A characterization of paracompactness

by

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C. H. Dowker [3] proved that the following condition is equivalent to the countable paracompactness in a normal space:

(1) For each descending sequence $\{F_n\}$ of closed sets with empty intersection, there is a sequence $\{G_n\}$ of open sets with empty intersection such that $F_n \subset G_n$ for each n.

F. Ishikawa [4] characterized the countable paracompactness by the similar condition given below.

(2) For each descending sequence $\{F_n\}$ of closed sets with empty intersection, there is a descending sequence $\{G_n\}$ of open sets such that $\bigcap \{\bar{G}_n\} = 0$ and $F_n \subset G_n$ for each n.

Our purpose of the present paper is to characterize the paracompactness in a similar fashion.

A family $\mathfrak{F} = \{F_\alpha |\ \alpha \in A\}$ of subsets with a well ordered index set A with no last element will be called a chain provided $\bigcap \{F_\alpha |\ \alpha < \beta\} \neq 0$ for each $\beta \in A$. We shall introduce a relation $\lessapprox$, called dominancy, on the family of chains. If $\mathfrak{F} = \{F_\alpha |\ \alpha \in A\}$ and $\mathfrak{G} = \{G_\alpha |\ \alpha \in A\}$ are two descending chains, then $\mathfrak{F} \lessapprox \mathfrak{G}$ if $\bigcap \{\bar{F}_\alpha |\ \alpha < \beta\} \subset \mathrm{Int}(\bigcap \{G_\alpha |\ \alpha < \beta\})$ for each $\beta \in A$. In case of two subsets considered as constant chains, $F \lessapprox G$ if and only if $\mathrm{Cl}(F) \subset \mathrm{Int}(G)$. Let $\Phi(A)$ be the collection of all descending chains with the index set A and let $\Gamma(A)$ denote the collection of constant chains, then $\Gamma(A) \subset \Phi(A)$. Note that the normality is characterized by the fact that for each pair of constant chains $\mathfrak{F}, \mathfrak{G} \in \Gamma(A)$ such that $\mathfrak{F} \lessapprox \mathfrak{G}$, there is a constant chain $\mathfrak{H} \in \Gamma(A)$ such that $\mathfrak{F} \lessapprox \mathfrak{H} \lessapprox \mathfrak{G}$.

The main theorem is concerned with the characterization of paracompactness in terms of the separation condition on the family of descending chains $\Phi(A)$. $\mathfrak{F} = \{F_\alpha |\ \alpha \in A\}$ is said to be a free chain provided $\bigcap \{\bar{F}_\alpha |\ \alpha \in A\} = \emptyset$. Let $\Phi^*(A)$ denote the collection of all free descending chains with the index set A. Theorem 1 states that *a space* *is paracompact if and only if the following condition is satisfied for each index set A: For each $\mathfrak{F} \in \Phi^*(A)$ and for each $\mathfrak{G} \in \Phi(A)$ such that $\mathfrak{F} \prec \mathfrak{G}$, there is a chain $\mathfrak{H} \in \Phi^*(A)$ such that $\mathfrak{F} \prec \mathfrak{H} \prec \mathfrak{G}$.*

The proof of Theorem 1 is closely related to that of the author's earlier result asserting that the normality of $X \times \beta X$ implies the paracompactness of X. In this connection, the Stone-Čech compactification plays an important role in the following arguments. All spaces mentioned here will be completely regular and T_1 unless otherwise specified. We shall depend heavily on the well known theorem due to A. H. Stone [8] asserting that every pseudo-metric space is paracompact. This is actually the key lemma for the main theorem, as in the case of former result mentioned above.

Finally, we shall apply our theorem to obtain a new covering characterization of paracompactness, which is a simultaneous generalization of the results due to E. Michael [6] and the author [11]. We shall show that *a space is paracompact if and only if every open covering has a linearly cushioned open refinement.*

1. Chains. A family $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ of subsets of X with a well ordered index set A with no last element will be called a *chain* provided that $\bigcap \{F_\alpha|\ \alpha < \beta\} \neq \emptyset$ for each $\beta \in A$. $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ is a *descending* (*ascending*) *chain* if $F_\alpha \supset F_\beta$ ($F_\alpha \subset F_\beta$) whenever $\alpha < \beta$. In the following, $\mathrm{Cl}(E)$, $\mathrm{Int}(E)$ and $\mathrm{C}(E)$ denote respectively the closure, the interior and the complement of the set E. For convenience, we shall also use the usual notation $\bar{E}$ for $\mathrm{Cl}(E)$. The chain $\bar{\mathfrak{F}} = \{\bar{F}_\alpha|\ \alpha \in A\}$ ($\mathfrak{F}^0 = \{\mathrm{Int}(F_\alpha)|\ \alpha \in A\}$) will be called the *closure chain* (the *interior chain*) and $\mathfrak{F}^c = \{\mathrm{C}(F_\alpha)|\ \alpha \in A\}$ will be called the *complementary chain* of $\mathfrak{F}$. If each F_α has a property P, then we shall say that $\mathfrak{F}$ is a chain of P-sets or simply a *P-chain*. A chain $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ of P-sets is said to be *complete* if both $\bigcup \{F_\alpha|\ \alpha < \beta\}$ and $\bigcap \{F_\alpha|\ \alpha < \beta\}$ has the property P for each $\beta \in A$. A descending (ascending) chain of closed sets (open sets) is complete. A chain $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ is said to be a *free chain* if the closure chain has empty intersection. That is $\mathfrak{F}$ is free provided $\bigcap \{\bar{F}_\alpha|\ \alpha \in A\} = \emptyset$. We shall call $\mathfrak{F}$ a *chain covering* if the interior chain forms an open covering. The complementary chain of free chain is a covering chain. A chain $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ such that $F_\alpha = F$ for each $\alpha \in A$, for a fixed non empty subset F, is said to be a constant chain of length A. Let $\Omega(A)$ denote the family of all chains of length A (i.e. the chains with the index set A). $\Gamma(A)$ denotes the family of all constant chains of length A. By $\Phi(A)$ ($\Psi(A)$), we shall denote the family of all descending (ascending) chains of length A. The family of all descending free chains of length A will be denoted by $\Phi^*(A)$. Now, we shall define a relation on the family $\Omega(A)$ as follows:

