Tree-like matrix rings

by

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The main purpose of the present paper is to call attention to a certain simple module-theoretic method and, as an illustration of its value, to derive a representation of (left) torsion-free (semi) uniserial rings (with unity) of finite length and to give a full characterization and a "canonic" form of them in terms of so-called decorated (finite rooted) trees. These results (cf. also R. R. Colby and E. A. Ruther, Jr [1]) generalize Goldie's theorem on block-triangular matrix rings in [5] which is, in turn, a generalization of the Wedderburn–Artin structure theorem. Thus, our approach offers also a very lucid proof of the latter classical result. Another important aspect of our method is the fact that its application can easily be extended to more general classes of torsion-free rings (cf. [3]) to which the methods of [1] or [5] (restricted by the condition that components of the rings contain unique minimal ideals) cannot be applied. Moreover, as a consequence of our results, we get a complete description of indecomposable injective modules over these rings.

1. Preliminaries. Throughout the paper, R always denotes an (associative) ring with unity ε, and M a unital (left) R-module. In particular, write ${}_RR$ to point out the fact that the ring R is considered as an R-module.

A submodule N of M is said to be *essential* in M if

$$N \cap X \neq 0 \quad \text{for every non-zero submodule } X \text{ of } M\,.$$

If every non-zero submodule of M is essential in M, then the R-module M is called *uniform*. In the respective sense, we speak about (left) essential or uniform ideals of R.

An R-module M is said to be *torsion-free* if M contains no non-zero element of an essential order. Thus, a (left) torsion-free ring is just a ring with zero singular ideal in the terminology of R. E. Johnson [4].

Given an R-module M, define the *socle sequence*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_i \subseteq M_{i+1} \subseteq \ldots \subseteq M$$

of M by

$$M_{i+1}/M_i = \text{socle}(M/M_i) \quad \text{for } i = 0, 1, 2, \ldots$$

If $M = M_i$ for a certain i, then M is said to be of *finite length* and the least i with that property is called the *length* of M. Again, applying the latter definitions to ${}_RR$, we get the concept of a socle series of a ring and the concepts of a ring of finite length and the length of a ring.

An R-module M is said to be *uniserial* if there is a direct decomposition

$$M = \bigoplus_{\omega \in \Omega} M_\omega$$

of M such that, for every $\omega \in \Omega$, the submodules of M_ω form a chain (by inclusion). Notice that, in particular, all M_ω's are uniform. Thus, a ring R is (left) uniserial if every left principle indecomposable ideal L_i of R has the property that all left ideals of R contained in L_i form a chain (*).

By a *tree*, we shall understand throughout the paper a finite rooted tree, i.e. a (partially) ordered finite set $(T, \leqslant)$ such that

(i) T possesses the least element 0, called the root;

(ii) for every $t' \leqslant t''$, the order $\leqslant$ induces in the interval $\langle t', t''\rangle = \{x|\ x \in T \ \&\ t' \leqslant x \leqslant t''\}$ a linear order.

Obviously, a tree is a meet-semilattice; denote by $t' \wedge t''$ the meet of the elements t' and t'' of T. Furthermore, for every $t \in T$, denote by T_t the set of all upper neighbours of t, i.e.

$$T_t = \{x|\ x \in T \ \&\ t \leqslant x \ \&\ \langle t, x\rangle = \{t, x\}\};$$

moreover, put

$$T^* = T \setminus \{0\}.$$

Now, a *decorated tree* $(T, \leqslant, (n, D))$ is a system consisting of pairs (n_t, D_t) indexed by T^*, where n_t are natural numbers and D_t division rings such that

$$D_{t''} \subseteq D_{t'} \quad \text{for every } t' \leqslant t'' \text{ of } T^*.$$

In an obvious manner, two decorated trees $(T', \leqslant', (n', D'))$ and $(T'', \leqslant'', (n'', D''))$ are said to be *isomorphic* if there exists a one-to-one order-preserving mapping Φ of $(T', \leqslant')$ onto $(T'', \leqslant'')$ and a system $\{\Psi_t|\ t \in T'^*\}$ of ring isomorphisms $\Psi_t\colon D'_t \to D''_{t\Phi}$ such that

$$n'_t = n''_{t\Phi} \quad \text{for all } t \in T'^*,$$

and that

$$\Psi_{t'} \text{ is an extension of } \Psi_{t''} \quad \text{for every } t' \leqslant t'' \text{ of } T'^*.$$

2. Method. Our method consists in three steps belonging, in part, to the folklore of the module theory.

(*) Consequently, such a ring is necessarily (right) perfect (cf. [4]).

