Tree-like matrix rings

by

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The main purpose of the present paper is to call attention to a certain simple module-theoretic method and, as an illustration of its value, to derive a representation of (left) torsion-free (semi) uniserial rings (with unity) of finite length and to give a full characterization and a "canonic" form of them in terms of so-called decorated (finite rooted) trees. These results (cf. also R. B. Colby and E. A. Ruther, Jr [1]) generalize Goldie's theorem on block-triangular matrix rings in [5] which is, in turn, a generalization of the Wedderburn–Artin structure theorem. Thus, our approach offers also a very lucid proof of the latter classical result. Another important aspect of our method is the fact that its application can easily be extended to more general classes of torsion-free rings (cf. [3]) to which the methods of [1] or [5] (restricted by the condition that components of the rings contain unique minimal ideals) cannot be applied. Moreover, as a consequence of our results, we get a complete description of indecomposable injective modules over these rings.

1. Preliminaries. Throughout the paper, \( R \) always denotes an (associative) ring with unity \( e \), and \( M \) a unital (left) \( R \)-module. In particular, write \( aR \) to point out the fact that the ring \( R \) is considered as an \( R \)-module.

A submodule \( N \) of \( M \) is said to be essential in \( M \) if

\[ N \cap X \neq 0 \quad \text{for every non-zero submodule } X \text{ of } M. \]

If every non-zero submodule of \( M \) is essential in \( M \), then the \( R \)-module \( M \) is called uniform. In the respective sense, we speak about (left) essential or uniform ideals of \( R \).

An \( R \)-module \( M \) is said to be torsion-free if \( M \) contains no non-zero element of an essential order. Thus, a (left) torsion-free ring is just a ring with zero singular ideal in the terminology of R. E. Johnson [4].

Given an \( R \)-module \( M \), define the socle sequence

\[ 0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_i \subseteq M_{i+1} \subseteq \ldots \subseteq M \]

of \( M \) by

\[ M_{i+1}/M_i = \text{socle}(M/M_i) \quad \text{for } i = 0, 1, 2, \ldots \]
If \( M = M_i \) for a certain \( i \), then \( M \) is said to be of \textit{finite length} and the least \( i \) with that property is called the \textit{length} of \( M \). Again, applying the latter definitions to \( \mathcal{P}_R \), we get the concept of a socle series of a ring and the concepts of a ring of finite length and the length of a ring.

An \( R \)-module \( M \) is said to be \textit{uniserial} if there is a direct decomposition

\[
M = \bigoplus_{i \in I} M_i
\]

of \( M \) such that, for every \( \omega \in \Omega \), the submodules of \( M_\omega \) form a chain (by inclusion). Notice that, in particular, all \( M_\omega \)'s are uniform. Thus, a ring \( R \) is (left) uniserial if every left principle indecomposable ideal \( I \) of \( R \) has the property that all left ideals of \( R \) contained in \( I \) form a chain (*).

By a tree, we shall understand throughout the paper a finite rooted tree, i.e., a (partially) ordered finite set \((T, \leq)\) such that

(i) \( T \) possesses the least element \( 0 \), called the root;

(ii) for every \( t \leq t' \), the order \( \leq \) induces in the interval \( \langle t', t' \rangle = \{ x \in T : t \leq x \leq t' \} \) a linear order.

Obviously, a tree is a meet-semilattice; denote by \( t \wedge t' \) the meet of the elements \( t \) and \( t' \) of \( T \). Furthermore, for every \( t \in T \), denote by \( T_t \) the set of all upper neighbours of \( t \), i.e.

\[
T_t = \{ x \in T : t \leq x \} = \{ t, x \}.
\]

moreover, put

\[
T^* = T \setminus \{ 0 \}.
\]

Now, a \textit{decorated tree} \( (T, \leq, (n, D)) \) is a system consisting of pairs \((n, D)\) indexed by \( T^* \), where \( n_t \) are natural numbers and \( D_t \) division rings such that

\[
D_t \subseteq D_{t'}
\]

for every \( t \leq t' \) of \( T^* \).