DEFINITION 1. Let $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$, $\mathfrak{G} = \{G_\alpha|\ \alpha \in A\}$ be two chains of length A. We shall say that $\mathfrak{F}$ is *dominated by* $\mathfrak{G}$ (or $\mathfrak{G}$ *dominates* $\mathfrak{F}$) and write $\mathfrak{F} \prec \mathfrak{G}$ if the following conditions are satisfied for each $\beta \in A$:

$$\bigcap \{\mathrm{Cl}(F_\alpha)|\ \alpha < \beta\} \subset \mathrm{Int}(\bigcap \{G_\alpha|\ \alpha < \beta\}), \tag{3}$$

$$\mathrm{Cl}(\bigcup \{F_\alpha|\ \alpha < \beta\}) \subset \bigcup \{\mathrm{Int}(G_\alpha)|\ \alpha < \beta\}. \tag{4}$$

If $\mathfrak{F}$ and $\mathfrak{G}$ are constant chains with $F_\alpha = F$ and $G_\alpha = G$ for each $\alpha \in A$, then $\mathfrak{F} \prec \mathfrak{G}$ if and only if $\mathrm{Cl}(F) \subset \mathrm{Int}(G)$. In case that $\mathfrak{F}$ is a descending chain, $\mathfrak{F} \prec \mathfrak{G}$ if and only if (3) is satisfied. Similarly, if $\mathfrak{G}$ is an ascending chain, then (4) is equivalent to $\mathfrak{F} \prec \mathfrak{G}$. Note that the condition (4) is a modification of the condition that the family $\{F_\alpha|\ \alpha \in A\}$ is cushioned [6] in the family $\{\mathrm{Int}(G_\alpha)|\ \alpha \in A\}$, and that (3) is valid for the pair $\mathfrak{F}$. $\mathfrak{G} \in \Omega(A)$ if and only if (4) is satisfied for the pair $\mathfrak{G}^c, \mathfrak{F}^c \in \Omega(A)$.

A general theory concerning the separation of chains relative to the dominancy defined above will be given elsewhere. In the present paper, we shall be mainly concerned with the separation of descending chains.

2. The main theorem.

THEOREM 1. *A space X is paracompact if and only if the following condition is satisfied*

(P) *For each descending free chain $\mathfrak{F} \in \Phi^*(A)$ and for each chain $\mathfrak{G} \in \Omega(A)$ such that $\mathfrak{F} \prec \mathfrak{G}$, there is a descending free chain $\mathfrak{H} \in \Phi^*(A)$ such that $\mathfrak{F} \prec \mathfrak{H} \prec \mathfrak{G}$, for each index set A.*

Proof of the necessity. Since $\mathfrak{F} \prec \mathfrak{G}$ implies that $\bar{\mathfrak{F}} \prec \mathfrak{G}$, we may assume that $\mathfrak{F}$ is a closed chain. We shall show that there is a complete descending open chain $\mathfrak{H} \in \Phi^*(A)$ such that $\mathfrak{F} \prec \mathfrak{H} \prec \mathfrak{G}$.

For each point $x \in X$, choose the least index $\alpha(x)$ such that $x \notin F_{\alpha(x)}$. Put $F(\beta) = \bigcap \{F_\alpha|\ \alpha < \beta\}$ and $G(\beta) = \bigcap \{G_\alpha|\ \alpha < \beta\}$, then $x \in F(\alpha(x))$ and $G(\alpha(x))$ is a neighborhood of $F(\alpha(x))$ since $\mathfrak{F} \prec \mathfrak{G}$. Therefore, there is an open neighborhood $U(x)$ of x such that

$$\overline{U(x)} \cap F_{\alpha(x)} = \emptyset \tag{5}$$

and

$$U(x) \subset G(\alpha(x)). \tag{6}$$

Consider the open covering $\{U(x)|\ x \in X\}$ of X and let $\mathfrak{B} = \{V_\lambda|\ \lambda \in \Lambda\}$ be a locally finite open refinement. Put $W_\alpha = \bigcup \{V_\lambda|\ \bar{V}_\lambda \cap F_\alpha = \emptyset\}$, then $\bar{W}_\alpha = \bigcup \{\bar{V}_\lambda|\ \bar{V}_\lambda \cap F_\alpha = \emptyset\}$, since every subfamily of a locally finite family is closure preserving. Put $H_\alpha = \mathrm{C}(\bar{W}_\alpha)$, then $F_\alpha \subset H_\alpha$ for each $\alpha \in A$ and $\mathfrak{H} = \{H_\alpha|\ \alpha \in A\}$ is a descending complete open chain. In fact, $H(\beta) = \bigcap \{H_\alpha|\ \alpha < \beta\}$ is the complementary set of the union of a locally finite family $\{\bar{V}_\lambda|\ \bar{V}_\lambda \cap F_\alpha = \emptyset$ for some $\alpha < \beta\}$, and is hence open for each $\beta \in A$. Therefore, $\mathfrak{F} \prec \mathfrak{H}$. To prove $\mathfrak{H} \prec \mathfrak{G}$, we have to verify that $\bigcap \{\overline{H}_\alpha|\ \alpha < \beta\} \subset \mathrm{Int}(G(\beta))$ for each $\beta \in A$. We shall show that if $p \notin \mathrm{Int}(G(\beta))$ then $p \notin \bigcap \{\overline{H}_\alpha|\ \alpha < \beta\}$. Suppose $p \notin \mathrm{Int}(G(\beta))$ and let V_λ be a member of $\mathfrak{V}$ which contains p; then $V_\lambda \subset U(x)$ for some $x \in X$. Let $\alpha(x)$ be the least index for which $x \notin F_{\alpha(x)}$ then we have $\overline{U(x)} \cap F_{\alpha(x)} = \emptyset$ by (5), and therefore $\overline{V}_\lambda \cap F_{\alpha(x)} = \emptyset$. It follows that $V_\lambda \subset W_{\alpha(x)}$ and hence $V_\lambda \cap \overline{H}_{\alpha(x)} = \emptyset$. If $\alpha(x) \geqslant \beta$, then we have by (6) $p \in U(x) \subset G(\alpha(x)) \subset G(\beta)$ which is impossible. Therefore $\alpha(x) < \beta$ and $V_\lambda \cap \overline{H}_{\alpha(x)} = \emptyset$ implies that $V_\lambda \cap (\bigcap \{\overline{H}_\alpha|\ \alpha < \beta\}) = \emptyset$. Thus, we have $p \notin \bigcap \{\overline{H}_\alpha|\ \alpha < \beta\}$. Since the condition (4) is also easily seen to be satisfied by $\mathfrak{H}$ and $\mathfrak{G}$, it follows that $\mathfrak{H} \prec \mathfrak{G}$. Finally, it is easy to see that $\bigcap \{\overline{H}_\alpha|\ \alpha \in A\} \subset \bigcap \{\mathrm{C}(W_\alpha)|\ \alpha \in A\} = \mathrm{C}(\bigcup \{W_\alpha|\ \alpha \in A\}) = \emptyset$, in view of the fact that $\mathfrak{V}$ is a covering of X and $\overline{V}_\lambda \cap F_\alpha = \emptyset$ for some $\alpha \in A$. The proof is completed.

