\delta'/2}$. We cannot have $mf\bar{u} = m\bar{u}$ for all of these $\bar{u}$ and hence $mf\bar{u} > m\bar{u}$ for some $\bar{u}$ with $f\bar{u} \in Z^{\delta'/2}$. Setting $\bar{w} = f\bar{u}$ and $h = f^{-1}$ we see that the claim holds. Proposition 3.3 now implies that $mh\bar{v} > m\bar{v}$ for some $\bar{v} \in Z^{\delta'/2}$. Thus we have $mf\bar{x} < m\bar{x}$ and $mf\bar{y} > m\bar{y}$ for some $\bar{x}, \bar{y}$ such that $\{\bar{x}, \bar{y}\} = \{\bar{u}, \bar{v}\}$. Set $g(\bar{z}) = mf\bar{z} - m\bar{z}$ for all $z \in Z$. According to Lemma 3.7 $\bar{u}\overline{T^{\varepsilon'}}\bar{v}$ and thus by Lemma 3.13 $\#\{\bar{z} \in Z^{\varepsilon} |\ g(\bar{z}) = 0\} \geqslant \#0$ for some non-empty open subset of $X$. The conclusion of the theorem now follows from Lemma 2.13. This proves Case 1.



Consider Case 2. $V_p^n - GD_n(V_p)$ is disconnected for all $p \in X$. Let $\varepsilon$, $\delta$, $\varepsilon'$ and $\delta'$ be as above with the added easily satisfiable requirement that $4\varepsilon' <$ diameter $X$. Hence Lemmas 3.10, 3.11 and 3.12 apply. If $mf\bar{x} = m\bar{x}$ for continuum many $\bar{x} = (x_1, \ldots, x_n) \in Z^{\delta'/2}$ with $x_1 <_{x_1} x_2 <_{x_1} \ldots <_{x_1} x_n$ where $<_{x_1}$ is compatible with $<$ then the theorem follows from Lemma 2.13. So assume not. Proceeding as in Case 1 and by applying Lemma 3.12 to the appropriate element of $\{f, f^{-1}\}$ we obtain a $\bar{y} \in Z^{\delta'/2}$ such that either $\bar{y}\overline{T^{\varepsilon'}}\bar{x}$ or $\bar{y}\overline{T^{\varepsilon'}}\tau\bar{x}$ and $g(\bar{y})$ is positive (negative) if $g(\bar{x})$ is negative (positive) where $g$ is as in Case 1. Since $m$ is weakly symmetric $g(\tau\bar{x}) = g(\bar{x})$ and so the theorem now follows from Lemmas 3.13 and 2.13 as in Case 1. Q.E.D.

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INDIANA UNIVERSITY
Bloomington, Indiana

*Reçu par la Rédaction le 26. 12. 1968*