Elementary arcs in 3-space
that can be realized by squeezing 3-cells

by

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1. Introduction. If $C$ is a 3-cell in Euclidean 3-space $\mathbb{E}^3$, then it follows from [7, Theorem 3] that there exists a map $s$ of $\mathbb{E}^3$ onto itself squeezing $C$ to an arc. This paper deals with the converse question asking whether each arc in $\mathbb{E}^3$ can be realized by squeezing some 3-cell. The main result here, Theorem 1, indicates that if $A$ is such an arc which is locally tame modulo one point and which satisfies a finiteness condition at this point, then $A$ lies in the boundary of a disk in $\mathbb{E}^3$. Consequently, many almost tame arcs in $\mathbb{E}^3$ cannot be realized by squeezing 3-cells.

Let $A_4$ denote the set $\{(x, y, z) \in \mathbb{E}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ and $A_1$ the set $\{(x, 0, 0) \in A_4 \mid 1 < x \leq 1\}$. Define $\pi$ as the projection of $A_4$ onto $A_1$, sending $(x, y, z)$ to $(x, 0, 0)$.

Let $C$ be a 3-cell in the interior of a 3-manifold $M$. A map $s$ of $M$ onto itself is said to squeeze $C$ to an arc $A$ if and only if there exist homeomorphisms $h_0$ of $A_4$ onto $C$ and $h_1$ of $A_1$ onto $A = s(C)$ such that $h_0 = h_1 \pi$ and $s$ takes $M - C$ homeomorphically onto $M - s(C)$. In addition, we say that an arc $A$ in the interior of $M$ can be realized by squeezing a 3-cell if and only if there exists a 3-cell $C$ in $M$ and a map $s$ squeezing $C$ to $A$.

Although we could define similarly the concept of a map of $\mathbb{E}^3$ to itself squeezing a disk onto an arc, it would not enlarge the context of the question at hand, because any disk in the interior of $M$ can be thickened to a 3-cell that flattens back onto the same disk [7, Theorem 9]. Thus, an arc in the interior of $M$ can be realized by squeezing a 3-cell if it can be realized by squeezing a disk.

Let $A$ be an arc in $\mathbb{E}^3$ locally tame modulo an interior point $p$. The local enveloping genus of $A$ at $p$, written $\text{LEG}(A, p)$, is the smallest non-negative integer $r$ such that there exist arbitrarily small neighborhoods of $p$, each of which is bounded by a sphere with $r$ handles that intersects $A$ in precisely two points. If no such integer $r$ exists, we write

* Research supported in part by NSF Grant GP-8888.
LEG(A, p) = ∞. In the special case that LEG(A, p) = 0, we say that A is locally peripherally unknotted at p.

An arc A in B^3 is locally unknotted at a point p \in A if and only if there exist a disk D in B^3 and a neighborhood N of p such that N \cap A is contained in the boundary of D.

L. V. Keldysh [13] has shown that for any locally unknotted arc A in B^3 there exists a pseudo-isotopy of B^3 translating a straight line segment homeomorphically onto A. The property considered in this paper is almost as strong, for if A is an arc in B^3 that can be realized by squeezing a 3-cell, it follows from [16, Lemma 6] and [19] that there exists a pseudo-isotopy of B^3 (not necessarily satisfying the conditions of [13]) translating the 3-cell onto A. From this one can easily construct a pseudo-isotopy translating a straight line segment homeomorphically onto A. Loosely speaking, then, the results given in Section 5 represent converses, in severely limited cases, to Keldysh’s result.

If C is an n-cell, then Int(C) and Bd C denote the interior and boundary of C, respectively. The symbol C denotes the topological closure operator.

The attack on Theorem 1 involves three distinct procedures. First, we show that any 3-cell which squeezes to an almost tame arc can be altered so that the 3-cell is locally tame modulo the preimage of the wild point of the arc. Secondly, we show that the finiteness of the local enveloping genus implies that this preimage contains at most one troublesome point. Thirdly, beginning with a singular disk attached to the 3-cell, we enact the map approximations and disk tradings typical of 3-space topology to define a set whose image under the squeezing map is the desired disk.