Remark 1. In the above arguments, we have proved that the paracompactness implies the following condition which is slightly stronger than (P).

(P_1) For each $\mathfrak{F} \in \Phi^*(A)$ and for each $\mathfrak{G} \in \Omega(A)$ such that $\mathfrak{F} \prec \mathfrak{G}$, there is a complete open chain $\mathfrak{H} \in \Phi^*(A)$ such that $\mathfrak{F} \prec \mathfrak{H} \prec \mathfrak{G}$.

We shall discuss some other modifications of the condition (P) later. Note that if $\mathfrak{H} = \{H_\alpha|\ \alpha \in A\}$ is a descending complete open chain, then the family $\{\mathrm{C}(H_\alpha)|\ \alpha < \beta\}$ is a closure preserving collection of closed sets for each $\beta \in A$.

Proof of the sufficiency. Suppose that (P) is valid and we shall show by induction that the following statement $\mathrm{S}(\mathfrak{m})$ is true for each cardinal number $\mathfrak{m}$ which will imply that X is paracompact.

$\mathrm{S}(\mathfrak{m})$: Every open covering $\{U_\alpha|\ \alpha \in A\}$ with $\mathrm{Card}(A) = \mathfrak{m}$ has a locally finite open refinement. That is, X is $\mathfrak{m}$-paracompact [7].

(i) $\mathrm{S}(1)$ is obviously true.

(ii) Assuming that $\mathrm{S}(\mathfrak{m}')$ is true for each cardinal number $\mathfrak{m}' < \mathfrak{m}$, where $\mathfrak{m}$ is infinite,

we shall verify that $\mathrm{S}(\mathfrak{m})$ is true in the following five step (I)–(V).

(I) Normality. First of all, let us observe that the space X is normal if (P) is true. Let F be a closed set and let U be an open set containing F. Consider an open covering $\mathfrak{V}$ of X consisting of $\mathrm{C}(F)$ and the family $\{V(x)|\ x \in F\}$ such that $\overline{V(x)} \subset U$ for each $V(x)$. If $\mathfrak{V}$ has a finite subcovering, say $V_0 = \mathrm{C}(F), V_1, \ldots, V_n$, then $V = \bigcup_{k=1}^{n} V_k$ is an open set such that $F \subset V \subset \overline{V} \subset U$. If $\mathfrak{V}$ has no finite subcovering then we can construct a descending chain $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ of closed sets with $F_1 = F$ and $\bigcap \{F_\alpha|\ \alpha \in A\} = \emptyset$ in an obvious way. Consider the constant chain $\mathfrak{U} = \{U_\alpha|\ \alpha \in A\}$ with $U_\alpha = U$ for each $\alpha \in A$, then $\mathfrak{F} \prec \mathfrak{U}$. There is by (P) a chain $\mathfrak{W} = \{W_\alpha|\ \alpha \in A\}$ such that $\mathfrak{F} \prec \mathfrak{W} \prec \mathfrak{U}$. Then W_1 is a neighborhood of F such that $F \subset W_1 \subset \overline{W}_1 \subset U$. It follows that X is normal.

The following theorem due to Čech [1] will play the essential role in the step (IV).

LEMMA 1. *A space X is normal if and only if $\mathrm{Cl}_{\beta X}(F) \cap \mathrm{Cl}_{\beta X}(G) = \emptyset$ for each disjoint pair of closed sets F and G, where βX denotes the Stone–Čech compactification of X.*

This lemma will be used in the following form. Let us call open set $U^e = \mathrm{C}_{\beta X}(\mathrm{Cl}_{\beta X}(\mathrm{C}_X(U)))$ a proper extension of an open $U \subset X$. Then U^e is the largest open set of βX such that $U^e \cap X = U$ (Cf. [10]), and we have.

COROLLARY. *Let X be a normal space and let U be an open neighborhood of a closed subset F. Then the proper extension U^e of U is an open neighborhood of $\mathrm{Cl}_{\beta X}(F)$. That is, $F \subset U$ implies $\mathrm{Cl}_{\beta X}(F) \subset U^e$.*

(II) Construction of a descending chain. Let $\mathfrak{U} = \{U_\alpha|\ \alpha \in A\}$ be an open covering of X with $\mathrm{Card}(A) = \mathfrak{m}$. It is well known that a set A can be well ordered in such a way that $\mathrm{Card}(A(\alpha)) < \mathrm{Card}(A)$ for each $\alpha \in A$, where $A(\alpha) = \{\gamma \in A|\ \gamma < \alpha\}$.

Now, let us well order the index set A in this way. Then, we may assume that the subfamily $\mathfrak{U}_\alpha = \{U_\gamma|\ \gamma \in A(\alpha)\}$ is not a covering of X because otherwise $\mathfrak{U}_\alpha$ has a locally finite open refinement, by induction hypothesis, since $\mathrm{Card}(A(\alpha)) < \mathfrak{m}$. We can construct a descending free closed chain $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ by letting $F_1 = X$, $F_\alpha = \mathrm{C}_X(\bigcup \{U_\gamma|\ \gamma \in A(\alpha)\})$ for $\alpha > 1$. Put $\mathrm{Cl}_{\beta X}(F_\alpha) = C_\alpha$ and put $C = \bigcap \{C_\alpha|\ \alpha \in A\}$, then C is a compact set contained in $\beta X \setminus X$, where βX is the Stone–Čech compactification of X.

(III) Lemma.

