Approximating the standard model of analysis

by

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§ 1. Introduction. The class of $\beta$ models of analysis has been introduced by Mostowski ([6], [7]). These can be characterized as the models of analysis absolute for one-function-quantifier statements about sets in the model. (A more careful definition is given below.) About 1963 Putnam [8] and Gandy independently proved a conjecture of Cohen, that there is a smallest $\beta$ model, and that it coincided with the class of ramified analytical sets ([3], p. 60). (In contrast, there is no smallest $\omega$-model of analysis.) This result is helpful in indicating the extent and the boundary of the theory of $\beta$ models.

Let a $\beta_\alpha$ model of analysis [10] be one which is absolute for $\alpha$-function-quantifier statements about its sets. In 1969 J. Shilleto proved that there was a smallest $\beta_\alpha$ model, and that one could construct it by a procedure similar to the ramified analytical construction, but adding at each stage a segment of the sets $A^n_\alpha$ in sets already obtained [11].

In this paper we first give a simple argument which shows that there is a smallest $\beta_\alpha$ model. It gives a characterization of this smallest model in terms of the hierarchy of constructible sets. (For comparison, the smallest $\beta$ model can be characterized as the class of subsets of the natural numbers which are constructible with order less than $\alpha$, where $\alpha$ is the first ordinal for which this class forms a $\beta$ model.) Next we give a different construction of the smallest $\beta_\alpha$ model which is similar to the construction of the ramified analytical sets. It is simpler than Shilleto's construction in that at each stage all sets $A^n_\alpha$ in those already obtained are added. In §3 we extend these results to $\beta_n$ models for $2 < n < \omega$, assuming that a certain basis property holds.

We now attempt to explain our notation. In this paper a set is always a subset of the set $N$ of natural numbers. An $\omega$ model is identified with its class of sets, and so is considered to be a subclass of $\mathcal{F}N$. (See [4] for a discussion of $\omega$ models. In particular we demand that a model of analysis satisfy the full comprehension schema.) The standard model of analysis is of course $\mathcal{F}N$ itself. If $\mathcal{A}$ and $\mathcal{B}$ are subclasses of $\mathcal{F}N$, say that $\mathcal{A} \sqsubset^* \mathcal{B}$ if


and only if \( \mathcal{A} \subseteq \mathcal{B} \) and for any \( \Sigma_1^0 \) sentence \( \varphi \) with parameters for members of \( \mathcal{A} \)

\[
\models_\mathcal{A} \varphi \iff \models_\mathcal{B} \varphi .
\]

(This condition, that \( \Sigma_1^0 \) statements relativize downward, is clearly equivalent to saying that \( \Pi_1^1 \) statements relativize downward.) \( \mathcal{A} \) is defined to be a \( \beta_\alpha \) model if and only if \( \mathcal{A} \) satisfies comprehension and \( \mathcal{A} \models \Sigma_1^0 \) \( \mathcal{N} \).

Finally a \( \beta \) model is a \( \beta_\alpha \) model.

By a well-known argument ([11], Theorem 1.10, p. 87) we can also say that \( \mathcal{A} \models \Sigma_1^0 \) \( \mathcal{N} \) if and only if \( \mathcal{A} \) retains for classes which are \( \Sigma_1^0 \) relative to members of \( \mathcal{A} \). In particular an \( \alpha \) model is a \( \beta_\alpha \) model if and only if it is closed under relative \( \Delta^1_1 \)-ness.

Let \( \mathcal{L} \) be the class of constructible sets, and \( \mathcal{L}_\gamma \), those of order less than \( \alpha \). We will use several results from [1]. The standard \( \Sigma_1^1 \) definition of \( \mathcal{L} \) covers over a class \( \mathcal{M} \) of sets the class \( \mathcal{C}^{\mathcal{M}} \). If \( \mathcal{M} \) is a \( \beta \) model, then \( \mathcal{C}^{\mathcal{M}} \models \mathcal{L}_\gamma \), where \( \gamma \) is the least ordinal not represented in \( \mathcal{M} \) \( \mathcal{L} \) is a \( \beta_\alpha \) model. (See [13]; this fact is generalized in the theorem below.) If \( \mathcal{A} \models \mathcal{L} \) (i.e., \( \mathcal{A} \models \mathcal{L}_\gamma \) for all \( \gamma \)) then \( \mathcal{A} \) is also a \( \beta_\alpha \) model. This happens for example if \( \mathcal{A} \) is the class of constructibly analytical sets, and it happens if \( \mathcal{A} \models \mathcal{L}_\gamma \) for certain uncountably many countable ordinals \( \alpha \).

One last preliminary comment: Observe that a set belonging to a \( \beta_\alpha \) model \( \mathcal{M} \) is \( \Delta^1_1 \) in \( \mathcal{M} \) if and only if it is really \( \Delta^1_1 \).

