Approximating the standard model of analysis

by

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§ 1. Introduction. The class of β models of analysis has been introduced by Mostowski ([6], [7]). These can be characterized as the models of analysis absolute for one-function-quantifier statements about sets in the model. (A more careful definition is given below.) About 1963 Putnam [8] and Gandy independently proved a conjecture of Cohen, that there is a smallest β model, and that it coincided with the class of ramified analytical sets ([3], p. 60). (In contrast, there is no smallest ω-model of analysis.) This result is helpful in indicating the extent and the boundary of the theory of β models.

Let a β_n model of analysis [10] be one which is absolute for n-function-quantifier statements about its sets. In 1968 J. Shilleto proved that there was a smallest β_2 model, and that one could construct it by a procedure similar to the ramified analytical construction, but adding at each stage a segment of the sets Δ_2^1 in sets already obtained [11].

In this paper we first give a simple argument which shows that there is a smallest β_2 model. It gives a characterization of this smallest model in terms of the hierarchy of constructible sets. (For comparison, the smallest β model can be characterized as the class of subsets of the natural numbers which are constructible with order less than α, where α is the first ordinal for which this class forms a β model.) Next we give a different construction of the smallest β_2 model which is similar to the construction of the ramified analytical sets. It is simpler than Shilleto's construction in that at each stage all sets Δ_2^1 in those already obtained are added. In § 3 we extend these results to β_n models for $2 \leqslant n \leqslant \omega$, assuming that a certain basis property holds.

We now attempt to explain our notation. In this paper a set is always a subset of the set N of natural numbers. An ω model is identified with its class of sets, and so is considered to be a subclass of $\mathfrak{P}N$. (See [4] for a discussion of ω models. In particular we demand that a model of analysis satisfy the full comprehension schema.) The standard model of analysis is of course $\mathfrak{P}N$ itself. If $\mathcal{A}$ and $\mathcal{B}$ are subclasses of $\mathfrak{P}N$, say that $\mathcal{A} \preccurlyeq_n^1 \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and for any Σ_n^1 sentence φ with parameters for members of $\mathcal{A}$

$$\models_{\mathcal{B}} \varphi \Rightarrow \models_{\mathcal{A}} \varphi .$$

(This condition, that Σ_n^1 statements relativize downward, is clearly equivalent to saying that Π_{n+1}^1 statements relativize downward.) $\mathcal{A}$ is defined to be a β_n model if and only if $\mathcal{A}$ satisfies comprehension and $\mathcal{A} \prec_n^1 \mathfrak{P}N$. Finally a β model is a β_1 model.

By a well-known argument ([15], Theorem 1.10, p. 87) we can also say that $\mathcal{A} \prec_n^1 \mathfrak{P}N$ if and only if $\mathcal{A}$ is a basis for classes which are Σ_n^1 relative to members of $\mathcal{A}$. In particular an ω model is a β_2 model if and only if it is closed under relative Δ_2^1-ness.

Let $\mathfrak{L}$ be the class of constructible sets, and $\mathfrak{L}_\alpha$ those of order less than α. We will use several results from [1]. The standard Σ_2^1 definition of $\mathfrak{L}$ defines over a class $\mathcal{M}$ of sets the class $\mathfrak{L}^{\mathcal{M}}$. If $\mathcal{M}$ is a β model, then $\mathfrak{L}^{\mathcal{M}} = \mathfrak{L}_\mu$, where μ is the least ordinal not represented in $\mathcal{M}$. $\mathfrak{L}$ is a β_2 model. (See [13]; this fact is generalized in the theorem below.) If $\mathcal{A} \prec \mathfrak{L}$ (i.e., $\mathcal{A} \prec_n^1 \mathfrak{L}$ for all n) then $\mathcal{A}$ is also a β_2 model. This happens for example if $\mathcal{A}$ is the class of constructibly analytical sets, and it happens if $\mathcal{A} = \mathfrak{L}_\alpha$ for certain uncountably many countable ordinals α.

One last preliminary comment: Observe that a set belonging to a β_n model $\mathcal{M}$ is Δ_n^1 in $\mathcal{M}$ if and only if it is really Δ_n^1.

§ 2. β_2 models. We first have the following result, which is based on Shoenfield's absoluteness theorem [13].

THEOREM 1. *Let $\mathcal{M}$ be a β_2 model of analysis. Then $\mathfrak{L}^{\mathcal{M}} = \mathfrak{L} \cap \mathcal{M}$, $\mathfrak{L}^{\mathcal{M}} \prec_2^1 \mathcal{M}$, and $\mathfrak{L}^{\mathcal{M}}$ is a β_2 model.*

Proof. Since $\mathfrak{L}$ is a Σ_2^1 class, we have $\mathfrak{L}^{\mathcal{M}} \subseteq \mathfrak{L} \cap \mathcal{M}$ for any β model $\mathcal{M}$, and $\mathfrak{L} \cap \mathcal{M} \subseteq \mathfrak{L}^{\mathcal{M}}$ for any β_2 model. Now consider a Σ_2^1 sentence $\exists \alpha \forall \beta \theta$ with parameters for members of $\mathfrak{L}^{\mathcal{M}}$.