2.1. *A ring R with unity ε is isomorphic to the R-endomorphism ring of ${}_RR$: $R \cong \operatorname{End}_R({}_RR)$.*

Evidently, the mapping $\Phi\colon R \to \operatorname{End}_R({}_RR)$ defined by $\varrho\Phi = \varphi_\varrho$, where $\chi\varphi_\varrho = \chi\varrho$ for all $\chi \in R$, is an isomorphism (notice that, for $\varphi \in \operatorname{End}_R({}_RR)$, $\varphi = (\varepsilon\varphi)\Phi$).

2.2. *Let a (left) unital R-module M be a finite direct sum*

$$M = \bigoplus_{i=1}^{r} M_i.$$

Then, the endomorphism ring $\operatorname{End}_R(M)$ of M is isomorphic to the ring $\boldsymbol{M}(r, \operatorname{Hom}_R(M_i, M_j))$ of all $r \times r$ matrices (φ_{ij}) where, for every $1 \leqslant i, j \leqslant r$,

$$\varphi_{ij} \in \operatorname{Hom}_R(M_i, M_j).$$

(For a generalized version, see [2].)

Denoting by

$$\pi_i\colon M \to M_i \quad \text{and} \quad \iota_i\colon M_i \to M, \quad 1 \leqslant i \leqslant r,$$

the projections and injections associated with the decomposition $M = \bigoplus_{i=1}^{r} M_i$, we can see easily that the mapping

$$\Phi\colon \operatorname{End}_R(M) \to \boldsymbol{M}(r, \operatorname{Hom}_R(M_i, M_j))$$

defined by

$$\varphi\Phi = (\varphi_{ij}), \quad \text{where } \varphi_{ij} = \iota_i\varphi\pi_j \text{ for } 1 \leqslant i, j \leqslant r,$$

is an isomorphism (notice that, for $(\psi_{ij}) \in \boldsymbol{M}(r, \operatorname{Hom}_R(M_i, M_j))$, $\psi_{ij} = \big(\sum_{1 \leqslant i,j \leqslant r} \pi_i\psi_{ij}\iota_j\big)\Phi$).

2.3. *Let M_1 and M_2 be (left) unital R-modules and let*

$$\varphi \in \operatorname{Hom}_R(M_1, M_2).$$

If M_1 is uniform and M_2 torsion-free, then φ is either zero or a monomorphism, i.e. an embedding of M_1 into M_2. If, moreover, there is just one submodule N_2 of M_2 which is R-isomorphic to M_1, then $\operatorname{Hom}_R(M_1, M_2)$ can be endowed with a ring structure such that

$$\operatorname{Hom}_R(M_1, M_2) \cong \operatorname{End}_R(M_1)$$

is a division ring. If, in addition M_2 is uniform, then $\operatorname{End}_R(M_2)$ can be considered as a subring of $\operatorname{End}_R(M_1)$. All this happens, in particular, if M_2 is a uniserial R-module of finite length, unless $\operatorname{Hom}_R(M_1, M_2) = 0$.

Proof. The first part follows very easily (cf. [2]): If φ is not monic, then $\operatorname{Ker}\varphi \neq 0$ and thus, for every $m \in M$, the left ideal

$$E_m = \{\varrho|\ \varrho \in R \ \&\ \varrho m \subseteq \operatorname{Ker}\varphi\}$$

of R is essential in R. Now, since

$$(E_m m)\varphi = E_m(m\varphi) = 0\,,$$

we deduce that $m\varphi = 0$, i.e. φ is a zero morphism.