\section{2. \( \beta \) models}

We first have the following result, which is based on Shenfield's absoluteness theorem [13].

\textbf{Theorem 1.} Let \( \mathcal{M} \) be a \( \beta_\alpha \) model of analysis. Then \( \mathcal{C}^{\mathcal{M}} \models \mathcal{L} \cap \mathcal{M} \), \( \mathcal{C}^{\mathcal{M}} \models \Delta^1_1 \mathcal{M} \), and \( \mathcal{C}^{\mathcal{M}} \) is a \( \beta_\alpha \) model.

\textbf{Proof.} Since \( \mathcal{L} \) is a \( \Sigma_1^1 \) class, we have \( \mathcal{C}^{\mathcal{M}} \subseteq \mathcal{L} \cap \mathcal{M} \) for any \( \beta \) model \( \mathcal{M} \) and \( \mathcal{L} \cap \mathcal{M} \subseteq \mathcal{C}^{\mathcal{M}} \) for any \( \beta_\alpha \) model \( \mathcal{M} \). Now consider a \( \Sigma_1^1 \) sentence \( \exists \vartheta \forall \psi \beta \) with parameters for members of \( \mathcal{C}^{\mathcal{M}} \).

\[
\models_{\mathcal{M}} \exists \vartheta \forall \psi \beta \iff \models_{\mathcal{M}} \exists \vartheta \exists \psi \forall \varphi \beta \text{ by [13]}
\]

\[
\models_{\mathcal{M}} \exists \vartheta \exists \psi \forall \varphi \beta \iff \models_{\mathcal{M}} \exists \vartheta \forall \psi \exists \varphi \beta.
\]

Finally \( \mathcal{C}^{\mathcal{M}} \) satisfies comprehension since a set definable over \( \mathcal{C}^{\mathcal{M}} \) is definable also over \( \mathcal{M} \), and so is in \( \mathcal{M} \) as well as being in \( \mathcal{L} \).

Hence any \( \beta_\alpha \) model \( \mathcal{M} \) includes a \( \beta_\alpha \) model \( \mathcal{C}^{\mathcal{M}} \) which equals \( \mathcal{L}_\gamma \) for some \( \gamma \). We get the smallest \( \beta_\alpha \) model by simply choosing \( \gamma \) as small as possible. Thus we have:

\textbf{Corollary 2.} There is a smallest \( \beta_\alpha \) model, namely \( \mathcal{M}_\gamma \) for the first \( \alpha \) for which \( \mathcal{M}_\gamma \) is a \( \beta_\alpha \) model.

The \( \alpha \) referred to in this corollary is of course countable; in fact we can say much more. Carry out the above proof within the constructibles. The class of well-orderings \( \mathcal{W} \) such that \( \mathcal{C}^{\mathcal{W}} \) is a \( \beta_\alpha \) model is \( \Pi_1^1 \) and non-empty, so it contains a \( \Delta^1_1 \) element. Since anything which is constructibly a \( \beta_\alpha \) model is really one (by the last preliminary comment), we conclude that there is a constructively \( \Delta^1_1 \) (and hence really \( \Delta^1_1 \)) ordinal \( \alpha \) such that \( \mathcal{L} \) is a \( \beta_\alpha \) model. Similarly there is a constructively \( \Delta^1_1 \) set which encodes (in a natural way) a \( \beta_\alpha \) model.

A "ramified analytical" style construction of the smallest \( \beta_\alpha \) model can be given as follows: For a class \( \mathcal{A} \) of sets, let \( \mathcal{D} \mathcal{A} \) be the class of sets which are definable over \( \mathcal{A} \) by a formula containing parameters for members of \( \mathcal{A} \). Define by transfinite recursion:

\[
\mathcal{F}_0 = \emptyset,
\]

\[\mathcal{F}_{\alpha + 1} = \text{the class of sets } \mathcal{D} \mathcal{F}_\alpha \text{ relative to members of } \mathcal{D} \mathcal{F}_\alpha,\]

\[\mathcal{F}_\alpha = \bigcup \mathcal{F}_\beta \text{ for limit ordinals } \alpha.\]

This construction stabilizes at some ordinal, say \( \gamma \), at which a \( \beta_\alpha \) model is first obtained. (Clearly \( \mathcal{F}_\gamma = \mathcal{F}_{\alpha + 1} \) iff \( \mathcal{F}_\gamma \) is a \( \beta_\alpha \) model of analysis.) Let \( \mathcal{F} = \mathcal{F}_\gamma \).

\textbf{Theorem 3.} \( \mathcal{F} \) is the smallest \( \beta_\alpha \) model of analysis.

