Chains of simple closed curves and a dogbone space

by

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1. Introduction. R. H. Bing in [4] presented an example of an upper semicontinuous decomposition of $E^3$ into points and tame arcs, Bing's dogbone space, that is not topologically $E^3$. In [2], a second dogbone space, resulting from a simpler construction than that of Bing's dogbone space, was shown to be topologically different from $E^3$ but the proof could not be easily modified to apply to a third dogbone space, also presented in [2], resulting from an apparently minor change in the construction. In this paper, we prove some theorems about linking simple closed curves and use them to show that this third dogbone space is not topologically $E^3$.

It will be assumed where necessary or convenient that all embedded complexes are triangulated and polyhedral and any two are in relative general position and all homeomorphisms are piecewise linear.

The standard definitions and basic results employed will be those of Hocking and Young [6].

After Casler [3], if $N$ is a positive integer, $N\alpha$ will denote a sequence of positive integers $J(1), ..., J(N)$, and if $r$ is a positive integer, the sequence $J(1), ..., J(N)$, $r$ will be denoted by $N\alpha$, $r$. If $N = 0$, $N\alpha = 0$ and $N\alpha$, $r = r$.

2. Chains of simple closed curves. The concept of linking of simple closed curves will be that of [3], namely, two simple closed curves $Y_1$ and $Y_2$ link if and only if there is a two complex $Y_1\delta$ with boundary $Y_1$ and $Y_2$ intersects $Y_1\delta$ an odd number of times.

A simple chain $\zeta$ is a collection $L_1, ..., L_N$, $N \geq 3$, of simple closed curves which can be numbered so that $L_i$ links only $L_{i-1}$, $L_{i+1}$ links only $L_{i-2}$ and if $i \neq 1, N$, $L_i$ links only $L_{i-1}$ and $L_{i+2}$. A closed chain $\zeta$ is a collection $L_1, ..., L_N$, $N \geq 3$, of simple closed curves which can be numbered so that each $L_i$ links only $L_{i-1}$ and $L_{i+1}$, where subscripts are taken modulo $N$. A simple closed curve in a chain is called a link.
The following is well-known:

Theorem 1. Suppose $P$ is a topological cube in $E^3$ and $f$ is a homeomorphism of $S^2$ into $E^3$. Then each component of $P \cap f(S^2)$ separates $P$ into exactly two components.

We paraphrase Theorem 3 of [4] by Bing:

Theorem 2. Suppose $L_1$ and $L_2$ are two linking simple closed curves in the interior of a topological cube $P$ in $E^3$ and $f$ is a homeomorphism of $S^2$ into $E^3$. Then, for each component $M$ of $P \cap f(S^2)$, there is a component of $P - M$ that intersects both $L_1$ and $L_2$.

We prove:

Theorem 3. Suppose $\zeta$ is a simple chain in the interior of a topological cube $P$ in $E^3$, $f$ is a homeomorphism of $S^2$ into $E^3$, and some component $M$ of $P \cap f(S^2)$ separates two links of $\zeta$. Then some link of $\zeta$ intersects $M$.

Proof. Suppose $M$, a component of $P \cap f(S^2)$ separates links $L_i$ and $L_{i+1}$ of $\zeta$. Denote the two components of $P - M$ by $A$ and $B$ with $L_i \subset A$ and $L_{i+1} \subset B$. Let $T$ be the least integer such that $i < T < i + j$ and $L_T \cap B \neq \emptyset$. Then $L_{T-1} \subset A$. By Theorem 2, $L_T \cap A \neq \emptyset$. Thus, $L_T$ intersects $M$ and the proof of Theorem 3 is completed.

Since a closed chain $\zeta = \{L_1, ..., L_n\}$ may be expressed for each integer $j$ as the sum of two simple chains $\{L_1, ..., L_j\} \cup \{L_{j+1}, ..., L_n\}$, we apply Theorem 3 twice and obtain:

Theorem 4. Suppose $\zeta$ is a closed chain in the interior of a topological cube $P$ in $E^3$, $f$ is a homeomorphism of $S^2$ into $E^3$ and some component $M$ of $P \cap f(S^2)$ separates two links of $\zeta$. Then two links of $\zeta$ intersect $M$.

The proof of the following is inspired by the proof of Theorem 5 of [4] by Bing:

Theorem 5. Suppose $\zeta$ is a closed chain in the interior of a topological cube $P$ in $E^3$, $f$ is a homeomorphism of $S^2$ into $E^3$, and $\{U_i\}$, $1 < T$, is the set of components of $P - f(S^2)$, and $\zeta \cap U$. Then some $U_i$ intersects $L_1$ and two other elements of $\zeta$.