Assuming that there is a unique submodule N_2 of M_2 which is R-isomorphic to M_1, denote by $\psi\colon M_1 \to N_2$ a (fixed) R-isomorphism and by $\iota\colon N_2 \to M_2$ the embedding of N_2 in M_2. Define the mapping

$$\Phi\colon \mathrm{End}_R(M_1) \to \mathrm{Hom}_R(M_1, M_2)$$

by

$$\varphi\Phi = \varphi\Psi\iota \quad \text{for all } \varphi \in \mathrm{End}_R(M_1)\,.$$

It is a matter of routine to verify that Φ is a one-to-one mapping onto $\mathrm{Hom}_R(M_1, M_2)$ which respects the additive structure of $\mathrm{Hom}_R(M_1, M_2)$. In fact, every $a_i \in \mathrm{Hom}_R(M_1, M_2)$ can be written (in a unique way) in the form

$$a_i = \bar{a}_i\Psi\iota \text{ with } \bar{a}_i \in \mathrm{End}_R(M_1)\,;$$

thus, the multiplication

$$a_1 * a_2 = \bar{a}_1\bar{a}_2\Psi\iota \quad (= \bar{a}_1\bar{a}_2\Phi)$$

defined for every a_1 and a_2 of $\mathrm{Hom}_R(M_1, M_2)$ transforms $\mathrm{Hom}_R(M_1, M_2)$ into a ring isomorphic to $\mathrm{End}_R(M_1)$. Obviously, $\mathrm{End}_R(M_1)$ is a division ring. Furthermore, if M_2 is uniform, then $\theta\colon \mathrm{End}_R(M_2) \to \mathrm{End}_R(N_2)$ mapping every $\varphi \in \mathrm{End}_R(M_2)$ into its restriction φ_{N_2} to $N_2 \subseteq M_2$ is evidently an embedding of $\mathrm{End}_R(M_2)$ into $\mathrm{End}_R(N_2)$. And since $M_1 \simeq N_2$, $\mathrm{End}_R(M_2)$ can be embedded in $\mathrm{End}_R(M_1)$.

The rest of our assertion follows easily.

Now, a subsequent application of 2.1, 2.2 and 2.3 yields immediately that a (left) torsion-free uniserial ring of finite length is isomorphic to a ring of all $r \times r$ matrices (φ_{ij}), where the entries φ_{ij} are, for a fixed pair (i, j), elements of a division ring D_{ij} (which may, possibly, be trivial). As a matter of fact, on the basis of our simple observations, we can assert much more (cf. [1]). In particular, we obtain in this way a very lucid proof of the Wedderburn–Artin structure theorem. However, here we want to present an explicite description of our matrix representation and to give a full characterization of our rings by means of decorated trees.

3. Theorem.

THEOREM. *There is a one-to-one correspondence between the non-isomorphic (left) torsion-free uniserial rings of finite length and the non-isomorphic decorated trees. Every such ring of rank r can be represented as a ring of $r \times r$ matrices (x_{ij}) such that, for every $1 \leqslant i, j \leqslant r$, the entries x_{ij} belong to a division ring D_{ij} (equal, possibly, to 0) and satisfy:*

(i) *If $D_{kl} \neq 0$ for $r \geqslant k \geqslant l \geqslant 1$, then $D_{ij} = D_{kl}$ for all $l \leqslant i, j \leqslant k$.*

(ii) *If $D_{kl} \neq 0$ for $1 \leqslant k \leqslant l \leqslant r$, then $D_{kj} = D_{kl}$ for all $k \leqslant j \leqslant l$.*

(iii) *If $D_{kl} \neq 0$ for $1 \leqslant k \leqslant l \leqslant r$, then $D_{il} \subseteq D_{kl}$ for all $k \leqslant i \leqslant l$.*

(iv) *If $D_{kt} \neq 0$ and $D_{tl} \neq 0$ for $1 \leqslant k \leqslant t \leqslant l \leqslant r$, then $D_{kl} \neq 0$.*

(v) *$D_{ii} \neq 0$ for all $1 \leqslant i \leqslant r$.*

This representation is unique up to a certain simultaneous permutation of the rows and columns of all matrices (and, of course, isomorphic copies of D_{ij}'s).