\textbf{Proof.} First we claim that for every \( \alpha \leq \gamma \), \( \mathcal{F}_\alpha \subseteq \mathcal{F}_\gamma \). Any class \( \mathcal{A} \subseteq \mathcal{F} \) which is closed under \( \Delta^1_1 \)-ness must equal \( \mathcal{L}_\gamma \) for some \( \gamma \). Hence \( \mathcal{F}_\gamma = \mathcal{L}_\gamma \) for some function \( f \) and \( f \) is strictly increasing below \( \gamma \). Since \( \mathcal{F}_\gamma = \mathcal{L}_\gamma \), we have the \( \alpha \leq f(\alpha) \) for all \( \alpha \leq \gamma \), thus establishing the claim.

Now let \( \mathcal{M} \) be a \( \beta_\gamma \) model, and \( \mu \) the least ordinal not represented in \( \mathcal{M} \). Then by the above theorem \( \mathcal{C}^{\mathcal{M}} = \mathcal{L}_\mu \) is a \( \beta_\mu \) model. The least ordinal \( \gamma \) not represented in \( \mathcal{C}^{\mathcal{M}} \) may be smaller than \( \mu \), but \( \mathcal{L}_\gamma = \mathcal{L}_\gamma \) as defined within \( \mathcal{C}^{\mathcal{M}} \).

Since \( \Delta^1_1 \)-ness is a property absolute for \( \beta_\alpha \) models, \( \mathcal{F} \) coincides with the result of carrying out the construction of \( \mathcal{F} \) inside \( \mathcal{C}^{\mathcal{M}} \). (This is intuitively clear; the full details are in § 3.) Thus

\[
\mathcal{F} \subseteq \mathcal{F}_\gamma = \mathcal{L}_\gamma .
\]

If \( \gamma < \gamma \) then \( \mathcal{F}_\gamma \subseteq \mathcal{F}_\gamma \), whence equality holds and \( \gamma = \gamma \). Thus in any case \( \gamma < \gamma < \mu \) and \( \mathcal{F}_\gamma \subseteq \gamma \).

These methods do not extend (within ZF) to \( \beta_\alpha \) models for \( \alpha > 2 \).

For if there is a measurable cardinal, then \( \mathcal{L}_\gamma \) is never a \( \beta_\alpha \) model for any \( \alpha \),

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by a result of Solovay [14]. We can instead consider submodels of \( \mathcal{L} \). The above results generalize to \( \beta_n \) models, but in uninteresting ways. Let \( \mathcal{M} \) be a \( \beta_n \) model, \( n \geq 2 \). Then \( \mathcal{M}_\mu = \mathcal{L}_\mu \cap \mathcal{M} \), where \( \mu \) is the least ordinal not represented in \( \mathcal{M} \). The smallest \( \beta_n \) model is \( \mathcal{M}_a \), for the least possible \( a \). Everything in this smallest \( \beta_n \) model is constructibly \( \Delta^1_n \). The \( \mathcal{F}_a \) construction still works, but \( \Delta^1_n \)-ness must be interpreted in the sense of \( \mathcal{L} \). The union (over \( n \)) of the smallest \( \beta_n \) models is the class of constructively analytical sets, and this is the smallest class \( A \) such that \( A \prec \mathcal{L} \).

§ 3. \( \beta_n \) models. Although the methods of the preceding section do not extend to \( \beta_n \) models for \( n > 2 \), there are other methods which, assuming some basis properties, let us generalize the \( \mathcal{F}_a \) construction. These methods are similar to those used in the \( n = 1 \) case by Gandy and Putnam. Throughout this section, \( a \) is a fixed natural number, \( n \geq 2 \). Again we define the class \( \mathcal{F}_a \) by recursion; this time it will be slightly more convenient to begin with \( a = -1 \).

\[ \mathcal{F}_{-1} = \emptyset; \]
\[ \mathcal{F}_{a+1} = \text{the class of sets } \mathcal{L}_a \text{ relative to members of } \mathcal{DF}_a; \]
\[ \mathcal{F}_a = \bigcup_{\lambda \leq a} \mathcal{F}_\lambda, \text{ for limit ordinals } \lambda. \]

Let \( \mathcal{F} = \bigcup_{x \in \mathcal{F}_a} x \). Thus \( \mathcal{F} = \mathcal{G}_\gamma \), where \( \gamma \) is the least ordinal at which \( \mathcal{F}_{\gamma+1} = \mathcal{F}_\gamma \), the ordinal of closure.

**Theorem 4.** Assume that for any set \( A \), the class of sets \( \mathcal{L}_a \) in \( A \) forms a basis for the classes which are \( \Delta^1_2 \) in \( A \). Then \( \mathcal{F} \) is the smallest \( \beta_n \) model of analysis.