LEMMA 2. *Let C be a compact set contained in $\beta X \setminus X$. If there is a continuous function $F(x, z)$ on $X \times \beta X$ such that*

(7) $$F(x, z) = 0 \quad \text{if} \quad (x, z) \in \Delta$$

and

(8) $$F(x, z) = 1 \quad \text{if} \quad (x, z) \in X \times C,$$

then there is a locally finite open covering $\{O_\lambda|\ \lambda \in \Lambda\}$ of X such that $\mathrm{Cl}_{\beta X}(O_\lambda) \cap C = \emptyset$ for each $\lambda \in \Lambda$.

Proof. Let $F_x(z) \in \mathrm{C}(\beta X)$ denote the restriction of $F(x, z)$ on $\{x\} \times \beta X$. Define a pseudo-metric d on X as follows:

(9) $$d(x, y) = \|F_x(z) - F_y(z)\| = \mathrm{Sup}\{|F_x(z) - F_y(z)|\ |\ z \in \beta X\}.$$

Since βX is compact, $d(x, y)$ is continuous. That is, the pseudo-metric d determines a topology τ weaker than the original topology τ_0 of X. By a well known theorem due to A. H. Stone [8], the pseudo-metric space

(X, τ) is paracompact. Consider an open covering $\mathfrak{W} = \{W(x) \mid x \in X\}$ of (X, τ), where $W(x) = \{y \in X \mid d(x, y) < \frac{1}{2}\}$, and let $\mathfrak{O} = \{O_\lambda \mid \lambda \in \Lambda\}$ be a locally finite open refinement (with respect to τ) of $\mathfrak{W}$. We shall show that $\mathrm{Cl}_{\beta X}(W(x)) \cap C = \emptyset$ for each $x \in X$, which will imply that $\mathrm{Cl}_{\beta X}(O_\lambda) \cap C = \emptyset$ for each λ. It is clear by (9) that

$$|F_x(y)| = |F_x(y) - F_y(y)| \leqslant \|F_x(z) - F_y(z)\| = d(x, y) .$$

Put $U(x) = \{y \in X \mid |F_x(y)| < \frac{1}{2}\}$, then we have $W(x) \subset U(x)$ for each $x \in X$. Since $F_x(z)$ is a continuous function on βX and since $F_x(z) = 1$ on C, we have $\mathrm{Cl}_{\beta X}(U(x)) \cap C = \emptyset$, and it follows that $\mathrm{Cl}_{\beta X}(W(x)) \cap C = \emptyset$. Finally $\mathfrak{O}$ is a locally finite open covering of (X, τ_0) because it is a locally finite open covering of (X, τ) and τ is weaker than the original topology τ_0.

(IV) Construction of $F(x, z)$ on $X \times \beta X$. We shall define a continuous function $F(x, z)$ satisfying the condition of Lemma 2, for the compact set $C \subset \beta X \setminus X$ determined at the end of the step (II). Let $\mathfrak{F} = \{F_\alpha \mid \alpha \in A\}$ be the descending closed chain constructed in (II). Consider the constant chain $\mathfrak{G} = \{G_\alpha \mid \alpha \in A\}$ with $G_\alpha = X$ for each $\alpha \in A$, then $\mathfrak{F} \lesssim \mathfrak{G}$. There is by (P) a descending free chain $\mathfrak{W} = \{W_\alpha \mid \alpha \in A\}$ such that $\mathfrak{F} \lesssim \mathfrak{W} \lesssim \mathfrak{G}$. Since $\mathfrak{W} \lesssim \mathfrak{G}$ implies $\overline{\mathfrak{W}} \lesssim \mathfrak{G}$, we may choose $\mathfrak{W}$ to be a closed chain. Similarly, by applying (P) to the pair $\mathfrak{F}$, $\mathfrak{W}$ and observing that $\mathfrak{F} \lesssim \mathfrak{V} \lesssim \mathfrak{W}$ implies $\mathfrak{F} \lesssim \mathfrak{V}^0 \lesssim \mathfrak{W}$, we can see that there is a free open chain $\mathfrak{V}$ such that $\mathfrak{F} \lesssim \mathfrak{V} \lesssim \mathfrak{W}$.

We now define inductively a descending free open chain $\mathfrak{V}_r = \{V_\alpha(r) \mid \alpha \in A\}$ and a free closed (descending) chain $\mathfrak{W}_r = \{W_\alpha(r) \mid \alpha \in A\}$ for each diadic number $r = k/2^m \in [0, 1]$ so as to satisfy the following condition:

$$\text{(10)} \qquad \mathfrak{V}_{r'} \lesssim \mathfrak{V}_r \quad \text{and} \quad \mathfrak{W}_r \lesssim \mathfrak{W}_{r'} \quad \text{whenever} \quad r > r' .$$

Put $\mathfrak{V}_0 = \mathfrak{V}$, $\mathfrak{W}_0 = \mathfrak{G}$, $\mathfrak{V}_1 = \mathfrak{W}_0$ and $\mathfrak{W}_1 = \mathfrak{W}$. Suppose that both $\mathfrak{V}_r$ and $\mathfrak{W}_r$ are defined for each diadic number of the form $k/2^m$ with $0 \leqslant k \leqslant 2^m$, then we can define $\mathfrak{V}_r$ and $\mathfrak{W}_r$ for $r = (2k+1)/2^{m+1}$ as follows: Put $r_1 = k/2^m$ and $r_2 = (k+1)/2^m$, then $r_1 < r < r_2$. Since $\mathfrak{W}_{r_2} \lesssim \mathfrak{W}_{r_1}$, there is a descending free closed chain $\mathfrak{W}_r = \{W_\alpha(r) \mid \alpha \in A\}$ such that $\mathfrak{W}_{r_2} \lesssim \mathfrak{W}_r \lesssim \mathfrak{W}_{r_1}$. Similarly, there is a descending free open chain $\mathfrak{V}_r = \{V_\alpha(r) \mid \alpha \in A\}$ such that $\mathfrak{V}_{r_1} \lesssim \mathfrak{V}_r \lesssim \mathfrak{V}_{r_2}$. Thus, $\mathfrak{V}_r$ and $\mathfrak{W}_r$ are defined for each diadic number $r \in [0, 1]$.