Remark. Notice that, as a consequence of our Theorem, we get the following two statements on the (left) torsion-free uniserial ring R of finite length:

(a) $R/\mathrm{Soc}\,R$ is again (left) torsion-free (and, of course, uniserial of finite length).

(b) For every (left) minimal ideal V of R, the ring R possesses a (left) principal indecomposable ideal which is R-isomorphic to V.

Let us also remark that in course of our proof of Theorem, we shall describe the isomorphism and representation explicitely.

Proof of Theorem will be established in the following three steps:

A. Denote the class of all (left) torsion-free uniserial rings of finite length by $\mathfrak{R}$ and the class of all decorated trees by $\mathfrak{C}$. Define the mapping $\Phi\colon \mathfrak{R} \to \mathfrak{C}$ as follows. Given $R \in \mathfrak{R}$, consider a decomposition

$$R = \bigoplus_{i=1}^{r} L_i$$

of R into the (left) principal indecomposable ideals L_i, $1 \leqslant i \leqslant r$, and define on the set $\{L_i | 1 \leqslant i \leqslant r\}$ a preorder $\leqslant$ by

$$L_{i_1} \leqslant L_{i_2} \quad \text{if and only if} \quad \mathrm{Hom}_R(L_{i_1}, L_{i_2}) \neq 0\,.$$

The preorder $\leqslant$ defines a partition of the set $\{L_i | 1 \leqslant i \leqslant r\}$ into "equivalence classes" t; moreover, $\leqslant$ induces an order on the set T^* of all these classes. Adjoin to T^* an element 0 and define $0 \leqslant t$ for all $t \in T^*$. It is easy to see that $(T, \leqslant)$, where $T = T^* \cup \{0\}$, is a tree.

Now, for each $t \in T^*$, put

$$n_t = \mathrm{card}\,\{L_i | L_i \in t\}\,.$$

Furthermore, for each $t_0 \in T_0$, take

$$D_{t_0} \simeq \mathrm{End}_R(L_{i_0}) \quad \text{for } L_{i_0} \in t_0\,.$$

And, proceed by induction: Having chosen, for all $t \leqslant t$, division rings

$$D_t \simeq \mathrm{End}_R(L_i) \quad \text{with } L_i \in t$$

such that

$$D_{t'} \subseteq D_{t''} \quad \text{whenever } t'' \leqslant t'\,,$$

we can embed, for $t_* \in T_{\bar{i}}$,

$$D_{t_*} \cong \mathrm{End}_R(L_i) \quad \text{with } L_i \in t_*$$

into $D_{\bar{i}}$, in view of 2.3:

$$D_{t_*} \subseteq D_{\bar{i}} .$$

As a result, we get a decorated tree

$$R\Phi = \big(T, \leqslant, (n, D)\big) .$$

B. Denote by $\boldsymbol{M}(r, D_{ij})$ the ring of all $r \times r$ matrices (x_{ij}) described in Theorem and, furthermore, denote by $\mathcal{M}$ the class of all such matrix rings (with variable r, as well). Define the mapping $\Psi_{\lll} : \mathcal{C} \to \mathcal{M}$ as follows.

First, given $\big(T, \leqslant, (n, D)\big) \in \mathcal{C}$, choose, for each $t \in T$, a total order $\lll_t$ in T_t, and subsequently extend $\leqslant$ into a full order $\lll$ on T, defining for $t', t'' \in T$:

(1) If $t' \leqslant t''$, or $t' \geqslant t''$, then $t' \lll t''$, or $t' \ggg t''$, respectively.

(2) Otherwise, write $t = t' \wedge t''$, notice that $t' \neq t \neq t''$ and take

$$x' \in T_t \cap \langle t, t' \rangle \quad \text{and} \quad x'' \in T_t \cap \langle t, t'' \rangle ;$$

if $x' \lll_t x''$, or $x' \ggg_t x''$, then $t' \lll t''$, or $t' \ggg t''$, respectively.