It is clear that in any case \( \mathcal{F} \) is a model of analysis, since \( \mathcal{DF} \subseteq \mathcal{F}_{\gamma+1} \) (where \( \gamma \) is the ordinal of closure). And by the basis assumption, \( \mathcal{F} \prec \mathcal{L} \). It is the minimality that remains to be shown. The idea of the proof is as follows: Let \( \mathcal{M} \) be another \( \beta_n \) model. Then \( \mathcal{F} \mathcal{F}_{\mu} = \mathcal{F}_{\mu} \), where \( \mu \) is the least ordinal not represented in \( \mathcal{M} \). Then inside \( \mathcal{F}_{\mu} \) (indeed inside \( \mathcal{F} \) for any limit ordinal \( \lambda \)) we can define a well-ordering of the class and the construction of the \( \mathcal{F}_{\lambda} \) sets. This would, if comprehension failed in \( \mathcal{F}_{\lambda} \), permit us to define over \( \mathcal{F} \) a well-ordering of type \( \mu \) (see Lemma 6), which would then have to belong to \( \mathcal{M} \). The heart of the proof consists of verifying that the construction is correctly definable within \( \mathcal{F} \).

First we want to show how the construction of the class \( \mathcal{F}_{\mathcal{M}} \) can be described in second-order arithmetic, where \( \mathcal{M} \) is a well-ordering and \( |\mathcal{M}| \) is its order type. We initially set up a language involving ordinals; later the ordinals will be replaced by numbers in the field of \( \mathcal{M} \). The symbols are:

**Numerical variables:** Denumerably many; in what follows \( x, x_1, x_2, \ldots \) are number variables.

Set variables: Denumerably many; in what follows \( X, X_1, X_2, \ldots \) are set variables.

Function symbols: \( O, S, +, \times \).

Equality: \( = \).

Sentential connectives and numerical quantifiers: As usual.

Set quantifier symbols: \( \forall x \) for an ordinal \( a \).

Operator symbols: \( \land, \lor \).

The numerical terms are defined as usual. The set terms and the formulas are defined simultaneously:

1. Any set variable \( X \) is a set term. All the other set terms will be closed, i.e., no variables will occur free in them.
2. For numerical terms \( t_1, t_2 \) and a set term \( T, t_1 \approx t_2 \) and \( T_0 \) are formulas.
3. The sentential connective symbols and numerical quantifier symbols can be applied to formulas to form new formulas.
4. \( \forall x \varphi \) is a formula, where \( \varphi \) is a formula such that (a) all set quantifier symbols inside set terms occurring in \( \varphi \) are subscripted by ordinals strictly less than \( a \), and (b) all other set quantifier symbols in \( \varphi \) are subscripted \( a \).
5. \( \approx \) is a (closed) set term, where \( \approx \) is a formula in which no variable other than \( x \) occurs free. (This term is read, "the set of all \( x \) such that \( \varphi(x) \)."
6. \( A \varphi x \ldots x \varphi y \) is a (closed) set term, where \( \varphi \) and \( y \) are arithmetical formulas (i.e., formulas without set quantifiers except as may occur inside closed set terms) in which no variables occur free other than \( x, \ldots, x \). (This term is to denote a relatively \( \Delta^1_n \) set if possible, and is to denote \( \emptyset \) otherwise.)

We now proceed to give a (basically syntactical) definition of truth. The essential feature is that the definition is not relative to some universe for the set variables; but instead the set variables range over the denotations of closed set terms.

For a closed numerical term \( t \), let \( \mathcal{F}_t \) be the number it denotes. Truth for sentences is defined by recursion on the maximum subscript of a quantifier symbol, and, within one such maximum subscript \( a \), on the number of places at which \( \forall x, \exists x, \ldots \) occurs.

1. \( t \approx t' \) iff \( t \prec t' \). The sentential connective symbols and numerical quantifier symbols are treated in the natural way.
2. \( \varphi \) is true iff \( \mathcal{F}_t \varphi \), where \( \mathcal{F}_t \varphi \) is the result of replacing \( x \) in \( \varphi \) by the closed term \( t \) whenever \( x \) occurs free.
3. \( \forall x \varphi \) is true iff \( \mathcal{F}_x \varphi \) for every closed set term \( T \) containing only quantifiers subscripted by ordinals strictly smaller than \( a \).
4. Finally we come to the case of $\Delta x_1 \ldots x_n y$. Nothing is lost if we impose, for some large but fixed $k$, the additional restriction on set terms of this form that the arithmetical formula $\varphi$ must be of the form

$$\forall y_0 \exists z_0 \ldots \forall y_2 \theta$$

where $\theta$ has no quantifiers aside from those inside closed set terms [9]. On $\psi$ we impose the same restriction. Then we define:

$$\models \forall x_1 \ldots x_n y \psi \text{ iff there is a set } A \text{ such that}$$

(i) $t^* \in A$;

(ii) A natural number $n$ belongs to $A$ iff for every $B_1$ there exists some $B_2$ such that for every $B_3$, we have $\forall a \exists b_2 \ldots \forall a_2$

$$\models \forall x_1 \forall z_1 \ldots \forall x_n \varphi$$

when $X_{tu}$ (for a numerical term $u$) is replaced by $O \bowtie O$ if $u^* \notin B_1$, and by $O \bowtie O$ if $u^* \notin B_1$.