$$\begin{aligned} \models_{\mathcal{M}} \exists \alpha \forall \beta \theta &\Rightarrow \models_{\mathfrak{P}N} \exists \alpha \forall \beta \theta \\ &\Rightarrow \models_{\mathfrak{P}N} (\exists \alpha \in \mathfrak{L}) \forall \beta \theta \quad \text{by} \quad [13] \\ &\Rightarrow \models_{\mathcal{M}} (\exists \alpha \in \mathfrak{L}) \forall \beta \theta \\ &\Rightarrow \models_{\mathcal{M}} (\exists \alpha \in \mathfrak{L}) (\forall \beta \in \mathfrak{L}) \theta \\ &\Rightarrow \models_{\mathfrak{L}^{\mathcal{M}}} \exists \alpha \forall \beta \theta . \end{aligned}$$

Finally $\mathfrak{L}^{\mathcal{M}}$ satisfies comprehension since a set definable over $\mathfrak{L}^{\mathcal{M}}$ is definable also over $\mathcal{M}$, and so is in $\mathcal{M}$ as well as being in $\mathfrak{L}$. ■

Hence any β_2 model $\mathcal{M}$ includes a β_2 model $\mathfrak{L}^{\mathcal{M}}$ which equals $\mathfrak{L}_\mu$ for some μ. We get the smallest β_2 model by simply choosing μ as small as possible. Thus we have:

COROLLARY 2. *There is a smallest β_2 model, namely $\mathfrak{L}_\alpha$ for the first α for which $\mathfrak{L}_\alpha$ is a β_2 model.*

The α referred to in this corollary is of course countable; in fact we can say much more. Carry out the above proof within the constructibles. The class of well-orderings W such that $\mathfrak{L}_{|W|}$ is a β_2 model is Π_2^1 and non-empty, so it contains a Δ_3^1 element. Since anything which is constructibly a β_2 model is really one (by the last preliminary comment), we conclude that there is a constructibly Δ_3^1 (and hence really Δ_3^1) ordinal α such that $\mathfrak{L}_\alpha$ is a β_2 model. Similarly there is a constructibly Δ_3^1 set which encodes (in a natural way) a β_2 model.

A "ramified analytical" style construction of the smallest β_2 model can be given as follows: For a class $\mathcal{A}$ of sets, let $\boldsymbol{D}\mathcal{A}$ be the class of sets which are definable over $\mathcal{A}$ by a formula containing parameters for members of $\mathcal{A}$. Define by transfinite recursion:

$$\mathcal{F}_0 = \emptyset .$$

$$\mathcal{F}_{\alpha+1} = \text{the class of sets } \Delta_2^1 \text{ relative to members of } \boldsymbol{D}\mathcal{F}_\alpha .$$

$$\mathcal{F}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{F}_\alpha \text{ for limit ordinals } \lambda .$$

This construction stabilizes at some ordinal, say γ, at which a β_2 model is first obtained. (Clearly $\mathcal{F}_\alpha = \mathcal{F}_{\alpha+1}$ iff $\mathcal{F}_\alpha$ is a β_2 model of analysis.) Let $\mathcal{F} = \mathcal{F}_\gamma$.

THEOREM 3. *$\mathcal{F}$ is the smallest β_2 model of analysis.*

Proof. First we claim that for every $\alpha \leqslant \gamma$, $\mathfrak{L}_\alpha \subseteq \mathcal{F}_\alpha$. Any class $\mathcal{A} \subseteq \mathfrak{L}$ which is closed under Δ_2^1-ness must equal $\mathfrak{L}_\alpha$ for some α. Hence $\mathcal{F}_\alpha = \mathfrak{L}_{f(\alpha)}$ for some function f, and f is strictly increasing below γ. Since $\mathcal{F}_0 = \mathfrak{L}_0$, we have the $\alpha \leqslant f(\alpha)$ for $\alpha \leqslant \gamma$, thus establishing the claim.

Now let $\mathcal{M}$ be a β_2 model, and μ the least ordinal not represented in $\mathcal{M}$. Then by the above theorem $\mathfrak{L}^{\mathcal{M}} (= \mathfrak{L}_\mu)$ is a β_2 model. The least ordinal ν not represented in $\mathfrak{L}^{\mathcal{M}}$ may be smaller than μ, but

$$\begin{aligned} \mathfrak{L}_\nu &= \mathfrak{L} \quad \text{as defined within } \mathfrak{L}^{\mathcal{M}} \\ &= \mathfrak{L} \cap \mathfrak{L}^{\mathcal{M}} = \mathfrak{L}_\mu . \end{aligned}$$

Since Δ_2^1-ness is a property absolute for β_2 models, $\mathcal{F}_\nu$ coincides with the result of carrying out the construction of $\mathcal{F}$ inside $\mathfrak{L}^{\mathcal{M}}$. (This is intuitively clear; the full details are in § 3.) Thus

$$\mathcal{F}_\nu \subseteq \mathfrak{L}_\nu = \mathfrak{L}_\mu .$$

If $\nu \leqslant \gamma$ then $\mathfrak{L}_\nu \subseteq \mathcal{F}_\nu$, whence equality holds and $\nu = \gamma$. Thus in any case $\gamma \leqslant \nu \leqslant \mu$ and $\mathcal{F} \subseteq \mathfrak{L}_\nu \subseteq \mathcal{M}$. ■

These methods do not extend (within ZF) to β_n models for $n > 2$. For if there is a measurable cardinal, then $\mathfrak{L}_\alpha$ is never a β_3 model for any α, by a result of Solovay [14]. We can instead consider submodels of $\mathfrak{L}$. The above results generalize to $\beta_n^{\mathfrak{L}}$ models, but in uninteresting ways. Let $\mathcal{M}$ be a $\beta_n^{\mathfrak{L}}$ model, $n \geqslant 2$. Then $\mathcal{M} = \mathfrak{L}^{\mathcal{M}} = \mathfrak{L}_\mu$, where μ is the least ordinal not represented in $\mathcal{M}$. The smallest $\beta_n^{\mathfrak{L}}$ model is $\mathfrak{L}_a$ for the least possible a. Everything in this smallest $\beta_n^{\mathfrak{L}}$ model is constructibly Δ^1_{n+1}. The $\mathcal{F}_a$ construction still works, but Δ^1_n-ness must be interpreted in the sense of $\mathfrak{L}$. The union (over n) of the smallest $\beta_n^{\mathfrak{L}}$ models is the class of constructibly analytical sets, and this is the smallest class $\mathcal{A}$ such that $\mathcal{A} \precsim \mathfrak{L}$.