In an obvious manner, two decorated trees \( (T, \leq, (n, D)) \) and \( (T', \leq, (n', D')) \) are said to be \textit{isomorphic} if there exists a one-to-one order-preserving mapping \( \Phi \) of \((T, \leq)\) onto \((T', \leq)\) and a system \((\Phi_t, t \in T^*)\) of ring isomorphisms \( \Phi_t : D_t \to D'_{\Phi_t} \) such that

\[
\Phi_t = \Phi_{t'}
\]

for all \( t \leq t' \), and that

\[
\Phi_t
\]

is an extension of \( \Phi_t \) for every \( t \leq t' \) of \( T^* \).

2. \textbf{Method.} Our method consists in three steps belonging, in part, to the folklore of the module theory.

(*) Consequently, such a ring is necessarily (right) perfect (cf. [4]).
of \( R \) is essential in \( R \). Now, since

\[ (E_{m} \varphi) \psi = E_{m} (\varphi \psi) = 0 , \]

we deduce that \( \psi \varphi = 0 \), i.e. \( \varphi \) is a zero morphism.

Assuming that there is a unique subalgebra \( N_{1} \) of \( M_{2} \) which is

\( R \)-isomorphic to \( M_{2} \), denote by \( \varphi : M_{2} \to N_{1} \) a (fixed) \( R \)-isomorphism and by \( \iota : N_{1} \to M_{2} \) the embedding of \( N_{1} \) in \( M_{2} \). Define the mapping

\[ \Phi : \text{End}_{R}(M_{2}) \to \text{Hom}_{R}(M_{1}, M_{2}) \]

by

\[ \varphi \Phi = \varphi \Psi \quad \text{for all } \varphi \in \text{End}_{R}(M_{2}) . \]

It is a matter of routine to verify that \( \Phi \) is a one-to-one mapping onto \( \text{Hom}_{R}(M_{1}, M_{2}) \) which respects the additive structure of \( \text{Hom}_{R}(M_{1}, M_{2}) \). In fact, every \( \alpha_{1} \in \text{Hom}_{R}(M_{1}, M_{2}) \) can be written (in a unique way) in the form

\[ \alpha_{1} = \tilde{a}_{1} \Psi_{1} \text{ with } \tilde{a}_{1} \in \text{End}_{R}(M_{2}) ; \]

thus, the multiplication

\[ \alpha_{1} * \alpha_{2} = \tilde{a}_{1} \bar{a}_{2} \Psi_{1} = (\tilde{a}_{1} \bar{a}_{2}) \Phi \]

defined for every \( \alpha_{1} \) and \( \alpha_{2} \) of \( \text{Hom}_{R}(M_{1}, M_{2}) \) transforms \( \text{Hom}_{R}(M_{1}, M_{2}) \)

into a ring isomorphic to \( \text{End}_{R}(M_{2}) \). Obviously, \( \text{End}_{R}(M_{1}, M_{2}) \) is a division ring. Furthermore, if \( M_{2} \) is uniform, then \( \Theta : \text{End}_{R}(M_{2}) \to \text{End}_{R}(N_{1}) \)

mapping every \( \varphi \in \text{End}_{R}(M_{2}) \) into its restriction \( \varphi \mid N \) is evidently an embedding of \( \text{End}_{R}(M_{2}) \) into \( \text{End}_{R}(N_{1}) \). And since \( M_{2} \supseteq N_{1} \), \( \text{End}_{R}(M_{2}) \) can be embedded in \( \text{End}_{R}(M_{1}) \).

The rest of our assertion follows easily.

Now, a subsequent application of 2.1, 2.2 and 2.3 yields immediately that a (left) torsion-free universal ring of finite length is isomorphic to a ring of \( r \times r \) matrices \( (p_{ij}) \), where the entries \( p_{ij} \) are, for a fixed pair \((i, j)\), elements of a division ring \( D_{ij} \) (which may, possibly, be trivial). As a matter of fact, on the basis of our simple observations, we can assert much more (cf. [1]). In particular, we obtain in this way a very lucid proof of the Wedderburn–Artin structure theorem. However, here we want to present an explication description of our matrix representation and to give a full characterisation of our rings by means of decorated trees.