(iii) [The dual to (ii), using $\Sigma^R_0$ form and $\psi$.]

The English-language set theory above has been italicized; we will later need to consider restricting them to classes smaller than $\mathbb{N}$.

For a closed set term $T$, define its denotation $T^*$ by

$$T^* = \{n : \models T_n\}$$

We can then correlate set terms with the $T_n$ classes. A set is definable over $F_\alpha$ iff it is of the form

$$(\exists x \varphi)^*$$

where the set quantifier symbols inside set terms occurring in $\varphi$ are subscripted by ordinals less than $\alpha$, and all other set quantifier symbols in $\varphi$ are subscripted $\alpha$. And

$$\sim \{T_n : T \text{ a closed set term in which all set quantifier symbols are subscripted by ordinals less than } \beta\}$$

These two statements are easily verified (together) by induction.

As things now stand, formulas and terms may involve ordinal numbers. Ordinals themselves are lacking in analysis, but consider a well-ordering $W$ of some subset of the natural numbers. Then $W$ provides notations for the ordinals less than $|W|$. We obtain $W$-formulas and $W$-terms by using these notations in place of the ordinals themselves. And now we can assign G"odel numbers to these, or better yet take the $W$-formulas and $W$-terms to be themselves natural numbers. The definition of truth applies as well to $W$-expressions as to the original kind. Let

$$\models W = \text{the set of true } W\text{-sentences}.$$
We now can see the full details necessary to establish the claim made in the proof of Theorem 3. Let $\mathcal{M}$ be a $\beta_n$ model, and $\mu$ the least ordinal not represented in $\mathcal{M}$. The claim is that $\mathcal{F}^{\leftarrow}_A = \mathcal{F}_\mu$. The definition of $\mathcal{F}$ in analysis is:

- $A \in \mathcal{F}$ if there is a well-ordering $W$ and a truth set $V$ such that $\tau_W(V)$, and for some closed $W$-term $T$, a natural number $n$ belongs to $A$ iff $\tau_n \in V$.

When the $W$ quantifier is restricted to $\mathcal{M}$, we obtain $\mathcal{F}_\mu$, and nothing is lost when the $V$ quantifier is also restricted to $\mathcal{M}$.

In order to prove the minimality of $\mathcal{F}_\mu$, it will be helpful to know that over $\mathcal{F}_\mu$ we can define an ordering of type $\beta$, for $\beta$ less than the ordinal of closure. Our strategy is to take the least ordinal for which this fails, and to show that closure has occurred by that ordinal.

**Lemma 6.** For each $a$ less than the ordinal of closure, there is a well-ordering of type $a$ in $\mathcal{D} \mathcal{F}_\mu$. For any such well-ordering $W$ in $\mathcal{D} \mathcal{F}_\mu$, we have $V_W$ in $\mathcal{D} \mathcal{F}_{\alpha+1}$. (We assume here that the basis property stated in Theorem 4 holds.)

**Proof.** Let $\gamma$ be the least ordinal such that in $\mathcal{D} \mathcal{F}_\mu$, there is no well-ordering of type $\gamma$. We will show that $\mathcal{F}_\mu$ satisfies comprehension, whence $\gamma$ is at least as large as the ordinal of closure. (It then follows that equality holds. If $\mathcal{F}_\mu$ contains a well-ordering $W$ of order type greater than the ordinal of closure, we could diagonalize to construct a set in $\mathcal{D} \mathcal{F}_\mu$.)

For any $\alpha < \gamma$, then we have some well-ordering $W$ in $\mathcal{D} \mathcal{F}_\alpha$ of type $\alpha$. We first show that for any ordering $W$ of type $\alpha$ in $\mathcal{F}_{\alpha+1}$, we have $V_W$ in $\mathcal{D} \mathcal{F}_{\alpha+1}$ (where $\alpha < \gamma$).