§ 3. β_n models. Although the methods of the preceding section do not extend to β_n models for $n > 2$, there are other methods which, assuming some basis properties, let us generalize the $\mathcal{F}_a$ construction. These methods are similar to those used in the $n = 1$ case by Gandy and Putnam. Throughout this section, n is a fixed natural number, $n \geqslant 2$. Again we define the class $\mathcal{F}_a$ by recursion; this time it will be slightly more convenient to begin with $a = -1$.

$\mathcal{F}_{-1} = \emptyset$;

$\mathcal{F}_{a+1} =$ the class of sets Δ^1_n relative to members of $\boldsymbol{D}\mathcal{F}_a$;

$\mathcal{F}_\lambda = \bigcup_{a<\lambda} \mathcal{F}_a$ for limit ordinals λ.

Let $\mathcal{F} = \bigcup_{a \in On} \mathcal{F}_a$. Thus $\mathcal{F} = \mathcal{F}_\gamma$ where γ is the least ordinal at which $\mathcal{F}_{\gamma+1} = \mathcal{F}_\gamma$, the *ordinal of closure.*

THEOREM 4. *Assume that for any set A, the class of sets Δ^1_n in A forms a basis for the classes which are Σ^1_n in A. Then $\mathcal{F}$ is the smallest β_n model of analysis.*

It is clear that in any case $\mathcal{F}$ is a model of analysis, since $\boldsymbol{D}\mathcal{F} \subseteq \mathcal{F}_{\gamma+1} = \mathcal{F}$ (where γ is the ordinal of closure). And by the basis assumption, $\mathcal{F} \precsim^1_n \mathfrak{P}N$. It is the minimality that remains to be shown. The idea of the proof is as follows: Let $\mathcal{M}$ be another β_n model. Then $\mathcal{F}^{\mathcal{M}} = \mathcal{F}_\mu$, where μ is the least ordinal not represented in $\mathcal{M}$. Then inside $\mathcal{F}_\mu$ (indeed inside $\mathcal{F}_\lambda$ for any limit ordinal λ) we can define a well-ordering of the class and the construction of the $\mathcal{F}_a$ sets. This would, if comprehension failed in $\mathcal{F}_\mu$, permit us to define over $\mathcal{F}_\mu$ a well-ordering of type μ (see Lemma 6), which would then have to belong to $\mathcal{M}$. The heart of the proof consists of verifying that the construction is correctly definable within $\mathcal{F}_\lambda$.

First we want to show how the construction of the class $\mathcal{F}_{|W|}$ can be described in second-order arithmetic, where W is a well-ordering and $|W|$ is its order type. We initially set up a language involving ordinals; later the ordinals will be replaced by numbers in the field of W. The symbols are:

Numerical variables: Denumerably many; in what follows $x, x_1, x_2, \ldots$ are number variables.

Set variables: Denumerably many; in what follows $X, X_1, X_2, \ldots$ are set variables.

Function symbols: $\boldsymbol{O}, \boldsymbol{S}, +, \cdot$.

Equality: $\approx$.

Sentential connectives and *numerical quantifiers*: As usual.

Set quantifier symbols: $\forall_a$ for an ordinal a.

Operator symbols: λ, Δ.

The numerical terms are defined as usual. The set terms and the formulas are defined simultaneously:

1. Any set variable X is a set term. All the other set terms will be closed, i.e., no variables will occur free in them.

2. For numerical terms t_1, t_2 and a set term T, $t_1 \approx t_2$ and Tt_1 are formulas.

3. The sentential connective symbols and numerical quantifier symbols can be applied to formulas to form new formulas.

4. $\forall_a X\varphi$ is a formula, where φ is a formula such that (a) all set quantifier symbols inside set terms occurring in φ are subscripted by ordinals strictly less than a, and (b) all other set quantifier symbols in φ are subscripted a.

5. $\lambda x\varphi$ is a (closed) set term, where φ is a formula in which no variable other than x occurs free. (This term is read, "the set of all x such that φ".)

6. $\Delta x X_1 \ldots X_n \varphi\psi$ is a (closed) set term, where φ and ψ are arithmetical formulas (i.e., formulas without set quantifiers except as may occur inside closed set terms) in which no variables occur free other than $x, X_1, \ldots, X_n$. (This term is to denote a relatively Δ^1_n set if possible, and is to denote $\emptyset$ otherwise.)

We now proceed to give a (basically syntactical) definition of truth. The essential feature is that the definition is not relative to some universe for the set variables; but instead the set variables range over the denotations of closed set terms.

For a closed numerical term t, let t^* be the number it denotes. Truth for sentences is defined by recursion on the maximum subscript of a quantifier symbol, and, within one such maximum subscript a, on the number of places at which $\forall_a$, λ, or Δ occur.

1. $\models t_1 \approx t_2$ iff $t_1^* = t_2^*$. The sentential connective symbols and numerical quantifier symbols are treated in the natural way.