Proof. Let $Z$ be a continuum such that

(a) $Z$ is composed of closures of elements of $\{U_i\}$, $1 < T$.
(b) $Z$ intersects $L_1$ and two other elements of $\zeta$.
(c) no proper subcontinuum of $Z$ satisfies (a) and (b).

We show $Z$ contains exactly one element of $\{U_i\}$, $1 < T$. For, suppose $Z$ contains two elements of $\{U_i\}$, $1 < T$. Then, $Z$ is the sum of two proper subcontinua $Z_1$ and $Z_2$, both composed of closures of elements of $\{U_i\}$, $1 < T$, and $Z_1 \cap Z_2 = M$ for some component $M$ of $P \cap f(S^2)$. Suppose $Z$ intersects $L_1$ and $L_2$. We show a contradiction when we show that the assumption that $M$ does not separate any pair of $L_1$, $L_2$ and $L_3$ violates (c) and the assumption that $M$ separates some pair of $L_1$, $L_2$ and $L_3$ violates (c).

Suppose $M$ does not separate any pair of $L_1$, $L_2$ and $L_3$. Then, each of $L_1$, $L_2$ and $L_3$ intersects $M$ or some one of $L_1$, $L_2$ and $L_3$ does not intersect $M$. If each of $L_1$, $L_2$ and $L_3$ intersects $M$, then each of $L_1$, $L_2$ and $L_3$ intersects both $L_2$ and $Z_1$, a violation of (c). If some one of $L_1$, $L_2$ and $L_3$, say $L_1$, does not intersect $M$, then $L_1$ intersects exactly one of $Z_1$ and $Z_2$, say $Z_1$. Then, each of $L_2$ and $L_3$ must intersect $Z_1$ since, if $L_1$ intersects $Z_1$, only $L_3$ would not intersect $M$ and $M$ would separate $L_1$ and $L_3$. Thus, each of $L_1$, $L_2$ and $L_3$ would intersect $Z_1$, a violation of (c). Thus, the assumption that $M$ does not separate any pair of $L_1$, $L_2$ and $L_3$ violates (c).

Suppose $M$ separates some pair of $L_1$, $L_2$ and $L_3$. If $M$ separates $L_1$ and $L_2$, and, say, $L_3$, by Theorem 4 two links $L_2$ and $L_3$ of $\zeta$ intersect $M$ and $L_2 \neq L_3$, $L_3$ since $L_1$ does not intersect $M$. Thus, $L_1$, $L_2$ and $L_3$ intersect one of $Z_1$ and $Z_2$ and $Z_2$, a violation of (c). If $M$ separates $L_1$ and $L_3$, by Theorem 4 two links $L_1$ and $L_3$ of $\zeta$ intersect $M$ and one of them, say, $L_1$, is not $L_2$, then $L_2$, $L_3$ and one of $L_1$ and $L_3$ would intersect one of $Z_1$ and $Z_2$, a violation of (c). Thus, the assumption that $M$ separates some pair of $L_1$, $L_2$ and $L_3$ violates (c).

Thus, the promised contradiction has been demonstrated and the proof of Theorem 5 is complete.

Theorem 6 is the best result obtainable since for every integer $n > 3$, it is possible to construct a closed chain $\zeta$ of $n$ elements in the interior of a topological cube $P$ in $E^3$ and a homeomorphism $f$ of $S^2$ into $E^3$ such that every component of $P - f(S^2)$ intersects at most three elements of $\zeta$.

A result needed later is:

Theorem 6. Suppose $\zeta$ is a closed chain in the interior of a topological cube $P$ in $E^3$ and $f$ is a homeomorphism of $S^2$ into $E^3$. Then either

(i) some component of $P - f(S^2)$ separates two elements of $\zeta$; or,
(ii) some component of $P - f(S^2)$ intersects each element of $\zeta$.

Proof. We suppose (i) is false and show (ii) is true. If (i) is false, then no component of $P - f(S^2)$ separates any two elements of $\zeta$ and, hence, for any two elements of $\zeta$, there is a component of $P - f(S^2)$ intersecting both. Thus, we complete the proof of Theorem 6, by applying the following theorem by Bing [4, Theorem 5]:

Suppose $U$ is the interior of a topological cube, $Y$ is a collection of bounded continua in $U$, and $M$ is a compact 2-manifold with boundary such that for each pair of elements of $Y$, there is a component of $U - M$ intersecting both of those elements. Then there is a component of $U - M$ intersecting each element of $Y$.
3. Topological Figure Eights and Property R. An arc \( l \) is the image of the unit interval \( I = [0, 1] \) under a homeomorphism which will also be denoted by \( l \). The end-points of an arc \( l \) are \( l(0) \) and \( l(1) \) and \( l \) may be written \( l(0)(1) \). A \( p \)-od \( k \) is the union of \( p \) arcs \( l_i \) such that if \( i \neq j \), \( l_i \cap l_j = l_i(0) = l_j(0) \); the center of \( k \) is \( l_i(0) \) and the set of end-points of \( k \) is \( \{ l_i(1) : i = 1, ..., p \} \).