3. Theorem.

Theorem. There is a one-to-one correspondence between the non-isomorphic (left) torsion-free universal rings of finite length and the non-isomorphic decorated tree. Every such ring of rank \( r \) can be represented as a ring of \( r \times r \) matrices \( (a_{ij}) \) such that, for every \( 1 \leq i, j \leq r \), the entries \( a_{ij} \) belong to a division ring \( D_{ij} \) (equal, possibly, to 0) and satisfy:

(i) If \( D_{1} \neq 0 \) for \( r \geq k \geq r \), then \( D_{ij} = D_{1} \) for all \( l \leq i, j \leq k \).

(ii) If \( D_{1} \neq 0 \) for \( 1 \leq k \leq r \), then \( D_{ij} = D_{1} \) for all \( k \leq j \leq l \).

(iii) If \( D_{1} \neq 0 \) for \( 1 \leq k \leq l \), then \( D_{ij} \subseteq D_{1} \) for all \( k \leq i \leq l \).

(iv) If \( D_{1} \neq 0 \) and \( D_{1} \neq 0 \) for \( 1 \leq k \leq l \leq r \), then \( D_{ij} \neq 0 \).

(v) \( D_{1} \neq 0 \) for all \( 1 \leq i \leq r \).

This representation is unique up to a certain simultaneous permutation of the roots and columns of all matrices (and, of course, isomorphic copies of \( D_{1} \)).

Remark. Notice that, as a consequence of our Theorem, we get the following two statements on the (left) torsion-free universal ring \( R \) of finite length:

(a) \( R/\text{Soc} R \) is again (left) torsion-free (and, of course, uniserial of finite length).

(b) For every (left) minimal ideal \( V \) of \( R \), the ring \( R \) possesses a (left) principal indecomposable ideal which is \( R \)-isomorphic to \( V \).

Let us also remark that in course of our proof of Theorem, we shall describe the isomorphism and representation explicitly.

Proof of Theorem will be established in the following three steps:

A. Let us first consider the class \( \mathcal{A} \) of all universal rings of finite length by \( A \) and the class of all decorated trees by \( T \). Define the mapping \( \Phi : \mathcal{A} \times \mathcal{T} \) as follows. Given \( R \in \mathcal{A} \), consider a decomposition

\[ R = \bigoplus_{i} L_{i} \]

of \( R \) into the (left) principal indecomposable ideals \( L_{i} \), \( 1 \leq i \leq r \), and define on the set \( \{ L_{i} \mid 1 \leq i \leq r \} \) a preorder \( \leq \)

by

\[ L_{i} \leq L_{j} \quad \text{if and only if } \text{Hom}_{R}(L_{i}, L_{j}) \neq 0 . \]

The preorder \( \leq \) defines a partition of the set \( \{ L_{i} \mid 1 \leq i \leq r \} \) into "equivalence classes" \( t \); moreover, \( \leq \) induces an order on the set \( T^{*} \) of all these classes. Adjoin to \( T^{*} \) an element 0 and define 0 \( \leq t \) for all \( t \in T^{*} \). It is easy to see that \( (T^{*}, \leq) \), where \( T = T^{*} \cup \{ 0 \} \), is a tree.

Now, for each \( t \in T^{*} \), put

\[ n_{t} = \text{card}(\{ L_{i} \mid L_{i} \leq t \}) . \]

Furthermore, for each \( t_{e} \in T^{*} \), take

\[ D_{t} \cong \text{End}_{R}(L_{i}) \quad \text{for } L_{i} \leq t . \]

And, proceed by induction: Having chosen, for all \( t \leq t' \), division rings \( D_{t} \cong \text{End}_{R}(L_{i}) \) with \( L_{i} \leq t \) such that

\[ D_{t} \subseteq D_{t'} \quad \text{whenever } t' \leq t , \]

then

\[ D_{t} \cong \text{End}_{R}(L_{i}) \quad \text{for } L_{i} \leq t . \]

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we can embed, for \( t_* \in T_1 \),
\[
D_{t_*} \cong \text{End}_k(L_t) \quad \text{with} \quad L_t \in t_*,
\]
into \( D_{t_*} \), in view of 2.3:
\[
D_{t_*} \subseteq D_{t_*}.
\]
As a result, we get a decorated tree
\[
E(\Phi) = (T, \leq) \cdot \text{End}_k(M_1).
\]
B. Denote by \( M(r, D_1) \) the ring of all \( r \times r \) matrices (\( \mathcal{M}_D \)) described in Theorem A and, furthermore, denote by \( \mathcal{M}_D \) the class of all such matrix rings (with variable \( t_r \), as well). Define the mapping \( \Psi : \mathcal{M} \rightarrow \mathcal{M}_D \) as follows.