2. $\models \lambda x\varphi t$ iff $\models \varphi^x_t$, where φ^x_t is the result of replacing x in φ by the closed term t wherever x occurs free.

3. $\models \forall_a X\varphi$ iff $\models \varphi^X_T$ for every closed set term T containing only quantifiers subscripted by ordinals strictly smaller than a.

4. Finally we come to the case of $\Delta x X_1 \ldots X_n \varphi\psi$. Nothing is lost if we impose, for some large but fixed k, the additional restriction on set terms of this form that the arithmetical formula φ must be of the form

$$\forall x_1 \exists x_2 \ldots \forall x_k\, \theta$$

where θ has no quantifiers aside from those inside closed set terms [9]. On ψ we impose the same restriction. Then we define:

$\models \Delta x X_1 \ldots X_n \varphi\psi t$ iff *there is a set* A such that

(i) $t^* \in A$;

(ii) A natural number n belongs to A iff *for every* B_1 *there exists some* B_2 such that *for every* $B_3 \ldots$ we have $\forall a_1 \exists a_2 \ldots \forall a_k$

$$\models \theta^{x x_1 x_2 \ldots x_k}_{n a_1 a_2 \ldots a_k}$$

when $X_i u$ (for a numerical term u) is replaced by $\boldsymbol{O} \approx \boldsymbol{O}$ if $u^* \in B_i$, and by $\boldsymbol{O} \not\approx \boldsymbol{O}$ if $u^* \notin B_i$.

(iii) [The dual to (ii), using Σ^1_n form and ψ.]

The English-language set quantifiers above have been italicized; we will later need to consider restricting them to classes smaller than $\mathfrak{I}N$.

For a closed set term T, define its denotation T^* by

$$T^* = \{n\colon \models \boldsymbol{T n}\}.$$

We can then correlate set terms with the $\mathcal{F}_a$ classes. A set is definable over $\mathcal{F}_a$ iff it is of the form

$$(\lambda x \varphi)^*$$

where the set quantifier symbols inside set terms occurring in φ are subscripted by ordinals less than a, and all other set quantifier symbols in φ are subscripted a. And

$\mathcal{F}_\beta = \{T^*\colon T$ is a closed set term in which all set quantifier symbols are subscripted by ordinals less than $\beta\}$.

These two statements are easily verified (together) by induction.

As things now stand, formulas and terms may involve ordinal numbers. Ordinals themselves are lacking in analysis, but consider a well-ordering W of some subset of the natural numbers. Then W provides notations for the ordinals less than $|W|$. We obtain W-formulas and W-terms by using these notations in place of the ordinals themselves. And now we can assign Gödel numbers to these, or better yet take the W-formulas and W-terms to be themselves natural numbers. The definition of truth applies as well to W-expressions as to the original kind. Let

$$V_W = \text{the set of true } W\text{-sentences}.$$

Then V_W is a set of natural numbers, and is definable (from W) in analysis. For if we take the inductive definition of truth, and replace "$\models \varphi$" by "$\varphi \in V$" and replace ordinals by notations, we get a formula $\tau_W(V)$ having the free set variable V and a name for W. For a well ordering W, V_W is the unique set which satisfies τ_W (over $\mathfrak{I}N$).

Now assume that $\mathcal{M} \preccurlyeq^1_n \mathfrak{I}N$ and W is a well-ordering in $\mathcal{M}$. First we claim that over $\mathcal{M}$, τ_W can be satisfied by no set other than V_W. This requires verifying the absoluteness of everything in clause 4 of the definition of truth. The A quantifier can be restricted to $\mathcal{M}$ since the desired set will be Δ^1_n in V and hence will belong to $\mathcal{M}$. (The fact that $\mathcal{M} \preccurlyeq^1_n \mathfrak{I}N$ implies that $\mathcal{M}$ is closed under relative Δ^1_n-ness.) The B_i quantifiers can also be restricted to $\mathcal{M}$ because $\mathcal{M} \preccurlyeq^1_n \mathfrak{I}N$. The same argument shows that if $V_W \in \mathcal{M}$, then it satisfies τ_W over $\mathcal{M}$.

If we further assume that $\mathcal{M}$ satisfies comprehension, then we can conclude that $V_W \in \mathcal{M}$. This can be seen by induction on $|W|$. If $|W|$ is a limit ordinal then we can define V_W as the union of the sets $V_{W \upharpoonright a}$ where $W \upharpoonright a$ is the restriction of W to points smaller (in the sense of W) than a. By applying the inductive hypothesis we obtain $V_W \in \boldsymbol{D}\mathcal{M} = \mathcal{M}$. For the successor ordinal case, we use the following lemma:

Lemma 5. *Assume that* $\mathcal{M} \preccurlyeq^1_n \mathfrak{I}N$, W *is a well-ordering in* $\mathcal{M}$, *and* V_W *is in* $\mathcal{M}$. *Let* W^+ *be obtained from* W *by adding one new largest point to the ordering. Then* V_{W^+} *is explicity definable over* $\mathcal{M}$ *from* W.

Proof. Since V_W is definable over $\mathcal{M}$ from W (as the unique solution to τ_W), it suffices to show that V_{W^+} is definable from W and V_W. Let m be the new largest point in W^+. Then V_{W^+} is the union of V_W and the set of true W^+-sentences which contain m. The idea of the proof is that because $\mathcal{M} \preccurlyeq^1_n \mathfrak{I}N$, the definition of true sentences containing m performs correctly in $\mathcal{M}$. Consider then a W^+-sentence θ containing m.