Put $C_X(W_\alpha(r)) = W_\alpha^c(r)$ and $C_{\beta X}(\mathrm{Cl}_{\beta X}(C_X(V_\alpha(r)))) = V_\alpha^e(r)$, then the proper extension $V_\alpha^e(r)$ is a neighborhood of $C \subset \beta X \setminus X$, for each α and for each r, by the corollary of Lemma 1. Clearly $\{W_\alpha^c(r) \mid \alpha \in A\}$ is an open covering of X for each r. Therefore

$$\text{(11)} \qquad N(r) = \bigcup \{W_\alpha^c(r) \times V_\alpha^e(r) \mid \alpha \in A\}$$

is an open neighborhood of $X \times C$ in $X \times \beta X$ for each r. In view of the fact that $\mathfrak{V}_1 = \mathfrak{W}_0$ and $\mathfrak{W}_1 = \mathfrak{W}$, we have $V_\alpha(1) \subset W_\alpha(1)$ for each $\alpha \in A$ and follows that $N(1) \cap \Delta = \emptyset$. Next, we shall show that

$$\text{(12)} \qquad \overline{N(r')} \subset N(r) \quad \text{whenever} \quad r' < r .$$

To this end, we shall verify that if $(x, z) \notin N(r)$, then there is a neighborhood of (x, z) which does not meets $N(r')$. Let β be the least index such that $x \notin W_\beta(r)$. Then, $x \in \bigcap \{W_\alpha(r) \mid \alpha < \beta\}$ and $\bigcap \{W_\alpha(r') \mid \alpha < \beta\}$ is a neighborhood of x, since $\mathfrak{W}_r \lesssim \mathfrak{W}_{r'}$. There is a neighborhood $U(x)$ of x such that

$$\text{(13)} \qquad U(x) \subset W_\alpha(r') \quad \text{for each} \quad \alpha < \beta .$$

(If β is the first ordinal then define that $U(x) = X$.) On the other hand, we have $z \notin V_\beta^e(r)$ from $(x, z) \notin N(r)$ and $x \notin W_\beta(r)$. Since $\mathfrak{V}_{r'} \lesssim \mathfrak{V}_r$, $\overline{V_\beta(r')} \subset V_\beta(r)$ and, by the corollary of Lemma 1, we have $\mathrm{Cl}_{\beta X}(V_\beta(r')) \subset V_\beta^e(r)$. There is a neighborhood $U(z)$ (in βX) of z such that $U(z) \cap V_\beta^e(r') = \emptyset$. Since $\mathfrak{V}_{r'}$ is a descending chain, we have

$$\text{(14)} \qquad U(z) \cap V_\alpha^e(r') = \emptyset \quad \text{for each} \quad \alpha \geqslant \beta .$$

It follows from (13) and (14) that $U(x) \times U(z) \cap W_\alpha^c(r') \times V_\alpha^e(r') = \emptyset$ for each $\alpha \in A$. Thus, we have $U(x) \times U(z) \cap N(r') = \emptyset$. It follows that $\overline{N(r')} \subset N(r)$.

Now, define a continuous function $F(x, z)$ on $X \times \beta X$ by letting

$$\text{(15)} \qquad F(x, z) = \begin{cases} 1 - \mathrm{Inf}\{r \mid (x, z) \in N(r)\} & \text{if} \quad (x, z) \in N(1) , \\ 0 & \text{if} \quad (x, z) \notin N(1) . \end{cases}$$

Then, $F(x, z)$ is the desired continuous function on $X \times \beta X$. In fact, $F(x, z) = 1$ on $X \times C$, since $N(r) \supset X \times C$ for each r, and $F(x, z) = 0$ on Δ because $N(1) \cap \Delta = \emptyset$. The continuity of $F(x, z)$ follows from (12).

(V) Construction of locally finite open refinement. There is by Lemma 2 a locally finite open covering $\{O_\lambda \mid \lambda \in \Lambda\}$ such that $\mathrm{Cl}_{\beta X}(O_\lambda) \cap C = \emptyset$ for each $\lambda \in \Lambda$. Since $C = \bigcap \{C_\alpha \mid \alpha \in A\}$ and since $\{C_\alpha \mid \alpha \in A\}$ is a descending family and since $\mathrm{Cl}_{\beta X}(O_\lambda)$ is compact, there is a $\beta \in A$ such that $\mathrm{Cl}_{\beta X}(O_\lambda) \cap C_\beta = \emptyset$. This means that $\mathrm{Cl}_X(O_\lambda)$ is covered by the subfamily $\{U_\alpha \mid \alpha \in A(\beta)\}$ of the given covering $\{U_\alpha \mid \alpha \in A\}$, where $A(\beta) = \{\alpha \in A \mid \alpha < \beta\}$. Consider a covering $\mathfrak{U}(\lambda)$ of X consisting of $C_X(\mathrm{Cl}_X(O_\lambda))$ and $\{U_\alpha \mid \alpha \in A(\beta)\}$; then, $\mathfrak{U}(\lambda)$ has a locally finite open refinement $\mathfrak{V}(\lambda)$, by the fact that $\mathrm{Card}(A(\beta)) < \mathfrak{m}$ and the induction hypothesis. Let $\{V_\sigma \mid \sigma \in \Sigma_\lambda\}$ be the family consisting of all members of $\mathfrak{V}(\lambda)$ intersecting O_λ, and put $O_\lambda \cap V_\sigma = R_\sigma$ for each $\sigma \in \Sigma_\lambda$. Then $\mathfrak{R}(\lambda) = \{R_\sigma \mid \sigma \in \Sigma_\lambda\}$ is a locally finite family of open sets (locally finite at each point of X) and $\bigcup \{R_\sigma \mid \sigma \in \Sigma_\lambda\} = O_\lambda$. Construct a locally finite family of open sets

$\mathfrak{R}(\lambda)$ for each $\lambda \in \Lambda$ in this way, and put $\Sigma = \bigcup \{\Sigma_\lambda | \lambda \in \Lambda\}$. Then $\mathfrak{R} = \{R_\sigma | \sigma \in \Sigma\}$ is a locally finite open refinement of the given open covering $\mathfrak{U} = \{U_\alpha | \alpha \in A\}$. In fact, $\mathfrak{R}$ is obviously an open refinement, and each point $x \in X$ has a neighborhood U_0 which meets only a finite number of O_λ's, say $O_{\lambda_1}, O_{\lambda_2}, \ldots, O_{\lambda_n}$. For each λ_i, there is an open neighborhood U_i of x which intersects only finitely many members of $\mathfrak{R}(\lambda_i)$. Put $U = \bigcap_{i=0}^{n} U_i$; then U is a neighborhood of x such that $U \cap R_\sigma = \emptyset$ for all but a finite number of members of $\mathfrak{R}$. Thus $S(\mathfrak{m})$ is valid.

