identity by an isotopy which is fixed on the complement of Interior(A_0) and such that for each $i = 1, \ldots, 4$, $g(a_i)$ intersects at most one of $P_1 \cup P_2$ and $P_3 \cup P_4$. By the second of the above definitions and Lemma 2, L cannot be greater than zero since if L is any integer greater than zero and F is a homeomorphism satisfying the requirements of the second of the above definitions, there is a homeomorphism h such that $L-1$ and h also satisfy the requirements of the second of the above definitions. The contradiction that L is not zero nor greater than zero completes the proof of Theorem 8.

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Some characterizations of paracompactness in k-spaces*

by

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1. Introduction. Paracompact spaces and k-spaces both have the distinction of being simultaneous generalizations of metric and compact spaces. The purpose of this paper is to present some of the interactions between these seemingly unrelated notions. Throughout this paper the underlying topological structures will be the k-spaces (e.g. first countable, Fréchet, sequential, locally compact, k'-space, and k-space). Specifically, for spaces within the class of k-spaces, those with the paracompact property are characterized. For this purpose, four generalizations of paracompactness are introduced. These generalizations are defined in terms of refinements which have some finiteness condition on the elements of a given collection of subsets. With the additional structure of the k-spaces these refinements have the properties required for the characterizations. These characterizations are given in § 3 and are summarized in the implication diagram which appears in Figure 3.2.

The fundamental notions used in this study are developed in § 2. Applications of these concepts to metrizability of spaces are given in § 4. Some examples are presented in § 5. The term "space" will mean a Hausdorff topological space and the term "family" will mean a family of subsets.

2. Preliminaries. The fundamental notions involved in this work will be developed in the general setting of F-hereditary collections and weak topology in the sense of Whitehead. A family $\mathcal{K} = \{K_\alpha\colon \alpha \in A\}$ in a space X is said to be an *F-hereditary collection* provided: (i) $\mathcal{K}$ is a covering of X and (ii) for each closed set $F \subset X$, $F \cap K_\alpha \in \mathcal{K}$ for each $\alpha \in A$. Some mapping properties of collections with property (ii) were investigated by Rønnow [11]. For all F-hereditary collections of interest, the singletons are in $\mathcal{K}$, and (i) is satisfied. For instance, the collection of all compact

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sets and the collection of all sets which are the closure of images of convergent sequences are F-hereditary collections. If $\mathcal{K} = \{K_\alpha: \alpha \in A\}$ is an F-hereditary collection in a space X, then a family $\mathcal{F} = \{F_\beta: \beta \in B\}$ is said to be *$\mathcal{K}$-finite* if for each $\alpha \in A$, $K_\alpha \cap F_\beta \neq \emptyset$ for at most finitely many $\beta \in B$. If $\mathcal{K}$ is the F-hereditary collection of all compact sets (closures of convergent sequences) then $\mathcal{F}$ is called a *compact-finite* (*cs-finite*) family. A space X is said to have the *W-weak topology* with respect to a family $\mathcal{K}$ provided: $G \subset X$ is open if and only if $G \cap H$ is open in H for each $H \in \mathcal{K}$. A space X is a *k-space* if and only if X has the W-weak topology with respect to the F-hereditary collection of all compact sets in X. Also, a space X is a *sequential space* (Franklin [5]) if and only if X has the W-weak topology with respect to the F-hereditary collection of closures of convergent sequences in X. The first step in the characterization of paracompactness in k-spaces is given in the following fundamental lemma.

LEMMA 2.1. *If a space X has the W-weak topology with respect to an F-hereditary collection $\mathcal{K} = \{K_\alpha: \alpha \in A\}$, then every $\mathcal{K}$-finite closed family $\mathcal{F} = \{F_\beta: \beta \in B\}$ in X is locally finite.*

Proof. $\mathcal{F}$ is a closure preserving family. To prove this, let $B' \subset B$, and let α be any element of A. Since $\mathcal{F}$ is $\mathcal{K}$-finite there exists $\beta 1, \beta 2, \ldots, \beta n \in B'$ such that $(\bigcup\{F_\beta: \beta \in B'\}) \cap K_\alpha = (\bigcup\{F_{\beta i}: 1 \leqslant i \leqslant n\}) \cap K_\alpha$. Hence, since $F_{\beta i} \cap K_\alpha$ is closed in K_α and the finite union of closed sets is closed, $(\bigcup\{F_\beta: \beta \in B'\}) \cap K_\alpha$ is closed in K_α. Thus $\cup\{F_\beta: \beta \in B'\}$ is closed in X, because X has the W-weak topology with respect to $\mathcal{K}$. Hence, $\mathcal{F}$ is a closure preserving family. Accordingly, for each $p \in X$, $G = X - \bigcup\{F_\beta: p \notin F_\beta\}$ is an open neighborhood of p which intersects only those $F_\beta \in \mathcal{F}$ such that $p \in F_\beta$. Thus, since $\mathcal{F}$ is point-finite, G is the required neighborhood. Hence $\mathcal{F}$ is locally finite.