Case 1. θ does not contain set terms of the form $\Delta x X_1 \ldots X_n \varphi\psi$ with $\forall_m$ in φ or ψ. By the usual "truth is hyperarithmetical" argument, the set of true sentences of this form is Δ^1_1 in W and V_W. So the set is in $\mathcal{M}$ and is definable in $\mathcal{M}$ from W and V_W.

Case 2. θ is $\Delta x X_1 \ldots X_n \varphi\psi t$, where φ and ψ are as in Case 1. We need to formalize clause 4 of the definition of truth, with the set quantifiers relativized to $\mathcal{M}$. The quantifier on A can be restricted to $\mathcal{M}$, since the only possible solution for A is a set Δ^1_n in the denotations of the set terms in φ and ψ (and hence Δ^1_n in V_W). The B_i quantifiers can be restricted to $\mathcal{M}$ because $\mathcal{M} \preccurlyeq^1_n \mathfrak{I}N$. And then truth of φ and ψ is definable as in Case 1.

Case 3. Other sentences, for example those obtained from Case 2 sentences by numerical quantification or iteration of the Δ operation, are reducible to Case 2 by the transitivity of Δ^1_n-ness. ∎

We now can see the full details necessary to establish the claim made in the proof of Theorem 3. Let $\mathcal{M}$ be a β_n model, and μ the least ordinal not represented in $\mathcal{M}$. The claim is that $\mathcal{F}^{\mathcal{M}} = \mathcal{F}_\mu$. The definition of $\mathcal{F}$ in analysis is:

> $A \in \mathcal{F}$ iff there is a well-ordering W and a truth set V such that $\tau_W(V)$, and for some closed W-term T, a natural number n belongs to A iff $T\boldsymbol{n} \in V$.

When the W quantifier is restricted to $\mathcal{M}$ we obtain $\mathcal{F}_\mu$, and nothing is lost when the V quantifier is also restricted to $\mathcal{M}$.

In order to prove the minimality of $\mathcal{F}$, it will be helpful to know that over $\mathcal{F}_\lambda$ we can define an ordering of type λ, for λ less than the ordinal of closure. Our strategy is to take the least ordinal for which this fails, and to show that closure has occurred by that ordinal.

LEMMA 6. *For each α less than the ordinal of closure, there is a well-ordering of type α in $\boldsymbol{D}\mathcal{F}_\alpha$. For any such well-ordering W in $\boldsymbol{D}\mathcal{F}_\alpha$, we have V_W in $\boldsymbol{D}\mathcal{F}_{\alpha+1}$. (We assume here that the basis property stated in Theorem 4 holds.)*

Proof. Let γ be the least ordinal such that in $\boldsymbol{D}\mathcal{F}_\gamma$, there is no well-ordering of type γ. We will show that $\mathcal{F}_\gamma$ satisfies comprehension, whence γ is at least as large as the ordinal of closure. (It then follows that equality holds. If $\mathcal{F}$ contained a well-ordering W of order type greater than the ordinal of closure, we could diagonalize to construct a set in $\boldsymbol{D}\mathcal{F} - \mathcal{F}$.)

For any $\alpha < \gamma$ then, we have some ordering W in $\boldsymbol{D}\mathcal{F}_\alpha$ of type α. We first show that for any ordering W of type α in $\mathcal{F}_{\alpha+1}$, we have V_W in $\boldsymbol{D}\mathcal{F}_{\alpha+1}$ (where $\alpha < \gamma$).

Case 1. $\alpha = \beta+1$ and the ordering in question is W^+ where $W \in \mathcal{F}_\alpha = \mathcal{F}_{\beta+1}$. Then by inductive hypothesis $V_W \in \boldsymbol{D}\mathcal{F}_{\beta+1}$. Apply Lemma 5 with $\mathcal{M} = \mathcal{F}_{\beta+2}$ to obtain $V_{W^+} \in \boldsymbol{D}\mathcal{F}_{\beta+2} = \boldsymbol{D}\mathcal{F}_{\alpha+1}$.

Case 2. $\alpha = \beta+1$ and the ordering in question is W^+ (in $\mathcal{F}_{\alpha+1}$) but $W \notin \mathcal{F}_{\beta+1}$. We have in $\mathcal{F}_{\beta+1}$ another ordering U of type β. There is a unique isomorphism between W^+ and U^+. The isomorphism is implicitly arithmetically definable from the orderings, and so also belongs to $\mathcal{F}_{\beta+2}$. And V_{W^+} is recursive in V_{U^+} and the isomorphism (since the isomorphism induces a truth preserving map from W^+-sentences to U^+-sentences). By Case 1, V_{U^+} is in $\boldsymbol{D}\mathcal{F}_{\beta+2}$, and V_{W^+} must also be in this class.

Case 3. α is a limit ordinal. For any segment $W \upharpoonright a$ of the ordering W in $\mathcal{F}_{\alpha+1}$, there is a well-ordering U in $\mathcal{F}_{|a|_W+1}$ with V_U in $\mathcal{F}_{|a|_W+2}$. Thus in $\mathcal{F}_{\alpha+1}$ we have: $W \upharpoonright a$, U, the isomorphism between $W \upharpoonright a$ and U, V_U, and hence $V_{W\upharpoonright a}$. Then

$$f \in V_W \iff \exists a \exists V[\tau_{W\upharpoonright a}(V) \ \& \ f \in V]$$

and the V quantifier can be restricted to $\mathcal{F}_{\alpha+1}$. Thus V_W is in $\boldsymbol{D}\mathcal{F}_{\alpha+1}$.