2. Improving the 3-cell.

Lemma 1. Let C be a 3-cell in B^3, s a map of B^3 onto itself squeezing C to an arc that is locally tame modulo its endpoints e1 and e2, and A an open subset of B^3 containing C \times [0, 1] \cup \{e_1, e_2\}. Suppose gA is a homeomorphism of A onto a 3-cell C' in s[A] \cup e_1 \cup e_2 \cup Int C such that for each x \in Int C there exists a \in Int C, satisfying gA|\{a\} = s(x) \cap C. Then there exists a map s' squeezing C' onto s(U) and satisfying s'(C') \cup (B^3 - U) = s(U) \cup (B^3 - U).

Proof. Let K be a 3-cell in s(U) obtained by thickening s(U); that is, K is obtained so that there exists a homeomorphism of (K, s(U)) onto (K, s(U)).

Since s^{-1}Bd K is 1-1, s^{-1}(Bd K) must be a 2-sphere. Applying 3, Theorem 1, we find a homeomorphism G of B^3 onto itself taking s^{-1}(K) onto K and satisfying G|s^{-1}(B^3 - K) = s|s^{-1}(B^3 - K). For simplicity we suppress the homeomorphism G and assume that K = s^{-1}(K) and s|B^3 - K = identity.

It follows from (9, 12) that Bd C' is locally tame modulo (e1, e2).

For applications of this section this argument can be avoided by hypothesizing that Bd C' be locally tame modulo (e1, e2) Thus, it is easy to construct a map f of B^3 onto itself such that

1. f(s) \subset C',
2. f|B^3 - K = identity,
3. there exists a homeomorphism gA of A onto C' such that gA|\{a\} = f|a.

Note that the arc f(C') is locally tame modulo its endpoints e1 and e2 [5, Theorem 1]. The crucial fact to be established is that the pair (K, f(C')) is homeomorphic to the standard pair (A_1, A_2).

In this paragraph we prove that \pi_1(K - f(C')) is infinite cyclic. Note that \pi_1(K - C) is infinite cyclic; therefore, a loop in K - C is null homotopic if and only if it is null homologous. First, let L be a loop in K - C that is null homotopic in K - C'. It is a simple matter to show that L is null homotopic in K - Int C, and it then follows from (4, Theorem 3) that L is homologous in K - C to the sum of finitely many small loops near Bd C. Since C is a cell, these small loops can be obtained to be homologously trivial in K - C. As a consequence, L is both null homologous and null homotopic in K - C. Secondly, let L be a loop in K - C. Then L can be deformed in K - C' to a loop in K - Int C, and afterwards can be pushed off Bd C to a loop in K - C. These two properties imply that \pi_1(K - C) is isomorphic to \pi_1(K - C'). But K - f(C') and K - C are homeomorphic, so \pi_1(K - f(C')) is infinite cyclic.

Now we show that f(C') can be straightened in K at its endpoints. Let H be an embedding of K in the 3-sphere S^3 such that the closure K of K - H(K) is a 3-cell. Extend H|f(C') to a simple closed curve J such that J \cap K is a tame, unknotted spanning arc of K'. Using an argument like the one given in the preceding paragraph, we find that S^3 - J has uniformly abelian local fundamental groups at the points J \cap Bd K'.

Hence, J is tame [14, Corollary to Theorem 1].

Finally, it follows from [11, Theorem 2] that J \cup Bd K' is tame. In particular, H|f(C') \cup Bd K' is tame. As a consequence, the pair (K, f(C')) is homeomorphic to (A_1, A_2), and therefore, (K, f(C')) is homeomorphic to (K, s(U)). It is now a simple matter to obtain a homeomorphism f' of B^3 onto itself such that

1. f'|B^3 - K = identity,
2. f'|f(C') = id(U),
3. f|f(C') = s|U.

The required map s' is defined as s' = f|f(C').
LEMMA 2. Let $C$ be a 3-cell in $E^3$ and $s$ a map of $E^3$ onto itself squeezing $C$ to an arc $s(C)$ that is locally tame modulo an interior point $p$. Then there exists a 3-cell $C'$ contained in $C$ and there exists a map $t$ of $E^3$ onto itself squeezing $C'$ onto $s(C)$ such that $C' \cap s(C) = t^{-1}(p)$.