First, given \( [T, \leq] \cdot \text{End}_k(M_1) \), choose, for each \( t \in T \), a total order \( \leq_t \) in \( T_t \), and subsequently extend \( \leq_t \) into a full order \( \leq \) on \( T_t \), defining for \( t_*, t' \in T_t \):

1. If \( t < t' \), or \( t' < t'' \), then \( t < t' \), or \( t' < t'' \), respectively.
2. Otherwise, write \( t = t' \cup t'' \), notice that \( t_0 \neq t' \neq t'' \) and take
\[
\xi' \in T_t \cap \langle \xi, \xi' \rangle \\
\xi'' \in T_t \cap \langle \xi, \xi'' \rangle;
\]
if \( \xi' \leq \xi'' \), or \( \xi'' \leq \xi'' \), then \( t' \leq t_0 \), or \( t' \leq t'' \), respectively.

It is easy to see that \( \leq_t \) is a total order on the set \( T_t \). Thus, if \( T_t \) has \( s \) elements, we can write
\[
T_t = \{ t_1, \ldots, t_s \} \quad \text{with} \quad t_{i+1} \leq t_i \quad \text{for} \ 1 \leq i \leq s - 1.
\]
Observe also, that for every \( t_1, t_2 \), and \( t_3 \), such that
\[
t_1 \leq t_2 \quad \text{and} \quad t_2 \leq t_3,
\]
we have either \( z = t_1 \leq t_2 \) or \( z = t_3 \leq t_2 \).

Now define the matrix ring \( M(r, D_1) \) in the following way: First,
\[
\tau = \sum_{i=1}^w n_i.
\]
Secondly, given a pair \( (i, j) \) with \( 1 \leq i, j \leq r \), there is a unique \( w, 1 \leq w \leq s \) such that
\[
\sum_{i=1}^{w-1} n_i + 1 \leq i \leq \sum_{i=1}^w n_i \\
\text{(for} \ w = 1 \text{, put} \ \sum_{i=1}^{w-1} n_i = 0);\]
and then put \( D_{ij} = D_{n_i} \) for all \( i \) such that
\[
\sum_{i=1}^{w-1} n_i + 1 \leq j \leq \sum_{i=1}^w n_i + \sum_{i=1}^{w-1} n_i,
\]
and \( D_{ij} = 0 \) otherwise.

It is easy to see that \( \{ T, \leq \} \cdot \text{End}_k(M_1) \) \( \Psi : \mathcal{M} \rightarrow M(r, D_1) \) is an image under \( \Psi \) of a suitable decorated tree. And, more importantly, that two non-isomorphic decorated trees produce under \( \Psi \) two non-isomorphic matrix rings. This follows immediately from the fact that all the decorated tree "variables" can be read from the socle series
\[
0 = M_1 \subseteq M_r \subseteq \cdots \subseteq M_{r-1} \subseteq M_r \subseteq \cdots \subseteq M_r = M(r, D_1)
\]
of the matrix rings: The non-zero elements \( t \) of the tree correspond to the homogeneous components \( P_{tr}(M_1) \) of \( M(r, D_1) \). The order \( \leq \) relates to the inclusion of the column ideals of \( M(r, D_1) \) and the numbers \( w \); and the division rings \( D_t \) correspond to the ranks of \( P_{tr} \) and to the endomorphism rings of the minimal direct summands of \( P_{tr} \), respectively.