From the preceeding lemma, from Michael's [7] characterization of paracompactness by means of locally finite closed refinements and from the shrinkability of point-finite open coverings of a normal space, the validity of the following theorem is immediate.

THEOREM 2.2. *If a regular (normal) space X has the W-weak topology with respect to an F-hereditary collection $\mathcal{K}$ and if every open covering of X has a $\mathcal{K}$-finite closed (open) refinement, then X is paracompact.*

An interesting relationship, which depends heavily on the notion of F-hereditary collection, exists between the W-weak topology and the weak topology in the sense of Morita [10]. A space X is said to have the *M-weak topology* with respect to a closed covering $\mathcal{F} = \{F_\beta: \beta \in B\}$ if (i) $\mathcal{F}$ is a closure preserving family and (ii) for each subset $B' \subset B$, if $H \subset \bigcup\{F_\beta: \beta \in B'\}$ and $H \cap F_\beta$ is open (closed) in F_β for each $\beta \in B'$, then H is open (closed) in the subspace $\bigcup\{F_\beta: \beta \in B'\}$. This relationship

is displayed in the following theorem which is a consequence of Lemma 2.1 and a lemma by Morita [10, Lemma 1].

THEOREM 2.3. *If a space X has the W-weak topology with respect to an F-hereditary collection $\mathcal{K}$, then X has the M-weak topology with respect to every $\mathcal{K}$-finite closed covering of X.*

COROLLARY 2.4. *A k-space (sequential space) has the M-weak topology with respect to every closed compact-finite (cs-finite) covering.*

3. Main results. A family $\mathcal{F} = \{F_\beta: \beta \in B\}$ will be called *strongly compact-finite* (*strongly cs-finite*) if $\{\mathrm{Cl}(F_\beta): \beta \in B\}$ is compact-finite (cs-finite). A space X is said to be (*strongly*) *mesocompact* if every open covering of X has a (strongly) compact-finite open refinement. Also, a space X is called (*strongly*) *sequentially mesocompact* if every open covering of X has a (strongly) cs-finite open refinement. The relationships between these concepts are presented in the following implication diagram.

$$\begin{tikzcd}
 & \text{metacompact (point-finite)} \\
\text{mesocompact (compact-finite)} \arrow[r, Rightarrow] & \text{sequentially mesocompact (cs-finite)} \arrow[u, Rightarrow] \\
\text{strongly mesocompact (strongly compact-finite)} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{strongly sequentially mesocompact (strongly cs-finite)} \arrow[u, Rightarrow] \\
\text{paracompact (locally finite)} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Figure 3.1

Since every point-finite open covering of a normal space is shrinkable, every normal mesocompact (sequentially mesocompact) space is strongly mesocompact (sequentially mesocompact). Hence, from Theorem 2.2 the following theorems can be stated.

THEOREM 3.1. *A sequential space X is paracompact if and only if X is strongly sequentially mesocompact.*

THEOREM 3.2. *A k-space X is paracompact if and only if X is strongly mesocompact.*

Arhangelśkiĭ [1, Theorem 11] originally stated Theorem 3.2 in a slightly different form.

To this point, it has been necessary to consider only closed families because non-closed, compact-finite families may not be locally finite in a k-space. In fact, as will be shown in Example 5.3, a non-closed compact-finite family may fail to be closure preserving even in a space belonging to the narrower class of sequential spaces. This restriction to closed families may be removed by requiring the space to be a k'-space (Fréchet) instead of a k-space (sequential). Arhangel'skiĭ introduced both the k'-spaces [1] and the Fréchet spaces [2]. A space X is a *k'-space* if for each non-closed set $H \subset X$ and for each point $p \in X$ such that $p \in \mathrm{Cl}(H) - H$ there exists a compact set $K \subset X$ such that $p \in \mathrm{Cl}(H \cap K)$. Also, a space X is a *Fréchet space* provided: for each set $H \subset X$ and for each point $p \in \mathrm{Cl}(H)$ there exists a sequence $\{p_i\} \subset H$ such that $\{p_i\}$ converges to p.