Suppose \( l \) is an arc and \( A \) and \( B \) are sets. The integer \( N \) is an Intersection Number of \( l \) with respect to \( A \) and \( B \) if and only if there are \( N+1 \) points \( t_0, ..., t_N \) in \( I \) such that \( 0 = t_0 < ... < t_N = 1 \) and for each \( x \), \( l([V_x, V_{x+1}]) \) intersects at most one of \( A \) and \( B \).

A topological figure eight has Property R with respect to sets \( A \) and \( B \) if and only if for every two points \( p \) and \( q \) in opposite loops there is an arc \( l \) in it from \( p \) to \( q \) and 2 is an Intersection Number of \( l \) with respect to \( A \) and \( B \).

![Fig. 1](image)

For the remainder of this section, we adopt the notation of Figure 1. As in Figure 1, let \( P \) be a topological cube in \( F \) and \( L_1, ..., L_4 \) a collection of simple closed curves in \( F \) linked as shown. For each \( i = 1, ..., 4 \), \( L_i \) is the sum of two arcs from \( a_i \) to \( b_i \); which intersect only at their endpoints; to distinguish these arcs, we arbitrarily designate one \( +a_i b_i \) and the other \( -a_i b_i \). \( \alpha_i \) is an arc with \( c_i \) only on Boundary \( P \). The arcs \( \alpha_i \) and \( \alpha_4 \) are in the complement of \( \text{Interior}(P) \) with end-points only on Boundary \( P \).

The main result of this section is

**Theorem 4.** Suppose \( f \) is a homeomorphism of \( S^2 \) into \( F \), \( A \) and \( B \) are closed disjoint subsets of \( f(S^2) \), \( P \cap f(S^2) \subset A \cup B \), for each \( i = 1, ..., 4 \), each arc \( \pm a_i b_i \) intersects at most one of \( A \) and \( B \), and \( f(S^2) \cap (\alpha_1 \cup \alpha_3) \cup (\alpha_2 \cup \alpha_4) \cup \Omega = \emptyset \). Then there is a topological figure eight \( \Phi \) in Interior \( P \) such that \( \alpha_1 \cup \alpha_3 \cup \alpha_2 \cup \alpha_4 \cup \Omega \) such that \( \alpha_i \) and \( \alpha_4 \) are in opposite loops of \( \Phi \) and \( \Phi \) has Property R with respect to \( A \) and \( B \).

**Proof.** From Theorem 6 we have

(i) some component of \( P \cap f(S^2) \) separates two of \( L_1, ..., L_4 \) or

(ii) some component of \( P \cap f(S^2) \) intersects each of \( L_1, ..., L_4 \).

We begin the argument for (i) by supposing that some component \( M \) of \( P \cap f(S^2) \) separates two of \( L_1, ..., L_4 \). The component \( M \) cannot separate \( L_i \) from \( L_j \) or \( L_k \) from \( L_l \) or \( L_m \). Thus, we assume that \( M \) separates \( L_i \) from \( L_j \). Then, by Theorem 5, some component \( U_i \) of \( P \cap f(S^2) \) intersects \( L_i \), \( L_j \) and \( L_k \) and some component \( U_j \) of \( P \cap f(S^2) \) intersects \( L_i \), \( L_j \) and \( L_k \).

Select a point \( p_i \) in \( U_i \) and for \( i = 1, 3, 4 \), construct an arc \( p_i \alpha_i \) by constructing an arc in \( U_i \) from \( p_i \) to \( L_i \); then along \( L_i \) to \( a_i \) so as to intersect at most one of \( A \) and \( B \) and finally along \( a_i c_i \). There results a 3-od \( k_i \) in \( P \) such that \( p_i \alpha_i \) is the center of \( k_i \), the end-points of \( k_i \) are \( a_i \alpha_i \) and \( a_i \), and each arc \( p_i \alpha_i \), \( i = 1, 3, 4 \), intersects at most one of \( A \) and \( B \). Similarly, select a point \( p_j \) in \( U_j \) and construct a 3-od \( k_j \) with center \( p_j \) and end-points \( a_j \alpha_j \) and \( a_j \) such that each arc \( p_j \alpha_j \), \( i = 2, 3, 4 \), in \( b_i \) intersects at most one of \( A \) and \( B \). Let \( K = k_1 \cup k_3 \cup k_2 \cup k_4 \).