It follows by induction that $S(\mathfrak{m})$ is valid for each cardinal number $\mathfrak{m}$ and therefore X is paracompact. The proof is completed.

Remark 2. Let $P(\mathfrak{m})$ be the statement asserting that the condition (P) in Theorem 1 is true if $\mathrm{Card}(A) \leqslant \mathfrak{m}$. In the step (I) of the above proof, we have observed that X is normal if $P(\mathfrak{m})$ is true for each cardinal number $\mathfrak{m}$. However, the condition $P(\mathfrak{m})$ for a fixed $\mathfrak{m}$ does not imply the normality of X in general. For example, $P(\aleph_0)$ is valid in a countably compact space, but a countably compact space need not be normal. From the proof of the above theorem, it is not difficult to see that $P(\mathfrak{m})$ is equivalent to $S(\mathfrak{m})$ in case that X is a normal space. (Proof of the necessity is not trivial.)

Remark 3. In Lemma 2, we can replace βX by any compactification BX of X. This theorem was originally appeared in [10]. If $X \times BX$ is normal, then the existence of the continuous function satisfying the condition of Lemma 2 is a consequence of Urysohn's lemma. It may also be worthwhile to note that $F(x, z)$ determines a continuous mapping of X into the Banach space $C(BX)$ of continuous function by mapping x to $F_x(z)$.

In the following, we shall give some other expression of the main theorem. It is clear that $\mathfrak{F} \prec \mathfrak{G}$ if and only if $\mathfrak{F}^c \succ \mathfrak{G}^c$ and that the complementary chain $\mathfrak{F}^c$ of a descending free chain is an ascending chain whose interior chain covers the space X. We call such a chain simply a *chain covering:*

THEOREM 1*. *A space is paracompact if and only if the following condition is satisfied:*

(P_2) (¹) *For each chain covering $\mathfrak{U} = \{U_\alpha | \alpha \in A\}$ and for each chain $\mathfrak{W} = \{W_\alpha | \alpha \in A\}$, such that $\mathfrak{W} \prec \mathfrak{U}$, there is a chain covering $\mathfrak{V} = \{V_\alpha | \alpha \in A\}$ such that $\mathfrak{W} \prec \mathfrak{V} \prec \mathfrak{U}$.*

Remark 4. Theorem 1* is not exactly the dual statement of Theorem 1. In fact, the case that $\mathfrak{U}$ is a finite chain covering is not excluded in Theorem 1*, and the condition (P_2) for finite index set A is equivalent to the normality.

(¹) The characterization of paracompactness by this condition was announced at the Topology Symposium held in Prague in 1966.

In Remark 1, we have mentioned that (P_1) ⇒ (P). It is obvious that (P_2) ⇒ (P). In view of the proof of Theorem 1, it is also clear that the following condition is equivalent to the paracompactness:

(P_3) (²) *For each descending free closed chain $\mathfrak{F} = \{F_\alpha | \alpha \in A\}$ and for each descending complete open chain $\mathfrak{G} = \{G_\alpha | \alpha \in A\}$ such that $\mathfrak{F} \prec \mathfrak{G}$, there is a descending complete open chain $\mathfrak{H} = \{H_\alpha | \alpha \in A\}$ such that $F_\alpha \subset H_\alpha \subset \bar{H}_\alpha \subset G_\alpha$ for each $\alpha \in A$ (i.e. $\mathfrak{F} \prec \mathfrak{H} \prec \mathfrak{G}$).*

It is however not known to the author if the following condition, which is a simultaneous weakning of (P_3) and (P), implies the paracompactness.

(P_4) *For each descending free chain $\mathfrak{F} = \{F_\alpha | \alpha \in A\}$ and for each (descending) complete open chain $\mathfrak{G} = \{G_\alpha | \alpha \in A\}$ such that $\mathfrak{F} \prec \mathfrak{G}$, there is a descending free chain $\mathfrak{H} = \{H_\alpha | \alpha \in A\}$ such that $\mathfrak{F} \prec \mathfrak{H} \prec \mathfrak{G}$.*

By putting $G_\alpha = X$ for each $\alpha \in A$, we can see that (P_4) implies the following condition:

(P_5) *For each descending free (closed) chain $\mathfrak{F} = \{F_\alpha | \alpha \in A\}$, there is a (descending) free (open) chain $\mathfrak{H} = \{H_\alpha | \alpha \in A\}$ such that $\mathfrak{F} \prec \mathfrak{H}$.*

In case that A is a countable index set, the condition (P_5) is nothing other than the condition (2) in Ishikawa's characterization of the countable paracompactness given at the top of this paper. Note that every countable sequence of open (closed) sets is a complete open (closed) chain. In view of this fact, the following problem seems to be of some interest:

PROBLEM 1. *Is the following condition equivalent to the paracompactness?*

(P^*) *For each descending free (closed) chain $\mathfrak{F} = \{F_\alpha | \alpha \in A\}$, there is a descending free complete open chain $\mathfrak{H} = \{H_\alpha | \alpha \in A\}$ such that $\mathfrak{F} \prec \mathfrak{H}$.*

This problem is concerned with the following problem in the sense that the affirmative answer to Problem 1 gives the affirmative answer to Problem 2.

PROBLEM 2. *Suppose that X is the union of a closure preserving family of compact subsets. Is it true that X is always paracompact?*

For each descending chain $\mathfrak{F} = \{F_\alpha | \alpha \in A\}$, put $C(\mathfrak{F}) = \bigcap \{\mathrm{Cl}_{\beta X}(F_\alpha) | \alpha \in A\}$ and call $C(\mathfrak{F})$ the *absolute center* of $\mathfrak{F}$. Let us call the intersection $\bigcap \{F_\alpha | \alpha \in A\}$ the *relative center* of $\mathfrak{F}$. Roughly speaking, a descending chain is characterized by its center and the mode of diminution of the members. In case of free descending chains, the relative centers are always

(²) This characterization of paracompactness was announced at the International Congress of Mathematicians held in Moscow in 1966, under the additional assumption that X is a normal space.

empty. On the other hand, the absolute centers distinguish the locations of descending chains. Thus, compactifications can be used to visualize one of the characteristic factors of descending chains. The dominancy is concerned with the description of the mode of diminution of a descending chain.