Proof. Let $h_i$ denote the homeomorphism of $\partial C_i$ onto $C_i$ and $h_i$ the homeomorphism of $A_i$ onto $s(A_i)$ such that $h_i \circ s_i \circ h_i^{-1}$. Define a homeomorphism $g_0$ of $A_0$ into $h_0(A_0 \cap \text{Int} A_0) \cup s^{-1}(p)$ such that $g_0(A_0)$ contains $s^{-1}(p) \cup h_0(A_0)$. Let $g_0 \pi = g_0$. Since the disk $s^{-1}(p)$ is cellular [16, Lemma 6], there exists a map $v$ of $E^3$ onto itself whose only inverse set is $s^{-1}(p)$. The map $s^{-1} \circ v$ transforms each of the two maximal 3-cells in $s^{-1}(A_0)$ onto a subarc of $s(C)$ having $p$ as an endpoint. Hence, it follows from two applications of Lemma 1 that there exists a map $v'$ squeezing the two arcs $s^{-1}(A_0)$ onto $s(C)$. Define $t$ as $v'$. The 3-cell $C' = g_0(A_0)$ is locally tame modulo $g_0(Bd A_0) \cup s^{-1}(p)$ (see the proof of Lemma 1). By hypothesis, however, $E^3 - s(C)$ is 1-loc at each endpoint of $s(C)$, so both $E^3 - C'$ and $E^3 - C$ are 1-loc at the points $h_0(Bd A_0) = g_0(Bd A_0)$. Consequently, Theorem 6 of [3] implies that $Bd C'$ is locally tame modulo $Bd C' \cap t^{-1}(p)$.

3. Isolating the bad point. It would be interesting to know whether the hypothesis $\beta G(s(C)) < \infty$ in the following lemma is truly necessary. If not, the corresponding hypothesis about the local enveloping genus could be eliminated from each of the results stated in Section 5.

LEMMA 3. Suppose $C$ is a 3-cell in $E^3$, $s$ a map of $E^3$ onto itself squeezing $C$ to an arc, and $p$ an interior point of $s(C)$ such that $Bd C$ is locally tame modulo the simple closed curve $J = Bd C \cap s^{-1}(p)$ and $\beta G(s(C)) < \infty$. Then there exists a point $b \in J$ such that for each neighborhood $V$ of $J$, there exists an open set $\mathbb{V}$ containing $J - \{b\}$ such that every loop in $V - C$ is null homotopic in $V - C$.

Proof. The argument parallels that of [6, Theorems 2 and 3]. Let $W$ be a neighborhood of $p$. By hypothesis there exists a sphere with $n$ handles $H$ such that $H$ separates $p$ from $W$ and $H$ intersects $s(C)$ in two points $x_1$ and $x_2$. Thus, the set $s^{-1}(H)$ intersects $C$ in the disks $s^{-1}(C_1)$ and $s^{-1}(C_2)$. Note that $s^{-1}(H)$ may fail to be a 2-manifold at points of $Bd s^{-1}(C)$ ($i = 1, 2$).

Because $s(C)$ is locally tame away from $p$, we can perform isotopies only moving points near $s(C)$ to show that if $A$ is a subarc of $\text{Int} s(C)$ containing $p$ in its interior and $W$ is a neighborhood of $s^{-1}(A)$, then there exists a sphere with $n$ handles $H$ such that $H \cap s(C) = Bd A$ and $W \supset s^{-1}(H)$.

Using the above we can reapply the techniques of [6, Theorem 2], in spite of the fact that the sets $s^{-1}(H)$ are not exactly closed 2-manifolds, to prove that there exists a finite set $Q$ in $J$ such that, for each neighborhood $U$ of $J$, each point of $J - Q$ has a neighborhood $N$ such that every loop in $N - C$ is null homotopic in $U - C$. The preimages under $s$ of these spheres with handles provide the means for shrinking near $J$, without allowing the image to stretch out towards $s^{-1}(Bd s(C))$.