LEMMA 3.3. *Every compact-finite family $\mathcal{F} = \{F_a: a \in A\}$ in a k'-space X is closure preserving.*

Proof. Let A' be any subset of A. It is sufficient to show that $\mathrm{Cl}(\bigcup\{F_a: a \in A'\}) \subset \bigcup\{\mathrm{Cl}(F_a): a \in A'\}$. Let p be any element of $\mathrm{Cl}(\bigcup\{F_a: a \in A'\})$. If $p \in \bigcup\{F_a: a \in A'\}$, then the condition is satisfied. Thus, consider the case where $p \notin \bigcup\{F_a: a \in A'\}$. Since X is a k'-space, there exists a compact set $K \subset X$ such that $p \in \mathrm{Cl}\big(K \cap (\bigcup\{F_a: a \in A'\})\big)$. Since $\mathcal{F}$ is compact-finite, there exists a finite subset $A(K) \subset A'$ such that $(\bigcup\{F_a: a \in A'\}) \cap K = \big(\bigcup\{F_a: a \in A(K)\}\big) \cap K$. Hence, $p \in \mathrm{Cl}\big(\bigcup\{F_a: a \in A(K)\}\big)$. Thus there exists an $a \in A'$ such that $p \in \mathrm{Cl}(F_a)$. Accordingly, $p \in \bigcup\{\mathrm{Cl}(F_a): a \in A'\}$, and this completes the proof.

By substituting a convergent sequence for the compact set in the preceeding proof the following lemma is proved.

LEMMA 3.4. *Every cs-finite family in a Fréchet space is closure preserving.*

From Lemmas 3.3, 3.4 and Michael's [8] characterization of paracompactness by means of a closure preserving refinement, the next theorem follows.

THEOREM 3.5. *A regular Fréchet space (k'-space) is paracompact if and only if every open covering has a cs-finite (compact-finite) refinement.*

COROLLARY 3.6. *A regular Fréchet space (k'-space) is paracompact if and only if it is sequentially mesocompact (mesocompact).*

In Example 5.4 it is shown that a compact-finite family in a Fréchet space, which is stronger than a k'-space, need not be locally finite. Further, strengthening of k'-spaces and Fréchet spaces will yield the local finiteness of the compact-finite and the cs-finite families.

Clearly, the strengthening of k'-space to locally compact space is sufficient to assure the local finiteness of a compact-finite family. Thus the following theorem is immediate.

THEOREM 3.7. *A locally compact space is paracompact if and only if it is mesocompact.*

This may be stated in a stronger form. From Bourbaki [4, Theorem 5, p. 96] the following theorem may be stated.

THEOREM 3.8. *A locally compact space X is mesocompact if and only if X is the sum of a family of locally compact, σ-compact spaces.*

It is clear that the star finite property holds in a locally compact mesocompact space (i.e. a locally compact mesocompact space is strongly paracompact). Thus, a connected locally compact mesocompact space is Lindelöf.

The strengthening of Frechét space to first countable space is sufficient to assure the local finiteness of a cs-finite family.

LEMMA 3.9. *In a first countable space X a family $\mathcal{F} = \{F_\beta: \beta \in B\}$ is locally finite if and only if it is cs-finite.*

Proof. The necessity is clear. To prove the sufficiency, note that a point-finite, closure preserving, closed family is locally finite. Hence, since $\mathcal{F}$ is closure preserving, it is sufficient to show $\{\mathrm{Cl}(F_\beta): \beta \in B\}$ is point-finite. Let p be any point of X, and let $\{U_i: i \in N\}$ be any countable base at p ordered in the natural way by set inclusion. Since $\mathcal{F}$ is cs-finite, thus point-finite, it is sufficient to show $B' = \{\beta \in B: p \in \mathrm{Cl}(F_\beta) - F_\beta\}$ is finite. Assume B' is infinite. Then let $\{\beta(i): i \in N\}$ be any countably infinite subset of B'. Then for each $i \in N$, $U_i \cap F_{\beta(i)} \neq \emptyset$. For each $i \in N$ choose a point p_i from $U_i \cap F_{\beta(i)}$. Then $\{p_i\}$ converges to p. But this is a contradiction, because $p_i \in F_{\beta(i)}$, for each $i \in N$ implies $\mathcal{F}$ is not cs-finite. Thus B' is finite, and this establishes that $\{\mathrm{Cl}(F_\beta): \beta \in B\}$ is point-finite. Hence, $\mathcal{F}$ is locally finite.

From this lemma the next theorem follows immediately.

THEOREM 3.10. *A first countable space is paracompact if and only if it is sequentially mesocompact.*

The preceeding characterizations of paracompactness in k-spaces are summarized in the following implication diagram. Since each of the structures indicated is a paracompact space, it is the additional structure of the k-spaces which require the implication arrows to assume the directions indicated.