We shall call $C \subset \beta X \setminus X$ a central cocover if $C = \mathrm{C}(\mathfrak{F})$ for some descending free chain $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$. C is said to be a central cocover of weight $\mathfrak{m}$, if $\mathrm{Card}(A) = \mathfrak{m}$. Given an open covering $\mathfrak{U} = \{U_\alpha|\ \alpha \in A\}$ of X, we can construct a compact set $\mathrm{C}^*(\mathfrak{U}) \subset \beta X \setminus X$ by letting $\mathrm{C}^*(\mathfrak{U}) = \bigcap \{C_{\beta X}(U_\alpha^s)|\ \alpha \in A\}$. Conversely, for each compact set $C^* \subset \beta X \setminus X$, there is an open covering $\mathfrak{U}$ such that $\mathrm{C}^*(\mathfrak{U}) = C^*$. Let $\mathrm{C}(\mathfrak{U})$ be the central cocover defined in the step (II) of the proof of Theorem 1. Then, it is clear that $\mathrm{C}^*(\mathfrak{U}) \supset \mathrm{C}(\mathfrak{U})$ for each open covering $\mathfrak{U}$ of X. However, $\mathrm{C}^*(\mathfrak{U})$ is not identical with $\mathrm{C}(\mathfrak{U})$ in general. Theorem $1^{\#}$ shows that the paracompactness is characterized by the property how central cocover are embedded in the Stone–Čech compactification βX.

Let $\mathfrak{V} = \{V_\alpha|\ \alpha \in A\}$ be a descending chain of βX. Let us agree to call $\mathfrak{V}$ a *free neighborhood chain* of $C \subset \beta X \setminus X$ (of weight $\mathfrak{m}$) provided that ($\mathrm{Card}(A) = \mathfrak{m}$ and) $\bigcap \{V_\alpha|\ \alpha < \beta\}$ is a neighborhood of C for each $\beta \in A$ and $\bigcap \{V_\alpha \cap X|\ \alpha \in A\} = \emptyset$. Then, it is easy to see that the following condition characterizes the paracompactness:

($\mathrm{P}^{\#}$) *For each central cocover $C \subset \beta X \setminus X$, it is true that*

(16) *There is a free neighborhood chain $\mathfrak{U}$ of C.*

(17) *Given a free neighborhood chain $\mathfrak{U}$ of C, there is a free neighborhood chain $\mathfrak{V}$ of C such that $\mathfrak{V} \prec \mathfrak{U}$.*

(18) *Given two free neighborhood chains $\mathfrak{U}$, $\mathfrak{V}$ of C such that $\mathfrak{V} \prec \mathfrak{U}$, there is a free neighborhood chain $\mathfrak{W}$ such that $\mathfrak{V} \prec \mathfrak{W} \prec \mathfrak{U}$.*

Here, the weights of $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ are identical but they need not coincide with the weight of the central cocover C.

We may abbreviate the above condition ($\mathrm{P}^{\#}$) as follows: *Every central cocover has a nested system of free neighborhood chains.* In fact, we can construct a system of free neighborhood chains $\{\mathfrak{V}_r\}$, where r run through the set of diadic numbers in the unit interval, by repeated use of the condition ($\mathrm{P}^{\#}$), in the same way as in the step (IV) of the proof of Theorem 1. Thus, we have the following

THEOREM $1^{\#}$. *A space is paracompact if and only if every central cocover has a nested system of free neighborhood chains.*

3. An application. The notion of paracompact spaces was originally introduced in [2] by the property of covering asserting that

(19) *Every open covering has a locally finite open refinement.*

E. Michael [5] proved that the following condition is equivalent to (19).

(20) *Every open covering has a σ-locally finite open refinement.*

Two types of weaker conditions which are equivalent to the paracompactness has been obtained by E. Michael [6] and the author [11].

(21) *Every open covering has a linearly locally finite open refinement.*

(22) *Every open covering has a σ-cushioned open refinement.*

In the following, we shall give a simultaneous generalization of these results.

DEFINITION 2. A family $\mathfrak{V} = \{V_\lambda|\ \lambda \in \Lambda\}$ with a well ordered index set Λ is said to be a *linearly cushioned refinement of* $\mathfrak{U} = \{U_\alpha|\ \alpha \in A\}$ if there is a mapping $\varphi\colon \Lambda \to A$ satisfying the following condition: $\overline{\bigcup \{V_\lambda|\ \lambda \in \Lambda^*\}} \subset \bigcup \{U_\alpha|\ \alpha \in \varphi(\Lambda^*)\}$ for each bounded subset Λ^* of Λ.

The mapping φ in the above definition will be called a cushion mapping of $\mathfrak{V}$. It is clear that $\overline{V}_\lambda \subset U_\alpha$ for each $\lambda \in \Lambda$ and for $\alpha = \varphi(\lambda) \in A$.

THEOREM 2. ([3]) *A space is paracompact if and only if every open covering has a linearly cushioned open refinement.*

Proof. Let $\mathfrak{F} = \{F_\alpha|\ \alpha \in A\}$ be a descending free closed chain and let $\mathfrak{U} = \{U_\alpha|\ \alpha \in A\}$ be a chain such that $\mathfrak{F} \prec \mathfrak{U}$. Put $O_\alpha = \mathrm{Int}(U(\alpha)) \cap C(F_\alpha)$, where $U(\alpha) = \bigcap \{U_\gamma|\ \gamma < \alpha\}$. Then $\mathfrak{O} = \{O_\alpha|\ \alpha \in A\}$ is an open covering of the space X. Let $\mathfrak{V} = \{V_\lambda|\ \lambda \in \Lambda\}$ be a linearly cushioned open refinement of $\mathfrak{O}$, and let $\mathfrak{W} = \{W_\sigma|\ \sigma \in \Sigma\}$ be a linearly cushioned open refinement of $\mathfrak{V}$. Let $\varphi\colon \Lambda \to A$ and $\psi\colon \Sigma \to \Lambda$ denote the cushion mapping of $\mathfrak{V}$ and $\mathfrak{W}$ respectively. For each $x \in X$, choose an index $\sigma(x) \in \Sigma$ such that $x \in W_{\sigma(x)}$ and let $\lambda(x) \in \Lambda$ be the least index for which $x \in V_{\lambda(x)}$. If $\varphi(\lambda) = \alpha$, then $\overline{V}_\lambda \subset \mathrm{Int}(U(\alpha)) \cap C(F_\alpha)$. Therefore $\overline{V}_\lambda \subset U_\gamma$ for each $\gamma < \alpha$. If $\beta \geqslant \alpha$, then $\overline{V}_\lambda \cap F_\beta = \emptyset$ since $\mathfrak{F}$ is a descending chain. Thus we have