**Case 1.** $a = \beta + 1$ and the ordering in question is $W^+$ where $W \in \mathcal{F}_\alpha$. We first show that for any ordering $W$ of type $\alpha$ in $\mathcal{F}_{\alpha+1}$, we have $V_W$ in $\mathcal{D} \mathcal{F}_{\alpha+1}$. Then by inductive hypothesis $V_W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Apply Lemma 5 with $\mathcal{F}_\mu = \mathcal{F}_{\alpha+1}$ to obtain $\forall W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Apply Lemma 5 with $\mathcal{F}_\mu = \mathcal{F}_{\alpha+1}$ to obtain $\forall W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Then by inductive hypothesis $V_W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Apply Lemma 5 with $\mathcal{F}_\mu = \mathcal{F}_{\alpha+1}$ to obtain $\forall W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Then by inductive hypothesis $V_W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Apply Lemma 5 with $\mathcal{F}_\mu = \mathcal{F}_{\alpha+1}$ to obtain $\forall W \in \mathcal{D} \mathcal{F}_{\alpha+1}$. Then by inductive hypothesis $V_W \in \mathcal{D} \mathcal{F}_{\alpha+1}$.

**Case 2.** $a = \beta + 1$ and the ordering in question is $W^+$ in $\mathcal{F}_{\alpha+1}$. We have in $\mathcal{F}_{\alpha+1}$ another ordering $U$ of type $\beta$. There is a unique isomorphism between $W^+$ and $U^*$. The isomorphism is arithmetic definable from the orderings, and so also belongs to $\mathcal{F}_{\alpha+1}$. And $V_W$ is recursive in $V_U$ and the isomorphism since the isomorphism induces a truth preserving map from $W^+$-sentences to $U^*$-sentences. By Case 1, $V_W$ is in $\mathcal{D} \mathcal{F}_{\alpha+1}$, and $V_W$ must also be in this class.

**Case 3.** $a$ is a limit ordinal. For any segment $W[a]$ of the ordering $W$ in $\mathcal{F}_{\alpha+1}$, there is a well-ordering $U$ in $\mathcal{F}_{\alpha+1}$ with $V_U$ in $\mathcal{F}_{\alpha+1}$. Thus in $\mathcal{F}_{\alpha+1}$ we have: $V[W \mid a, U]$, the isomorphism between $W[a]$ and $U$, $V_U$, and hence $V_W$. Then

$$f \epsilon V_W \iff \exists a \forall V[W \mid a](V) \& f \epsilon V$$

and the $V$ quantifier can be restricted to $\mathcal{F}_{\alpha+1}$. Thus $V_W$ is in $\mathcal{D} \mathcal{F}_{\alpha+1}$.

To prove Lemma 6, it remains to show that $\mathcal{F}_\mu$ satisfies comprehension. For that we use Lemma 7, below, with $\delta = \gamma$ and $\mathcal{A}_\mu = \mathcal{F}_\mu$. We must verify that the hypotheses of that lemma are satisfied. Clearly $\gamma$ is a limit ordinal and hypotheses (1) and (2) are met. Hypothesis (3) holds, since for any $a$, $\mathcal{F}_a$ is closed under $\alpha'$-ness. Hypothesis (4) follows at once from the definition of $\gamma$.

As for hypothesis (5), recall that a set $\mathcal{A}$ belongs to $\mathcal{F}_\mu$ if it is of the form $\{V \mid x \in V \wedge V \in \mathcal{F}_\mu\}$ for some $V$ where $V \in \mathcal{F}_\mu$. Thus we can let $a(i, x, W)$ be: $i$ is a $W$-term beginning with $\beta$ and for some $V$ for which $\tau(V)$, the sentence $a(i, x)$ is in $V$.

Finally for hypothesis (6) we need a definable well-ordering of $\mathcal{F}_\mu$. First well-order the closed set terms, ordering first according to the largest ordinal subscript, then by length, and then lexicographically. (Actually any reasonable well-ordering could be employed here.) Then define:

$$A \subseteq B \iff A \text{ is denoted by some closed term which is smaller than any closed term denoting } B.$$
6. There is a well-ordering of \( \mathcal{A}_\beta \) definable over \( \mathcal{A}_\delta \).

Then \( \mathcal{A}_\delta \) satisfies comprehension.

Proof. We will define, by recursion on the prenex formula \( \varphi \), a function \( f_\delta \) mapping \( \delta \) into \( \delta \) such that:

(i) \( f_\delta \) is non-decreasing and continuous, and \( a \leq f_\delta(a) \) for \( a < \delta \).

(ii) For any \( a < \delta \), any string \( B \) of sets from \( \mathcal{A}_\delta \), and string \( \bar{n} \) of natural numbers,

\[
|A_{1,0,0}| \bar{n}[\bar{n}, B] \iff |A_{1,0,0}| \bar{n}, B] .
\]

(iii) \( f_\delta \) is definable over \( \mathcal{A}_\delta \) in the sense that the relation which holds between \( \mathcal{A} \) and \( B \) iff both are well-orderings in \( \mathcal{A}_\delta \) and \( f_\delta(\mathcal{A}) = |B| \) is definable over \( \mathcal{A}_\delta \).

Once we have such functions, it easy to see that the comprehension axioms are satisfied. For

\[
|A_{1,0,0}| \bar{n}, B] \iff |A_{1,0,0}| \bar{n}, B] .
\]

by using assumption 2.