Let $U$ be a neighborhood of $J$. Since $s^{-1}(p)$ is cellular [16, Lemma 6], $C$ is cellular [19]; hence, $J$ contains at most one point $b$ such that $Bd C$ fails to be pierced by a tame arc at $b$ [15, Theorem 2]. Find a neighborhood $V$ of $J - \{b\}$ such that each loop in $V - C$ is null homotopic in $V - \{b\} \cup \text{Int} C$. With this, the argument of [6, Theorem 3] can be applied to finish the proof.

Remark. If the arc $s(C)$ of Lemma 3 is locally peripherally unknotted, then one can show by appealing to [6, Theorem 6] that the exceptional point $b$ can be deposed. Accordingly, the conclusion of the lemma must be changed to read: Then, for each neighborhood $U$ of $J$, each point of $J$ has a neighborhood $N$ such that each loop in $N - C$ is null homotopic in $V - C$.

4. Obtaining the disk. The proof of the following lemma could be shortened considerably if the simple closed curve $J$ pierced a disk at some point, for in that case we could readily produce a disk $D$ attached to $C$ such that the image $s(D)$ has the desired properties.

LEMMA 4. Suppose $C$ is a 3-cell in $E^3$, $s$ a map of $E^3$ onto itself squeezing $C$ to an arc, and $p$ an interior point of $s(C)$ such that $Bd C$ is locally tame modulo the simple closed curve $J = Bd C \cap s^{-1}(p)$ and $\beta G(s(C)) < \infty$. Suppose further that there exists an interval $T$ in $J$ such that for each neighborhood $U$ of $J$, there exists a neighborhood $V$ of $T$ such that every loop in $V - C$ is null homotopic in $V - C$. Then $s(C)$ lies in the boundary of a disk in $E^3$.

Proof. Let $W_1, W_2, \ldots$ be a decreasing sequence of open sets whose intersection equals $J$. Let $B$ denote an (abstract) disk, $E$ a subarc of $Bd B$, and $E'$ the closure of $Bd B - E$.

Use Theorem 6.3 of [4] to obtain a map $f$ of $B$ into $E^3$ such that:

1. $f(B)$ is a homeomorphism of $B$ onto $s(C)$,
2. $f(B - E') \subset E^3 - C$,
3. $f^{-1}([T \subset f(B)])$ is a 0-dimensional subset $X$ of $B$, $X$ is one-one.

It follows from standard methods in plane topology that there exists a null sequence of disks $D_1, D_2, \ldots$ in $Bd B$ such that:

1. $Bd D_1 \cap X = \emptyset$,
2. $f(D_1) \subset W_1$,
3. $D_i \cap D_{i+1}$ is a subarc of $Bd D_i \cap Bd D_{i+1}$.
(8) \( D_1 \cap D_1 = \emptyset \) if \(|i - j| > 1\),
(9) \( (X \cap R) \cup (\cup D_i) \) is a disk,
(10) \( X \subset \operatorname{Cl}(\cup D_i) \).

Extend the disk \( D_1 \) so that \( D_1 \cap \operatorname{Bd} R \) is a subarc of \( \operatorname{Bd} R - R \). Define \( F' = \operatorname{Cl}(R - \cup D_i) \). Then \( F' \) is the union of two disks \( F_1 \) and \( F_2 \) whose only intersection is the point \( f(X \cap R') \subset T \).

Now we begin to replace the sets \( f(D_i) \) with disks. Let \( U_i \) be a neighborhood of \( T \) such that each simple closed curve in \( U_1 \cap F_1 \) bounds a disk in \( W_1 \cap F_1 \) \((i = 1, 2)\). According to the hypothesis of this lemma, we can redefine the map \( f \) on very small subdisks of \( D_1 \) containing \( D_1 \cap X \), thereby obtaining a map \( f' : D_1 \rightarrow E' \) such that

(11) \( f'(D_1) \cap (G \cup f(R')) = \emptyset, \)
(12) \( f'(D_1) \subset f(D_1) \cup U_1, \)
(13) \( f' \cap \operatorname{Bd} D_1 = f' \cap \operatorname{Bd} D_1, \)
(14) \( f' \) is a homeomorphism in a neighborhood of \( \operatorname{Bd} D_1 \).