A copy of \( K \) is shown in Figure 2a (see p. 136). We show how to construct the desired figure eight \( \Phi \) by selecting, except for one case, arcs \( p_i \alpha_i \) in \( K \) or arcs each of which are so close to some arc \( p_i \alpha_i \) that the selected arc intersects \( A \) or \( B \) only if \( p_i \alpha_i \) intersects \( A \) or \( B \). It is always true that \( (\alpha_i \cup \alpha_4) \cap (A \cup B) = \emptyset \). The cases where some \( p_i \alpha_i \) does not intersect \( A \cup B \) may be neglected. Thus, we have six arcs of the form \( p_i \alpha_i \), each of which intersects at most one of \( A \) and \( B \), a total of 64 cases. However, we may assume \( p_i \alpha_3 \) intersects only \( A \) and \( p_i \alpha_4 \) intersects only \( B \) without loss of generality.

For the case 13, the solutions are obtainable from of arcs in \( K \) or arcs near \( K \). To solve case 13, we use a theorem by Bing [4, Theorem 6], paraphrased for our purposes:

**Suppose A and B are two mutually exclusive closed subset of a topological cube \( P \) and \( p_1 \alpha_1, q_1 \alpha_2 \) and \( q_2 \alpha_3 \) are homotopic arcs in \( P \) such that \( q_1 \alpha_1, q_2 \alpha_3 \subset A \cup B \). Then, there is an arc in \( P \) with end-points \( c_1 \) and \( c_1 \) such that \( 1 \cap (A \cup B) = \emptyset \).**
The arc $l$ allows the solution of case 13 and the argument when some component $M$ of $P - f(S^8)$ separates $L_1$ and $L_2$ is complete. The argument when some component of $P - f(S^8)$ separates $L_3$ and $L_4$ follows by symmetry, thus completing the argument for (i).

The argument for (ii) follows readily since it may be assumed that the centers $p_1$ and $p_2$ of the 3-ods $k_1$ and $k_2$ of (i) are in the same component of $P - f(S^8)$. Thus, the proof of Theorem 7 is complete.

4. A dogbone space that is not topologically $E^3$. To construct the dogbone space of this paper, let $A_4$ be a solid double torus in $E^3$, as in Figure 4. Embed a cube $U_1$ in the top of $A_4$ and cubes $D_1$ and $D_2$ in the bottom of $A_4$. Then, embed solid double tori $A_5, ..., A_4$, linked as indicated, in $A_4$, although each of $A_5, ..., A_4$ is shown as a finite graph, it is topologically equivalent to $A_4$. For each $i = 1, ..., 4$, Closure $\{A_i - (U_i \cup D_1 \cup D_2)\}$ is a topological cube and the intersection of Interior $(A_i)$ with any horizontal plane is an open disk or the sum of two disjoint open disks.

For each $i = 1, ..., 4$, cubes $U_{i1}, D_{i2}$ and $D_{i2}$ and solid double tori $A_{i1}, ..., A_{i4}$ are embedded in $A_i$ such that there is a homeomorphism of $E^3$ onto itself which is the identity on the complement of some open set containing $A_i$ and takes $A_i$ onto $A_i$, $U_i$ onto $U_{i1}$, and $D_i$ onto $D_{i2}$, $f = 1, 2$. Let this process be continued; succeeding steps of the construction may be described inductively.

Let $M = A_4 \cap \bigcup A_i \cap \bigcup A_{i1} \cap \bigcup A_{i2} \cap \bigcup ...$. Let $G$ be the set whose elements are components of $M$ and one-point subsets of $E^3 - M$. Then, $G$ is an upper semicontinuous decomposition of $E^3$ into tame arcs and one-point sets. Let $E^3/G$ denote the associated decomposition space, the dogbone space of this paper. We show:

**Theorem 8.** $E^3/G$ is not topologically $E^3$. 

![Image of chains of simple closed curves and a dogbone space]
Proof. To assist in the proof of Theorem 5, we state some definitions and prove some lemmas.

Let $C$ denote $U_1 \cup D_1 \cup D_1 \cup \sum A_i$, $i = 1, \ldots, 4$. Then $C$ is a topological cube with handles. As in Figure 5, let $I_n$ be a central curve of $C$ consisting of points $a_1$, $b_1$ and $c_1$ and arcs $a_2$, $b_2$, $c_2$, $a_3$, $b_3$, $c_3$, where the end-points of $a_i$ are $a_i$ and $b_i$ if $A_i$ intersects $U_i$ and $D_i$. Similarly, for a fixed sequence $N_{n+1}$, $N_{n+2} \cup D_N \cup D_{n+2} \cup \sum A_N$, $i = 1, \ldots, 4$, is a cube with handles with central curve $I_{n+2}$.