For arithmetical (i.e., elementary) formulas \( \varphi \) we take \( f_\delta \), to be the identity function on \( \delta \). This is definable over \( \mathcal{A}_\delta \) by assumption 3. For the negation \( \neg \varphi \) of \( \varphi \) we simply take \( f_\varphi(a) = f_\delta(a) \). The only other case is that of the quantified formula \( \exists \mathcal{X} \varphi \). Here we simply take:

\[
f_{\exists \mathcal{X} \varphi}(a) = \text{the least } \beta \geq a \text{ such that for any string } B \\
\text{from } \mathcal{A}_\delta, \text{any } \bar{n},
\]

\[
|A_{1,0,0}| \bar{n}, B] \iff |A_{1,0,0}| \bar{n}, B] .
\]

(*)

There are such \( \beta \)'s, c.g. \( \beta = \delta \). But we will first show that (for fixed \( \bar{n}, B \)), \( \beta \) satisfying (\( \alpha \)) which is definable over \( \mathcal{A}_\delta \) and hence is less than \( \delta \). (Later the dependence on \( \bar{n} \) and \( B \) will be eliminated.)

Case A. Suppose \( |A_{1,0,0}| \bar{n}, B] \). Thus for some \( C \) in \( \mathcal{A}_\delta = \bigcup_{\varphi \in \mathcal{A}_\delta} |A_{1,0,0}| \bar{n}, B] \).

Choose such a \( C \) in some \( \mathcal{A}_\gamma \); by assumption 1 we may suppose \( \gamma \geq \alpha \).

Then we may simply take \( \beta = f_\delta(\gamma) \), for

\[
|A_{1,0,0}| \bar{n}, B] \iff |A_{1,0,0}| \bar{n}, B] .
\]

and hence

\[
|A_{1,0,0}| \bar{n}, B] \iff |A_{1,0,0}| \bar{n}, B] .
\]

Case B. Suppose on the other hand

\[
|A_{1,0,0}| \bar{n}, B] \iff |A_{1,0,0}| \bar{n}, B] .
\]

as in the preceding paragraph. Again we string these orderings together to obtain a definable ordering longer than any one. Its order type is less than \( \delta \) and is the uniform \( \beta \) desired.

Finally we must verify that \( f_{\exists \mathcal{X} \varphi} \) meets conditions (i)-(iii). That \( f_{\exists \mathcal{X} \varphi} \) is non-decreasing follows from its definition and the fact that the classes \( \mathcal{A}_\delta \) are non-decreasing as \( a \) increases. Its continuity similarly follows from the fact that \( \mathcal{A}_\delta \) equals \( \bigcup_{\mathcal{A}} \) for a limit ordinal \( \lambda \). And \( f_{\exists \mathcal{X} \varphi}(a) \geq a \).
by definition. Condition (ii) is obviously satisfied. For condition (iii) we must show that \( f_{\mathcal{X}} \) is definable over \( \mathbb{A} \). We have

\[
\exists A \exists \mathcal{X} \exists \mathcal{Y} f(\mathcal{X}, \mathcal{Y}) \iff [\mathcal{X}] \subseteq [\mathcal{Y}] \text{ for any string } \mathcal{X} \text{ of sets from } \mathcal{A} \text{ and any } \mathcal{Y}.
\]

By using (primarily) assumption 5, this condition on \( \mathcal{A} \) and \( \mathcal{U} \) (in \( \mathbb{A} \)) is definable over \( \mathbb{A} \).

Finally we are able to conclude that \( \mathcal{F} \) is included in any other \( \mathcal{X} \), model \( \mathcal{M} \). Let \( \mu \) be the least ordinal not represented in \( \mathcal{M} \). As explained above, \( \mathcal{F} = \mathcal{F}_\mu \). If \( \mu \) were less than the ordinal of closure, there would be an ordering of type \( \mu \) definable over \( \mathcal{F} \). But since \( \mathcal{F}_\mu \) is a definable class in \( \mathcal{M} \), the ordering would be definable over \( \mathcal{M}_\mu \) and hence in \( \mathcal{M} \). This completes the proof of Theorem 4.

**Theorem 5.** Continue to assume that for any \( \mathcal{A} \), the class of sets \( \mathcal{A} \) in \( \mathcal{A} \) is a basis for the classes \( \Sigma^1_n \) in \( \mathcal{A} \). Then for any set in \( \mathcal{F} \), there is a formula (without set parameters) which correctly defines that set over any \( \beta_n \) model \( \mathcal{M} \).

**Proof.** Consider the set \( \mathcal{A} \in \mathcal{F} \). It suffices to show that \( \mathcal{A} \) is definable over \( \mathcal{F} \). For \( \mathcal{F} = \mathcal{F}_\mu \) is a definable class over \( \mathcal{M} \).