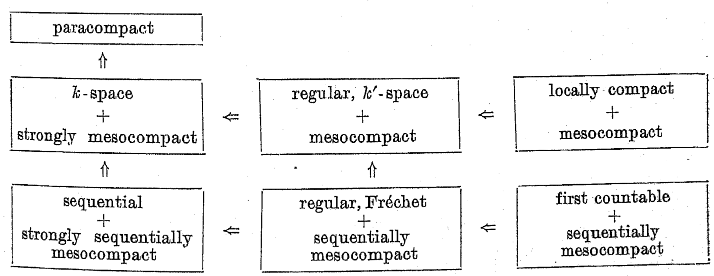

Figure 3.2

4. Applications. Some of the most important applications of paracompact spaces and locally finite families have been in the area of metrizability. The natural relationship between the metric property of first countability and the cs-finite families indicates that there should be some interesting applications of sequential mesocompactness to metrizability. For instance, it is well known that a paracompact, locally metrizable space is metrizable. Since a locally metrizable space is first countable, it can be seen that a locally metrizable space is metrizable if and only if it is sequentially mesocompact.

A translation of the Nagata–Smirnov Metrization Theorem, into the natural setting of a base with a particular property on the convergent sequences may be stated as follows.

THEOREM 4.1. *A space X is metrizable if and only if X is regular and has a σ-cs-finite base.*

The truth of this statement is clear, because the existence of a σ-cs-finite base implies the first countability of the space.

The metrizability of developable spaces has been an important motivation to research in topology since F. B. Jones inquired whether a normal developable (Moore) space is metrizable. R. W. Heath has supplied an important negative result in an example of a regular metacompact developable space which is not metrizable. The most important affirmative answer to this question has been supplied by Bing [3]. Bing has shown that a collectionwise normal, developable space is metrizable. By utilizing the important first countability of a developable space, an affirmative answer in the direction of the metacompact spaces is obtained.

THEOREM 4.2. *A developable space X is metrizable if and only if X is sequentially mesocompact.*

The validity of this statement follows from the paracompactness of a sequentially mesocompact developable space. Note that the first countability of the space and not the normality plays the essential role in this theorem.

5. Examples. In this section some examples are presented which clarify and sharpen the preceeding results.

EXAMPLE 5.1. *A normal mesocompact space which is not paracompact.*

As the space with the desired properties, consider the subspace G, defined by Michael [9], of Bing's space F [3, Example G]. The reader is referred to the cited papers for the definition of these spaces. Bing proved that F is normal and not collectionwise normal (F is not paracompact). For the purpose of this paper it is necessary to characterize

the compact subsets of F. Let $\mathcal{C}$ be the collection of all infinite closed subsets of F. For each $p \in P$, define the subcollection, $\mathcal{C}_p$, of $\mathcal{C}$ as

$$\mathcal{C}_p = \{H_p \in \mathcal{C}\colon f_p \text{ is the only cluster point of } H_p, f_t \notin H_p \text{ if } t \neq p\}.$$

For each $p \in P$, let

$$\mathcal{K}_p = \big\{K_p \in \mathcal{C}_p\colon \text{for each } q \in Q, \{f \in K_p\colon f(q) \neq f_p(q)\} \text{ is finite}\big\}.$$

Finally, let $\mathcal{F}^*$ be the collection of all finite subsets of F, and let $\mathcal{K} = \mathcal{F}^* \cup \cup (\bigcup \{\mathcal{K}_p\colon p \in P\})$.

It can be shown that for each $p \in P$, $\mathcal{K}_p$ is the collection of all infinite compact sets in F which contain only f_p as a cluster point. The existence of infinite compact sets can be established by considering the following subset $D(p)$ of F. Let p be an arbitrary but fixed point in P. Consider the subset $D(p)$ of F defined by,

$$D(p) = \{f_{[q]} \in F\colon q \in Q, f_{[q]}(q) \neq f_p(q), f_{[q]}(q') = f_p(q') \text{ for each } q' \neq q\}.$$

For each $q \in Q$, there exists a unique $f_{[q]} \in D(p)$ defined by,

$$f_{[q]}(q') = \begin{cases} |f_p(q')-1|, & \text{if } q = q', \\ f_p(q'), & \text{if } q \neq q'. \end{cases}$$

To see this, consider $q', q \in Q$ such that $q' \neq q$. Then $f_{[q']}(q') \neq f_p(q') = f_{[q]}(q')$. Since Q is infinite, $D(p)$ is infinite. The point f_p is the ideal point of a one point compactification of the discrete space $D(p)$. Also, it can be established that every compact set in F is a finite union of sets in $\mathcal{K}$.