Thus, it follows from Dehn's Lemma [18] that there exists a disk \( E_1 \) such that

(15) \( \operatorname{Bd} E_1 = f' \cap \operatorname{Bd} D_1, \)
(16) \( E_1 \subset f(D_1) \cup U_1, \)
(17) \( \operatorname{Int} E_1 \cap (G \cup f(R')) = \emptyset. \)

The only undesirable property is that \( \operatorname{Int} E_1 \) may meet \( F \), so we adjust \( F \) slightly to produce 1-manifolds of the components of \( \operatorname{Int} E_1 \cap F \), and then we either trade disks or perform isotopies on arcs of intersection, pushing the latter towards the hole between the components of \( F \), to remove all intersections. This leaves us with a disk \( E_1 \) such that

(15') \( \operatorname{Bd} E_1 = f' \cap \operatorname{Bd} D_1, \)
(16') \( E_1 \subset f(D_1) \cup W_1, \)
(17') \( E_1 \cap G = \emptyset, \)
(18) \( E_1 \cap F \subset \operatorname{Bd} E_1 \cap \operatorname{Bd} F'. \)

From Conditions 6 and 17' we see that \( E_1 \) intersects only finitely many of the sets \( f(D_i) \). We could collect the associated \( D_i \)'s in a larger disk, but to prevent notational complications, we simply assume that \( E_1 \cap f(D_1) = \emptyset \) \((i > 2)\). This implies that \( \operatorname{Int} E_1 \cap f(\operatorname{Bd} D_i) = \emptyset \). Adjust \( f \mid \operatorname{Int} D_1 \) slightly so that the components of \( E_1 \cap f(\operatorname{Int} D_i) \) are simple closed curves, and define \( Y_2 \) as the union of the disks of \( Y_2 \) bounded by the preimages of these curves. We redefine \( f \) on \( Y_2 \) so that \( f(D_2) \cap E_1 = \emptyset \), \( f(\operatorname{Bd} D_2 - (X - Y_2)) \) is one-one, and \( f(Y_2) \subset W_1. \)

Let \( U_1 \) be a neighborhood of \( T \) such that each simple closed curve in \( U_1 \cap F_2 \) bounds a disk in \( W_1 \cap F_1 \) \((i = 1, 2)\). Cover the points of \( X \cap (D_1 - Y_2) \) with pairwise disjoint disks \( G_1, ..., G_n \) in \( \operatorname{Int} D_1 - Y_2. \)

According to the hypothesis of this lemma, we can redefine the map \( f \) on very small subsets of \( G_1, ..., G_n \), thereby obtaining a map \( f' : D_1 \rightarrow E' \) such that

(19) \( f'(D_1) \cup G_1 = f(D_1) \cup G_1, \)
(20) \( f'(D_1) \cap (G \cup f(R') \cup f(D_1 - G_1)) = \emptyset, \)
(21) \( f'(G_1) \cap f'(R') = \emptyset, \)
(22) \( f' \cap \operatorname{Bd} G_1 = f' \cap \operatorname{Bd} G_1, \)
(23) \( f' \) is a homeomorphism in a neighborhood of \( \operatorname{Bd} G_1 \) \((i = 1, ..., n)\).

Thus, it follows from Dehn's Lemma [18] that \( f(D_k) \) may be replaced with a homeomorphism \( f' : D_k \rightarrow E' \) such that \( f' \) has the same properties as \( f' \) listed in Conditions 19–23 above.