To prove Lemma 6, it remains to show that $\mathcal{F}_\gamma$ satisfies comprehension. For that we use Lemma 7, below, with $\delta = \gamma$ and $\mathcal{A}_\alpha = \mathcal{F}_\alpha$. We must verify that the hypotheses of that lemma are satisfied. Clearly γ is a limit ordinal and hypotheses (1) and (2) are met. Hypothesis (3) holds, since for any α, $\mathcal{F}_\alpha$ is closed under Δ_n^1-ness. Hypothesis (4) follows at once from the definition of γ.

As for hypothesis (5), recall that a set A belongs to $\mathcal{F}_{|W|}$ iff it is of the form T^* where all set quantifier symbols in T are subscripted by ordinals less than $|W|$. Thus we can let $\varepsilon(i, x, W)$ be: "i is a W-term beginning with Δ and for some V for which $\tau_W(V)$, the sentence $\boldsymbol{ix}$ is in V".

Finally for hypothesis (6) we need a definable well-ordering of $\mathcal{F}_\gamma$. First well-order the closed set terms, ordering first according to the largest ordinal subscript, then by length, and then lexicographically. (Actually any reasonable well-ordering could be employed here.) Then define:

> $A < B \iff A$ is denoted by some closed term which is smaller than any closed term denoting B.

This relation well-orders $\mathcal{F}$; we claim that on $\mathcal{F}_\gamma$ it is definable over $\mathcal{F}_\gamma$. This is because:

> $A < B \iff$ There is a well-ordering W and a set V such that $\tau_W(V)$ and a W-term t_a denoting A such that for any W-term t_b denoting B we have $\langle t_a, t_b\rangle \in L_W$,

where L_W is the well-ordering (arithmetical in W) induced on W-terms by our ordering on terms. Here "t denotes A" means "$\forall n(n \in A \iff \boldsymbol{tn} \in V)$". The existential quantifier on W can be restricted to $\mathcal{F}_\gamma$ (for A in $\mathcal{F}_\gamma$). Furthermore the existential quantifier on V can be restricted to $\mathcal{F}_\gamma$, as observed above. This completes the proof of Lemma 6, except for verifying Lemma 7.

LEMMA 7. *Let δ be a limit ordinal, and assume that for $\alpha \leqslant \delta$ we have classes $\mathcal{A}_\alpha$ of sets such that the following conditions hold:*

1. *For $\alpha < \beta \leqslant \delta$, $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \subseteq \mathfrak{P}N$. For limit ordinals $\lambda \leqslant \delta$, $\mathcal{A}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$.*
2. *$\boldsymbol{D}\mathcal{A}_\alpha \subseteq \mathcal{A}_{\alpha+1}$, for $\alpha < \delta$.*
3. *$\mathcal{A}_\delta \prec_1^1 \mathfrak{P}N$.*
4. *Any ordinal $\alpha < \delta$ is represented in $\mathcal{A}_\delta$. Any ordinal represented in $\boldsymbol{D}\mathcal{A}_\delta$ is strictly less than δ.*
5. *There is a formula ε which, given a well-ordering W, defines over $\mathcal{A}_\delta$ an enumeration of $\mathcal{A}_{|W|}$. That is,*

$$\mathcal{A}_{|W|} = \{\{x\colon \models_{\mathcal{A}_\delta} \varepsilon[i, x, W]\}\colon i \in N\}.$$

for a well-ordering W.

6. *There is a well-ordering of* $\mathcal{A}_\delta$ *definable over* $\mathcal{A}_\delta$.

Then $\mathcal{A}_\delta$ *satisfies comprehension.*

Proof. We will define, by recursion on the prenex formula φ, a function f_φ mapping δ into δ such that:

(i) f_φ is non-decreasing and continuous, and $\alpha \leqslant f_\varphi(\alpha)$ for $\alpha < \delta$.

(ii) For any $\alpha < \delta$, any string $\vec{B}$ of sets from $\mathcal{A}_\alpha$, and string $\vec{n}$ of natural numbers

$$\models_{\mathcal{A}_{f_\varphi(\alpha)}} \varphi[\vec{n}, \vec{B}] \Longleftrightarrow \models_{\mathcal{A}_\delta} \varphi[\vec{n}, \vec{B}] .$$

(iii) f_φ is definable over $\mathcal{A}_\delta$ in the sense that the relation which holds between A and B iff both are well-orderings in $\mathcal{A}_\delta$ and $f_\varphi(|A|) = |B|$, is definable over $\mathcal{A}_\delta$.

Once we have such functions, it easy to see that the comprehension axioms are satisfied. For

$$\{n\colon \models_{\mathcal{A}_\delta} \varphi[n, \vec{B}]\} \in \boldsymbol{D}\mathcal{A}_{f_\varphi(\alpha)} \subseteq \mathcal{A}_\delta$$

by using assumption 2.

For arithmetical (i.e., elementary) formulas φ we take f_φ to be the identity function on δ. This is definable over $\mathcal{A}_\delta$ by assumption 3. For the negation $\neg\varphi$ of φ we simply take $f_{\neg\varphi} = f_\varphi$. The only other case is that of the quantified formula $\exists X\varphi$. Here we simply take:

$f_{\exists X\varphi}(\alpha) =$ the least $\beta \geqslant \alpha$ such that for any string $\vec{B}$ from $\mathcal{A}_\alpha$, any $\vec{n}$,

$$(*) \qquad \models_{\mathcal{A}_\beta} \exists X\varphi[\vec{n}, \vec{B}] \Longleftrightarrow \models_{\mathcal{A}_\delta} \exists X\varphi[\vec{n}, \vec{B}] .$$

There are such β's, e.g. $\beta = \delta$. But we will first show that (for fixed $\vec{n}, \vec{B}$), we can find a β satisfying $(*)$ which is definable over $\mathcal{A}_\delta$ and hence is less than δ. (Later the dependence on $\vec{n}$ and $\vec{B}$ will be eliminated.)