Michael has defined the subspace G of F as follows: $G = F_p \bigcup \{f \in F\colon f(q) = 0$ except for finitely many $q \in Q\}$. Since G is a closed subspace of F G is normal. Michael has shown that G is a metacompact space which, is not collectionwise normal (i.e. G is not paracompact). Since G is metacompact, to establish the mesocompactness of G it will suffice to show that every compact set in G is finite. To this end, let K be any compact set in G. From the characterization of the compact sets in F, K is the finite union of sets in $\mathcal{K}$. Recall that $\mathcal{K}$ is the union of the collection of all finite sets in F and the collection of all infinite compact sets having one and only one f_p as a cluster point. Thus to show K is finite, it is sufficient to show that if K_t is any infinite compact set in F having only f_t as a cluster point, then $K_t \cap G$ is finite. To this end, we assume that for some $t \in P$, $K_t \cap G$ is infinite. Let $\{f_i\colon i \in I\}$ be any countable collection of distinct elements of $K_t \cap G$. Since $\{f_i\colon i \in I\} \subset K_t$ and f_t is the only cluster point of K_t, $\{f_i\colon i \in I\} \cap \{f_t\}$ is a closed subset of a compact set K_t. Now for each $i \in I$, let $Q_i = \{q \in Q\colon f_i(q) = 1\}$. Then by the definition of G, for each $i \in I$, Q_i is finite. Thus $Q_I = \bigcup \{Q_i\colon i \in I\}$ is at most countable. Since $Q_t = \{q \in Q\colon t \in q\}$ is uncountable and Q_I is at most countable, there exists $q' \in Q_t - Q_I$. Let $R = \{q'\}$. Since $f_t(q') = 1$ and $f_i(q') = 0$ for each

$i \in I$, $f_i \notin R(f_i)$ for each $i \in I$. Thus $\{R(f_t)\} \cup \{\{f_i\}: i \in I\}$ is an open covering of $\{f_i: i \in I\} \cup \{f_t\}$ which has no finite subcovering. Thus $\{f_i: i \in I\} \cup \{f_t\}$ is a non-compact, closed subset of K_t. But this contradicts the compactness of K_t. Thus $K_t \cap G$ is finite for each $t \in P$. Hence, every compact set in G is finite. Thus G is a normal mesocompact space which is not paracompact.

EXAMPLE 5.2. *A regular metacompact developable space which is not sequentially mesocompact.*

R. W. Heath [6, Example 1] has presented an example of a metacompact Moore space which is not screenable—hence not metrizable. This space can not be sequentially mesocompact, because if it were, then by Theorem 4.2 it would be metrizable.

EXAMPLE 5.3. *A compact-finite (cs-finite) open family in a k-space (sequential space) which is not closure preserving.*

Let X be the space defined by Franklin [5, Example 1.8]. Franklin establishes that X is a sequential space (thus a k-space) which is not a Fréchet space. In fact X is not even a k'-space.

The compact sets in X which contain 0 as a cluster point must be of the form $K \cup (L \cup \{0\})$, where L is a subsequence of $\left\{\frac{1}{n}: n \in N\right\}$ and K is a compact set in X such that 0 is not a cluster point of K in the usual sense. Hence, $\left\{\left(\frac{1}{n+1}, \frac{1}{n}\right): n \in N\right\}$ is a compact-finite (cs-finite) open family in X which is not closure preserving, since

$$\bigcup\left\{\left[\frac{1}{n+1}, \frac{1}{n}\right]: n \in N\right\} = (0, 1] \neq [0, 1] = \mathrm{Cl}\left(\bigcup\left\{\left(\frac{1}{n+1}, \frac{1}{n}\right): n \in N\right\}\right).$$

EXAMPLE 5.4. *A compact-finite (cs-finite) open family in a k'-space (Fréchet space) which is not locally finite.*

Let Q' be the space defined by Franklin [5, Example 1.11]. The quotient mapping $f: Q \to Q'$ may be defined by $f(r) = r$ for each $r \in Q - I$ and $f(n) = 0$ for each $n \in I$. Q' is not first countable at the point 0 and is not locally compact at the point 0. Franklin states that Q' is a Fréchet space. Accordingly, Q' is a k'-space. The family $\{(n, n+1): n \in I\}$ is a compact-finite (thus cs-finite) family, and every open neighborhood of 0 intersects every set $(n, n+1)$. Hence, $\{(n, n+1): n \in I\}$ is not locally finite at 0.

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