By removing intersections between \( F \) and \( f(D_k) \) as before, we obtain a disk \( E_k \) such that

(24) \( \operatorname{Bd} E_k = f' \cap \operatorname{Bd} D_k, \)
(25) \( E_k \subset f(D_k) \cup W_1, \)
(26) \( E_k \cap G = \emptyset, \)
(27) \( E_k \cap F \subset \operatorname{Bd} E_k \cap \operatorname{Bd} F, \)
(28) \( E_k \cap f(D_1) \cap (i > 2) \) is contained in the union of subdisks of \( E_k \), each of which is contained in \( W_1. \)

Repeating the procedure outlined in the three preceding paragraphs, making certain to use the disks of Condition 28 in the initial step of each repetition, we obtain disks \( E_k, ..., E_k, ..., \) such that

(29) \( \operatorname{Bd} E_k = f(\operatorname{Bd} D_k), \)
(30) \( E_k \subset f(D_k) \cup W_k, \)
(31) \( E_k \cap G = \emptyset, \)
(32) \( E_k \cap F \subset \operatorname{Bd} E_k \cap \operatorname{Bd} F. \)

To complete the proof, let \( F' = F \cup (\cup E_k) \). Although \( F' \) itself may fail to be compact, the construction guarantees that \( \sigma(F') \) is the desired disk.

5. The main results.

THEOREM 1. Suppose the arc \( A \) in \( E^3 \) can be realised by squeezing a 3-cell \( C \) to locally tame modulo an interior point \( p \), and \( \operatorname{LEG}(A, p) < \infty \). Then \( A \) lies in the boundary of a disk in \( E^3. \)

Proof. By Lemma 2 there exist a 3-cell \( C \) in \( E^3 \) and a map \( f \) squeezing \( C \) to \( A \) such that \( \operatorname{Bd} C \) is locally tame modulo \( \operatorname{Bd} C \cap \gamma(A, p) \). Since \( \operatorname{LEG}(A, p) < \infty \), it follows from Lemmas 3 and 4 that \( A \) lies in the boundary of a disk.

COROLLARY 1. Suppose \( A \) is the union of two arcs \( A_1 \) and \( A_2 \) in \( E^3 \) that intersect only in a common endpoint \( p \), \( A \) is locally tame modulo \( p, \)

A_1 is tame, \( \text{LEG}(A, p) < \infty \), and \( A \) can be realized by squeezing some 3-cell. Then \( A \) is tame.

Proof. Theorem 1 implies that \( A \) is contained in the boundary of a disk \( D \). It follows from [2, Theorem 8] that \( D \) can be amended so that it is locally tame modulo \( p \). Since \( D \) is locally tame modulo the tame arc \( A \), \( A \) is tame [8, Theorem 1], and this implies that \( A \) is tame.

Corollary 2. Let \( A \) be an arc in \( F \) locally tame modulo an interior point \( p \). Then \( A \) is tame if and only if \( A \) can be realized by squeezing a 3-cell and \( A \) is locally peripherally unknotted at \( p \).

Proof. One implication is obvious, and the other is an immediate consequence of Theorem 1 and the characterization of tame arcs given by Theorem VI of [11].

Corollary 3. No Wilder arc [see 10] in \( E^3 \) can be realized by squeezing a 3-cell.

Theorem 2. If the arc \( A \) in \( E^3 \) can be realized by squeezing a 3-cell and \( p \) is an isolated wild point of \( A \) such that \( \text{LEG}(A, p) < \infty \), then \( A \) is locally unknotted at \( p \).

Proof. If \( p \) is an endpoint of \( A \), the theorem is a rather widely known piece of folklore, which we have already presumed in the proof of Lemma 1, and which we do not prove.

If \( p \) is an interior point of \( A \), let \( B \) be a subarc of \( A \) such that \( B \) is locally tame modulo \( p \). With the methods of [1] used to establish Lemma 2, one can find a 3-cell that squeezes onto \( B \), in which case Theorem 2 follows from Theorem 1.

Theorem 3. Suppose the arc \( A \) in \( E^3 \) can be realized by squeezing a 3-cell and \( A \) is locally tame modulo a finite set of points \( p_1, \ldots, p_n \) such that \( \text{LEG}(A, p_i) < \infty \) for those \( p_i \)'s in \( A \). Then \( A \) lies in the boundary of a disk in \( E^3 \).

Proof. Simply piece together those local disks promised by Theorem 2 to determine the required disk.

References


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Regis par la Redaction le 18. 5. 1970