Case A. Suppose $\models_{\mathcal{A}_\delta} \exists X\varphi[\vec{n}, \vec{B}]$. Thus for some C in $\mathcal{A}_\delta = \bigcup_{\nu<\delta} \mathcal{A}_\nu$,

$$\models_{\mathcal{A}_\delta} \varphi[\vec{n}, \vec{B}, C] .$$

Choose such a C in some $\mathcal{A}_\nu$; by assumption 1 we may suppose $\nu \geqslant \alpha$. Then we may simply take $\beta = f_\varphi(\nu)$, for

$$\models_{\mathcal{A}_{f_\varphi(\nu)}} \varphi[\vec{n}, \vec{B}, C]$$

and hence

$$\models_{\mathcal{A}_{f_\varphi(\nu)}} \exists X\varphi[\vec{n}, \vec{B}] .$$

Case B. Suppose on the other hand

$$\text{not} \models_{\mathcal{A}_\delta} \exists X\varphi[\vec{n}, \vec{B}] .$$

Then let $\beta = \lim_{k\in\omega} f_\varphi^k(\alpha)$, where $f_\varphi^0(\alpha) = \alpha$ and $f_\varphi^{k+1}(\alpha) = f_\varphi(f_\varphi^k(\alpha))$. Assuming for the moment that $\beta < \delta$, we then have $f_\varphi(\beta) = \beta$. (In any case $\beta \leqslant f(\beta)$; the other inequality holds since f_φ is non-decreasing and continuous.) For any set C in $\mathcal{A}_\beta$, from the fact that $\varphi[\vec{n}, \vec{B}, C]$ is false in $\mathcal{A}_\delta$ we conclude that it is false in $\mathcal{A}_{f_\varphi(\beta)}$, i.e., $\mathcal{A}_\beta$.·. $\exists X\varphi[\vec{n}, \vec{B}]$ is false in $\mathcal{A}_\beta$, as desired. To complete this argument we must verify that in fact $\beta < \delta$. The idea is that we can define over $\mathcal{A}_\delta$ a well-ordering of type β. We begin with an ordering W_0 of type α, assured by assumption 4. Then we use the definability of f_φ. Say W_k is the kth ordering iff for some chain $(W_0, \ldots, W_k)$ we have W_{i+1} equal to the least set in $\mathcal{A}_\delta$ (in the ordering of assumption 6) such that $f_\varphi(|W_i|) = |W_{i+1}|$. Then an ordering $\prec$ for which

$\langle a, i\rangle \prec \langle b, j\rangle$ iff $i \leqslant j$ and $a \in$ the ith ordering and $b \in$ the ith ordering and $(i = j \Rightarrow a < b$ there)

is definable over $\mathcal{A}_\delta$ from W_0 and has length at least β. This concludes the case B argument.

But to obtain $f_{\exists X\varphi}(\alpha) < \delta$ we still need a second fact: There is some $\beta < \delta$ which satisfies $(*)$ simultaneously for all $\vec{n}$ and all $\vec{B}$ in $\mathcal{A}_\alpha$. Observe that $(*)$ is a definable condition on β. That is, the condition on $\vec{n}$, $\vec{B}$, and a well-ordering W that

$$\models_{\mathcal{A}_{|W|}} \exists X\varphi[\vec{n}, \vec{B}] \Longleftrightarrow \models_{\mathcal{A}_\delta} \exists X\varphi[\vec{n}, \vec{B}]$$

hold, is a definable condition over $\mathcal{A}_\delta$. For by assumption 5 we can define from W a set of integers encoding $\mathcal{A}_{|W|}$. Then by the "truth is hyperarithmetical" argument and assumption 3, we can define truth in $\mathcal{A}_{|W|}$.

We now proceed to manufacture a uniform β. Begin with a fixed ordering W of type α. By assumption 5 we can define from W an (integer-indexed) enumeration of the k-tuples of sets in $\mathcal{A}_\alpha$. Then for each $\vec{n}$ and $\vec{B}$ we can take the least ordering V such that

$$\models_{\mathcal{A}_{|V|}} \exists X\varphi[\vec{n}, \vec{B}] \Longleftrightarrow \models_{\mathcal{A}_\delta} \exists X\varphi[\vec{n}, \vec{B}]$$

as in the preceding paragraph. Again we string these orderings together to obtain a definable ordering longer than any one. Its order type is less than δ and is the uniform β desired.