It is easy to see that $\lll$ is a total order on the set T. Thus, if T^* has s elements, we can write

$$T^* = \{t_z | 1 \leqslant z \leqslant s\} \quad \text{with } t_z \lll t_{z+1} \quad \text{for } 1 \leqslant z \leqslant s-1 .$$

Observe also, that for every z, z_1 and z_2 such that

$$t_z \leqslant t_{z_1} \quad \text{and} \quad t_z \nleqslant t_{z_2},$$

we have either $z \leqslant z_1 \leqslant z_2$ or $z_2 \leqslant z \leqslant z_1$.

Now define the matrix ring $\boldsymbol{M}(r, D_{ij})$ in the following way: First,

$$r = \sum_{z=1}^{s} n_{t_z} .$$

Secondly, given a pair (i, j) with $1 \leqslant i, j \leqslant r$, there is a unique w, $1 \leqslant w \leqslant s$ such that

$$\sum_{z=1}^{w-1} n_{t_z} + 1 \leqslant i \leqslant \sum_{z=1}^{w} n_{t_z} \quad \Big(\text{for } w = 1, \quad \text{put } \sum_{z=1}^{w-1} n_{t_z} = 0\Big);$$

and then put $D_{ij} = D_{t_w}$ for all j such that

$$\sum_{z=1}^{w-1} n_{t_z} + 1 \leqslant j \leqslant \sum_{z=1}^{w-1} n_{t_z} + \sum_{t \geqslant t_w} n_t ,$$

and $D_{ij} = 0$ otherwise.

It is easy to see that $\big(T, \leqslant, (n, D)\big)\, \Psi_{\lll} = \boldsymbol{M}(r, D_{ij})$ just defined is a (matrix) ring and that it satisfies the conditions (i)–(v) of Theorem. Furthermore, it is easy to see that every such matrix ring satisfying (i)–(v) is an image under Ψ of a suitable decorated tree. And, more importantly, that two non-isomorphic decorated trees produce under Ψ two non-isomorphic matrix rings. This follows immediately from the fact that all the decorated tree "variables" can be read from the socle series

$$0 = \boldsymbol{M}_0 \subset \boldsymbol{M}_1 \subset \ldots \subset \boldsymbol{M}_{i-1} \subset \boldsymbol{M}_i \subset \ldots \subset \boldsymbol{M}_l = \boldsymbol{M}_l = \boldsymbol{M}(r, D_{ij})$$

of the matrix rings: The non-zero elements t of the tree correspond to the homogeneous components $\boldsymbol{P}_{z(i)}$ of $\boldsymbol{M}_i/\boldsymbol{M}_{i-1}$, the order $\leqslant$ relates to the inclusion of the column ideals of $\boldsymbol{M}(r, D_{ij})$ and the numbers n_t and the division rings D_t correspond to the ranks of $\boldsymbol{P}_{z(i)}$ and to the endomorphism rings of the minimal direct summands of $\boldsymbol{P}_{z(i)}$, respectively.

C. If we re-order the left principal indecomposable ideals L_i, $1 \leqslant i \leqslant r$, according to the extension $\lll$ of the order $\leqslant$ defined on $\{L_i | 1 \leqslant i \leqslant r\}$ in the section A., and then apply our method described in 2.1, 2.2 and 2.3, we get immediately

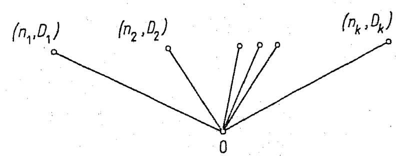
$$R \cong \boldsymbol{M}(r, D_{ij}) = R\Phi\Psi_{\lll} .$$

This completes the proof of our Theorem.

4. Remarks. Observe that we have not used in our proof the uniqueness of the decomposition $R = \bigoplus_{i=1}^{r} L_i$; as a matter of fact, this uniqueness is a consequence of our considerations.