C. If we re-order the left principal indecomposable ideals \( L_{1i} \) \( 1 \leq i \leq r \), according to the extension \( \leq \) of the order \( \leq \), defined on \( \{ L_{1i} \} \) \( 1 \leq i \leq r \); in the section \( \Delta \), and then apply our method described in 2.1, 2.2 and 2.3, we get immediately
\[
R \cong M(r, D_1).
\]
This completes the proof of our Theorem.

4. Remarks. Observe that we have not used in our proof the uniqueness of the decomposition \( R = \bigoplus_{i=1}^w L_{1i} \) as a matter of fact, this uniqueness is a consequence of our considerations.

Also, we can see readily that the ring \( M(r, D_1) \) considered as a right module is torsion-free and of finite length. Thus, from our Theorem, we deduce that a left torsion-free uniserial ring of finite length is right torsion-free and of finite length (cf. also Lemma below).

Notice that the decorated trees of the type
\[
\langle n_1, \xi_1 \rangle, \langle n_2, \xi_2 \rangle, \ldots, \langle n_r, \xi_r \rangle
\]
correspond to the semisimple rings (artinian rings \( R \) with \( \text{Rad} R = 0 \).
The trees of the type

\[
\begin{array}{c}
\text{Correspond to the artinian torsion-free uniserial rings of W. A. Goldie [3]. In contrast to this proof, we get the representation and its uniqueness very easily from the following simple lemma.}

\text{Lemma. Let } M(r, D_0) \text{ be a matrix ring satisfying (i)-(v) of Theorem. Then}

\[ M(r, D_0) = \bigoplus_{k=1}^{r} B_k, \]

\text{where } B_k, \ 1 \leq k \leq r, \text{ is the } k\text{-th row right ideal, i.e.}

\[ B_k = \{ (x_i) \mid x_i \in M(r, D_0) \text{ and } x_{ik} = 0 \text{ for all } i \neq k \}. \]

\text{All } B_k \text{ are uniserial as right } M(r, D_0)-\text{modules if and only if}

\text{(iii)'} \ D_{kl} \neq 0 \text{ for } 1 \leq k < l \leq r \text{ implies } D_{kl} = D_{kl} \text{ for all } k \leq i < l,

\text{i.e. if and only if } M(r, D_0) \text{ is right torsion-free uniserial ring of finite length.}

\text{Proof. In order to show (iii)’ indirectly assume that } i, k \text{ are the least indices for which}

\[ D_{ii} \subseteq D_{ki}, \quad k < i < l. \]

\text{Necessarily, } k = i - 1 \text{ and thus}

\[ D_{l-1,i-1} = D_{l-1,i-1} \supseteq D_{ii} \quad \text{and} \quad D_{l-1,i-1} = 0. \]

\text{Now, consider } D_{l-1,i-1} \text{ as a vector space over } D_{ii}, \text{ take two elements } g_1, g_2 \text{ of } D_{l-1,i-1} \text{ independent over } D_{ii}, \text{ and denote by } B^{(g_i)} \ p = 1, 2, \text{ the } r \times r \text{ matrix having } g_i \text{ in the } (i-1, i) \text{ position and } 0 \text{ elsewhere. It is easy to verify that the right (principal) ideals}

\[ B^{(g_i)} \cdot M(r, D_0) \quad p = 1, 2, \]

\text{are contained in } B_{l-1,i-1} \text{ and are, with respect to inclusion, incomparable.}

It is well known (for a simple proof see [3]) that the injective hull of } S_E \text{ is the full ring of } r \times r \text{ matrices over } D. \text{ Thus, the (indecomposable) injective hull } H^*_E(V) \text{ of a minimal left ideal } V \text{ of } E \text{ (or, for that matter, of a "column" ideal of } E) \text{ can be easily characterized by the tree } (**): \text{ the tree fully describes its } R\text{-submodule structure. In particular, one can see that the length of } H^*_E(V) \text{ equals to the maximum of the sums } 1 + \sum_{k} (D_{ik}; D_{ik}) \text{ where } 0 < t_1 < \ldots < t_k = t \text{ are all elements of the interval } (0, t) \text{ and } (D_{ik}; D_{ik}) \text{ denotes the dimension of } D_{ik} \text{ over } D_{kk}.\]

References


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