Finally we must verify that $f_{\exists X\varphi}$ meets conditions (i)–(iii). That $f_{\exists X\varphi}$ is non-decreasing follows from its definition and the fact that the classes $\mathcal{A}_\alpha$ are non-decreasing as α increases. Its continuity similarly follows from the fact that $\mathcal{A}_\lambda$ equals $\bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$ for a limit ordinal λ. And $f_{\exists X\varphi}(\alpha) \geqslant \alpha$

by definition. Condition (ii) is obviously satisfied. For condition (iii) we must show that $f_{\exists X\varphi}$ is definable over $\mathcal{A}_\delta$. We have

$f_{\exists X\varphi}(|A|) = |C|$ iff C is a well-ordering of the lowest order type such that $|A| \leqslant |C|$ and for any string $\vec{B}$ of sets from $\mathcal{A}_{|A|}$ and any $\vec{n}$

$$\models_{\mathcal{A}_{|C|}} \exists X\varphi[\vec{n}, \vec{B}] \Longleftrightarrow \models_{\mathcal{A}_\delta} \exists X\varphi[\vec{n}, \vec{B}] .$$

By using (primarily) assumption 5, this condition on A and C (in $\mathcal{A}_\delta$) is definable over $\mathcal{A}_\delta$. ■

Finally we are able to conclude that $\mathcal{F}$ is included in any other β_n model $\mathcal{M}$. Let μ be the least ordinal not represented in $\mathcal{M}$. As explained above, $\mathcal{F}^{\mathcal{M}} = \mathcal{F}_\mu$. If μ were less than the ordinal of closure, there would be an ordering of type μ definable over $\mathcal{F}_\mu$ (by Lemma 6). But since $\mathcal{F}_\mu$ is a definable class in $\mathcal{M}$, the ordering would be definable over $\mathcal{M}$, and hence in $\mathcal{M}$. This completes the proof of Theorem 4. ■

THEOREM 8. *Continue to assume that for any A, the class of sets Δ_n^1 in A forms a basis for the classes Σ_n^1 in A. Then for any set in $\mathcal{F}$, there is a formula (without set parameters) which correctly defines that set over any β_n model $\mathcal{M}$.*

Proof. Consider the set $A \in \mathcal{F}$. It suffices to show that A is definable over $\mathcal{F}$. For $\mathcal{F} = \mathcal{F}^{\mathcal{M}}$ is a definable class over $\mathcal{M}$.

The set A is definable over $\mathcal{F}$ from the set V_W for all sufficiently large well-orderings W. And V_W is definable over $\mathcal{F}$ from W. So it suffices to show that for every ordinal less than the ordinal of closure, there is a well-ordering of that type definable over $\mathcal{F}$.

Let λ be the least ordinal not represented by a well-ordering definable in $\mathcal{F}$. We claim that $\mathcal{F}_\lambda \precsim \mathcal{F}$. For suppose $A_1, \ldots, A_n$ are in $\mathcal{F}_\lambda$ and

$$\models_{\mathcal{F}} \exists X\psi[A_1, \ldots, A_n] .$$

Then the least (in our definable ordering of $\mathcal{F}$) well-ordering W such that for some B in $\mathcal{F}_{|W|}$,

$$\models_{\mathcal{F}} \psi[A_1, \ldots, A_n, B]$$

is definable in $\mathcal{F}$ from $A_1, \ldots, A_n$. But since A_i is in $\mathcal{F}_\lambda$, A_i is definable; so the W above is definable. Hence $|W| < \lambda$ and there is some B in $\mathcal{F}_\lambda$ which works. Hence $\mathcal{F}_\lambda \precsim \mathcal{F}$.

Consequently $\mathcal{F}_\lambda$ satisfies comprehension and so equals $\mathcal{F}$. So λ is the ordinal of closure. ■

The conclusion of this theorem can also be stated: A set is strongly representable (binumerable) in the theory of β_n models iff it belongs to $\mathcal{F}$.

§ 4. Further comments. In the preceding section n was a fixed number greater than one. If the basis hypothesis used there (that for a set A, the class of sets Δ_n^1 in A forms a basis for classes Σ_n^1 in A) holds for infinitely many values of n, then the class $\mathcal{A}$ of analytical sets is a basis for any analytical class. In this case (in fact equivalently) we have $\mathcal{A} \precsim \mathfrak{I}N$ since for analytical $\vec{B}$,

$$\exists A \models_{\mathfrak{I}N} \theta[A, \vec{B}] \Rightarrow (\exists A \in \mathcal{A}) \models_{\mathfrak{I}N} \theta[A, \vec{B}] .$$

And it is clear (without basis assumptions) that $\mathcal{A}$ must be included in any elementary submodel of $\mathfrak{I}N$. Thus we have the simple result:

THEOREM 9. *If the class $\mathcal{A}$ of analytical sets is a basis for analytical classes, then $\mathcal{A}$ is the smallest elementary submodel of $\mathfrak{I}N$.*

This is the analog to Theorem 4 for $n = \omega$, but its proof is vastly simpler. Also it is obvious for this class that if $\mathcal{A} \precsim \mathfrak{I}N$ then each element of $\mathcal{A}$ is definable (without set parameters) over $\mathcal{A}$.

The basis assumption of section 3 is a well-known consequence of the axiom of constructibility [1]. On the other hand, it has been shown that projective determinateness implies that for odd values of n, Δ_n^1 is not a basis for Σ_n^1; ([5], [2]). Martin and Solovay have conjectured that projective determinateness implies that the basis hypothesis does hold for even n; see [5], p. 156. In this event our results would at least hold for n even and $n = \omega$.

If we turn from truth to consistency, we have the following result by Silver (see [12]; cf. also Martin and Solovay [5]): If "ZF + there is a measurable cardinal" is consistent, then it remains consistent with the additional axioms:

1. $\Delta_3^{1,A}$ is a basis for $\Sigma_3^{1,A}$ for each set $A \subseteq N$.
2. There is a Δ_3^1 well-ordering of $\mathfrak{I}N$ in the order type of the least uncountable ordinal.

The second of these implies that our basis hypothesis holds for all $n > 3$ (cf. [1], pp. 350–351).

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