The set \( \mathcal{A} \) is definable over \( \mathcal{F} \) from the set \( \mathcal{V}_\mu \) for all sufficiently large well-orderings \( \mathcal{W} \). And \( \mathcal{V}_\mu \) is definable over \( \mathcal{F} \) from \( \mathcal{W} \). So it suffices to show that for every ordinal less than the ordinal of closure, there is a well-ordering of that type definable over \( \mathcal{F} \).

Let \( \lambda \) be the least ordinal not represented by a well-ordering definable in \( \mathcal{F} \). We claim that \( \mathcal{F}_\lambda \subseteq \mathcal{F} \). For suppose \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are in \( \mathcal{F}_\lambda \) and

\[
\exists \mathcal{X} \exists \mathcal{Y} dp(\mathcal{X}, \mathcal{Y}, \mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B})
\]

Then the least (in our definable ordering of \( \mathcal{F} \)) well-ordering \( \mathcal{W} \) such that for some \( \mathcal{B} \) in \( \mathcal{F}_\mu \),

\[
\exists \mathcal{X} \exists \mathcal{Y} f(\mathcal{X}, \mathcal{Y}, \mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B})
\]

is definable in \( \mathcal{F} \) from \( \mathcal{A}_1, \ldots, \mathcal{A}_n \). But since \( \mathcal{A}_1 \) is in \( \mathcal{F}_\lambda \), \( \mathcal{A}_1 \) is definable; so the \( \mathcal{V} \) above is definable. Hence \( |\mathcal{W}| < \lambda \) and there is some \( \mathcal{B} \) in \( \mathcal{F}_\lambda \) which works. Hence \( \mathcal{F}_\lambda \subseteq \mathcal{F} \).

Consequently \( \mathcal{F}_\lambda \) satisfies comprehension and so equals \( \mathcal{F} \). So \( \lambda \) is the ordinal of closure.

The conclusion of this theorem can also be stated: A set is strongly representable (biosnumerical) in the theory of \( \beta_n \) models if and only if it belongs to \( \mathcal{F} \).

**§ 4. Further comments.** In the preceding section \( n \) was a fixed number greater than one. If the basis hypothesis used there (that for a set \( \mathcal{A} \), the class \( \mathcal{A} \) of \( \mathcal{A} \) forms a basis for classes \( \Sigma^1_n \) in \( \mathcal{A} \)) holds for infinitely many values of \( n \), then the class \( \mathcal{A} \) of analytical sets is a basis for any analytical class. In this case (in fact equivalently) we have \( \mathcal{A} \subseteq \Delta \) since for analytical \( \mathcal{B} \),

\[
\exists A \exists \mathcal{X} \exists \mathcal{Y} dp(\mathcal{X}, \mathcal{Y}) \iff [\mathcal{X}] \subseteq [\mathcal{Y}] \text{ for } \gamma = \Delta_0 \neg \forall \mathcal{B} \rightarrow \exists \mathcal{A} \neg \forall \mathcal{B} \rightarrow \exists \mathcal{A} \neg \forall \mathcal{B}. \]

And it is clear (without basis assumptions) that \( \mathcal{A} \) must be included in any elementary submodel of \( \Delta \). Thus we have the simple result:

**Theorem 6.** If the class \( \mathcal{A} \) of analytical sets is a basis for analytical classes, then \( \mathcal{A} \) is the smallest elementary submodel of \( \Delta \).

This is the analog to Theorem 4 for \( n = 0 \), but its proof is vastly simpler. Also it is obvious for this class that if \( \mathcal{A} \subseteq \Delta_0 \) then each element of \( \mathcal{A} \) is definable (without set parameters) over \( \Delta_0 \).

The basis assumption of section 3 is a well-known consequence of the axiom of constructibility [1]. On the other hand, it has been shown that the basis hypothesis implies that for odd values of \( n \), \( \mathcal{A} \) is not a basis for \( \Sigma^1_n \) ([3], [2]). Martin and Solovay have conjectured that the basis hypothesis does hold for even \( n \); see [5], p. 156. In this event our results would at least hold for \( n \) even and \( n = \omega \).

If we turn from truth to consistency, we have the following result by Silver (see [12]; cf. also Martin and Solovay [5]): If \( \mathcal{L} \) there is a measurable cardinal" is consistent, then it remains consistent with the additional axioms:

1. \( \Delta^1_n \) is a basis for \( \Sigma^1_n \) for each set \( \mathcal{A} \subseteq \Delta \).
2. There is a \( \Delta^1_n \) well-ordering of \( \Delta \) in the order type of the least uncountable ordinal.

The second of these implies that our basis hypothesis holds for all \( n > 3 \) (cf. [1], pp. 350–351).

**References**