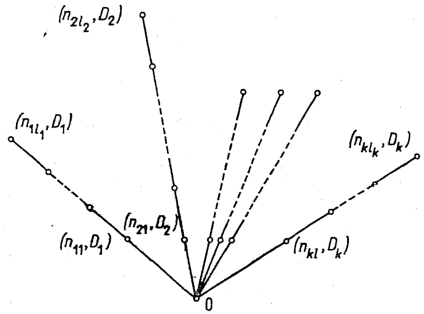
Also, we can see readily that the ring $\boldsymbol{M}(r, D_{ij})$ considered as a right module is torsion-free and of finite length. Thus, from our Theorem, we deduce that a left torsion-free uniserial ring of finite length is right torsion-free and of finite length (cf. also Lemma below).

Notice that the decorated trees of the type



correspond to the semisimple rings (artinian rings R with $\mathrm{Rad}\, R = 0$).

The trees of the type



correspond to the artinian torsion-free generalized uniserial rings of W. A. Goldie [3]. In contrast to this proof, we get the representation and its uniqueness very easily from the following simple

LEMMA. *Let* $M(r, D_{ij})$ *be a matrix ring satisfying* (i)–(v) *of Theorem. Then*

$$M(r, D_{ij}) = \bigoplus_{k=1}^{r} R_k ,$$

where R_k, $1 \leqslant k \leqslant r$, *is the* k*-th row right ideal, i.e.*

$$R_k = \{(x_{ij}) | \quad (x_{ij}) \in M(r, D_{ij}) \text{ and } x_{ij} = 0 \text{ for all } i \neq k\} .$$

All R_k *are uniserial as right* $M(r, D_{ij})$*-modules if and only if*

(iii)′ $D_{kl} \neq 0$ *for* $1 \leqslant k \leqslant l \leqslant r$ *implies* $D_{il} = D_{kl}$ *for all* $k \leqslant i \leqslant l$,

i.e. if and only if $M(r, D_{ij})$ *is right torsion-free uniserial ring of finite length.*

Proof. In order to show (iii)′ indirectly assume that i, k are the least indices for which

$$D_{il} \subsetneq D_{ik} , \quad k \leqslant i \leqslant l .$$

Necessarily, $k = i-1$ and thus

$$D_{i-1\,i-1} = D_{i-1\,i} \supsetneq D_{ii} \quad \text{and } D_{i\,i-1} = 0 .$$

Now, consider $D_{i-1\,i}$ as a vector space over D_{ii}, take two elements ϱ_1, ϱ_2 of $D_{i-1\,i}$ independent over D_{ii}, and denote by $E^{(\varrho_p)}$, $p = 1, 2$, the $r \times r$ matrix having ϱ_p in the $(i-1, 1)$ position and 0 elsewhere. It is easy to verify that the right (principal) ideals

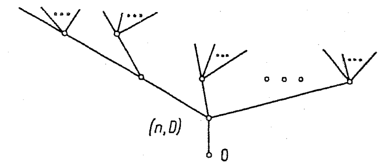
$$E^{(\varrho_p)} \cdot M(r, D_{ij}) , \quad p = 1, 2 ,$$

are contained in R_{i-1} and are, with respect to inclusion, incomparable.

The rest follows immediately from Theorem and the fact that the conditions (i), (ii), (iii)′, (iv) and (v) are symmetrical.

In conclusion, let us mention yet another application of our Theorem. Assume that a (left) torsion-free uniserial ring of finite length is simple, i.e. that R corresponds to a decorated tree with a single neighbour of 0, say to

(*)



It is well known (for a simple proof see [3]) that the injective hull of ${}_RR$ is the full ring of $r \times r$ matrices over D. Thus, the (indecomposable) injective hull $H_R(V)$ of a minimal left ideal V of R (or, for that matter, of a "column" ideal of R) can be easily characterized by the tree (*): the tree fully describes its R-submodule structure. In particular, one can see that the length of $H_R(V)$ equals to the maximum of the sums $1 + \sum_{i=2}^{k} (D_{t_{i-1}}: D_{t_i})$, where $0 < t_1 < \ldots < t_k = t$ are all elements of the interval $\langle 0, t \rangle$ and $(D_{t_{i-1}}: D_{t_i})$ denotes the dimension of $D_{t_{i-1}}$ over D_{t_i}.

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