(23) $$\overline{V}_\lambda \subset U_\gamma \quad \text{if} \quad \varphi(\lambda) > \gamma$$

and

(24) $$\overline{V}_\lambda \cap F_\beta = \emptyset \quad \text{if} \quad \varphi(\lambda) \leqslant \beta\,.$$

It follows from (23) and (24) that

(25) $$\overline{\bigcup \{V_\lambda|\ \varphi(\lambda) \leqslant \alpha,\ \lambda \in \Lambda^*\}} \subset \mathrm{C}(F_\alpha)$$

for any bounded set Λ^* and that

(26) $$\overline{\bigcup \{W_\sigma|\ \varphi\psi((\sigma)) \leqslant \alpha,\ \sigma \in \Sigma^*\}} \subset \bigcup \{V_\lambda|\ \varphi(\lambda) \leqslant \alpha\} \subset \mathrm{C}(F_\alpha)$$

for any bounded set Σ^*.

([3]) This characterization of paracompactness was announced at the Topology Conference at Arisona State University, held in 1967.

Put

(27) $W_\alpha(x)$

$$= W_{\sigma(x)} \setminus \Big[\overline{\bigcup \{V_\lambda |\ \lambda \leqslant \psi(\sigma(x)),\ \varphi(\lambda) \leqslant \alpha\}} \cup \overline{\bigcup \{W_\sigma |\ \sigma \leqslant \sigma(x),\ \varphi(\psi(\sigma)) \leqslant \alpha\}} \Big].$$

By (25) and (26), $W_\alpha(x)$ is a neighborhood of x if $x \in F_\alpha$. Put $N_\alpha = \bigcup \{W_\alpha(x)|\ x \in F_\alpha\}$ and consider an open chain $\mathfrak{N} = \{N_\alpha |\ \alpha \in A\}$. Put

(28) $W_{(\beta)}(x)$

$$= W_{\sigma(x)} \setminus \Big[\overline{\bigcup \{V_\lambda |\ \lambda \leqslant \psi(\sigma(x)),\ \varphi(\lambda) < \beta\}} \cup \overline{\bigcup \{W_\sigma |\ \sigma \leqslant \sigma(x),\ \varphi(\psi(\sigma)) < \beta\}} \Big].$$

Then, $W_{(\beta)}(x)$ is a neighborhood of $F(\beta) = \bigcap \{F_\alpha |\ \alpha < \beta\}$ and it is easy to see that $W_{(\beta)}(x) \subset W_\alpha(x)$ for each $\alpha < \beta$. It follows that $\mathfrak{F} \prec\!\!\!\!\!< \mathfrak{N}$. To prove $\mathfrak{N} \prec\!\!\!\!\!< \mathfrak{U}$, let y be a point such that $y \notin \operatorname{Int}(U(\beta))$, where $U(\beta) = \bigcap \{U_\alpha |\ \alpha < \beta\}$. We shall show that $y \notin \overline{\mathfrak{N}}_\alpha$ for some $\alpha < \beta$. Since $V_{\lambda(y)} \cap W_{\sigma(y)}$ is a neighborhood of y, there is an $\alpha < \beta$ such that $V_{\lambda(y)} \cap W_{\sigma(y)} \not\subset U_\alpha$, because otherwise $V_{\lambda(y)} \cap W_{\sigma(y)}$ is a neighborhood of y contained in $U(\beta)$ which is contradictory. From (23), we have the following relations:

$$\varphi(\lambda(y)) \leqslant \alpha, \tag{29}$$

$$\varphi(\psi(\sigma(y))) \leqslant \alpha. \tag{30}$$

Put $F_1 = \{x \in F_\alpha |\ \psi(\sigma(x)) \geqslant \lambda(y)\}$, $F_2 = \{x \in F_\alpha |\ \sigma(x) \geqslant \sigma(y)\}$ and $F_3 = \{x \in F_\alpha |\ \psi(\sigma(x)) < \lambda(y),\ \sigma(x) < \sigma(y)\}$ and put $N_i = \bigcup \{W_\alpha(x)|\ x \in F_i\}$ for $i = 1, 2$ and 3. Then $N_\alpha = \bigcup_{i=1}^{3} N_i$. In view of (27) and (29), we can see that $W_\alpha(x) \cap V_{\lambda(y)} = \emptyset$ for each $x \in F_1$, and hence we have $y \notin \overline{N}_1$. From (27) and (30), it follows that $W_\alpha(x) \cap W_{\sigma(y)} = \emptyset$ for each $x \in F_2$ and therefore $y \notin \overline{N}_2$. In case that $x \in F_3$,

$$\overline{\bigcup \{W_{\sigma(x)} |\ x \in F_3\}} \subset \overline{\bigcup \{W_\sigma |\ \sigma < \sigma(y),\ \psi(\sigma) < \lambda(y)\}} \subset \bigcup \{V_\lambda |\ \lambda < \lambda(y)\}.$$

Because of the minimality of $\lambda(y)$, $y \notin V_\lambda$ for each $\lambda < \lambda(y)$ and therefore $y \notin \overline{\bigcup \{W_{\sigma(x)} |\ x \in F_3\}}$. Thus, we have $y \notin \overline{N}_3$. Therefore $y \notin \overline{N}_\alpha$ and it follows that $\bigcap \{\overline{N}_\alpha |\ \alpha < \beta\} \subset \operatorname{Int}(U(\beta))$. Thus, we have proved that $\mathfrak{N} \prec\!\!\!\!\!< \mathfrak{U}$. It follows that X is paracompact by Theorem 1.

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