Fig. 5

Fig. 6

Also in Figure 5, let $P_1$, $P_2$, $P_3$, $P_4$ be disks in $A_4$ such that for each $i$, $P_i \cap \text{Boundary}(A_4) = \text{Boundary}(P_i)$. We may regard $P_i \cup P_3 \cup P_3 \cup P_i$ as the intersection of $A_4$ with a homeomorphic image of $S^3$.

The statement and proof of the following lemma is identical to that of the proof of Lemma 1 for Theorem 5 of [3].

**Lemma 1.** Suppose $g$ is a continuous function of $A_4$ into $A_4$ which is homotopic to the identity by a homotopy $G$ which is fixed on Boundary$(A_4)$. Then, for some $i = 1, \ldots, 4$, $g(a_i)$ intersects both $P_i \cup P_3$ and $P_3 \cup P_i$.

We prove:

**Lemma 2.** Suppose $N$ is a positive integer and $F$ is a homeomorphism of Boundary$(A_4) \cup \sum P_i$, $i = 1, \ldots, 4$, into $A_4$ which satisfies

(i) $F$ is the identity on Boundary$(A_4)$,

(ii) each $a_{N+2}$ in each $I_{N+2}$ intersects at most one on $F(P_i \cup P_3)$ and $F(P_3 \cup P_i)$.

Then there is a homeomorphism $h$ of Boundary$(A_4) \cup \sum P_i$, $i = 1, \ldots, 4$, into $A_4$ which satisfies

(i) $h$ is the identity on Boundary$(A_4)$,

(ii) each $a_{N+1}$ in each $I_{N+1}$ intersects at most one of $h(P_i \cup P_3)$ and $h(P_3 \cup P_i)$.

Proof. Suppose $N$ is a positive integer and $F$ is a homeomorphism which satisfy the hypotheses of the lemma. Let $(N-1)$ be a fixed sequence. The solid double torus $A_{N-1}$ is shown in Figure 6. For clarity, only the details of $A_{N-1}$ and $I_{N-1}$ are shown and possible intersections of $F(P_i \cup P_3 \cup P_3 \cup P_i)$ with $a_{N-1}$, $i = 1, \ldots, 4$, are indicated. It may be assumed that $F(P_i \cup P_3 \cup P_3 \cup P_i)$ does not intersect $a_{N-1}$, $i = 1, \ldots, 4$, since $F(P_i \cup P_3 \cup P_3 \cup P_i)$ could be adjusted in a neighborhood of $a_{N-1}$, $i = 1, \ldots, 4$, without adding intersections to any arc $a_{N-1}$. Thus, a cube $Y$ may be constructed in $A_{N-1}$ such that $Y$ contains $a_{N-1}$, $Y \cap I_{N-1}$ is a 4-od and $Y$ does not intersect $F(P_i \cup P_3 \cup P_3 \cup P_i)$. Replace $Y \cap I_{N-1}$ by two 3-ods with a single common end-point, expand $Y$ by a homeomorphism $h_1$ of $E^3$ onto $E^3$ which is the identity on the complement of Interior$(A_{N-1})$ and arrive at the situation of Figure 7 (see p. 140).

If a cube $Y'$ similar to $Y$ is constructed in $A_{N-1}$, $Y' \cap I_{N-1}$ is replaced by two 3-ods and $Y'$ is expanded by a homeomorphism $h_2$ of $E^3$ onto $E^3$ which is the identity on the complement of Interior$(A_{N-1})$, there results four simple closed curves which link in $D_{N-1}$ as shown in Figure 8 (see p. 140). For $i = 1, 2$ each pair of simple closed curves in $A_{N-1}$ is connected by an arc in $A_{N-1}$ which does not intersect $h_i h_j F(P_i \cup P_3 \cup P_3 \cup P_i)$ for $i \neq j$. Let $I_i$ denote the closure of each arc in the complement of $D_{N-1}$. Each simple closed curve in $D_{N-1}$ is the sum of two arcs each of which intersects at most one of $h_i h_j F(P_i \cup P_3)$ and $h_i h_j F(P_3 \cup P_i)$ in $W_i$. We apply Theorem 7 to obtain a topological figure of eight $F_i$, shown in Figure 9 (see p. 140), composed of 4-ods $k$ and the area $I_i$ and $I_i$ such that $I_i$ and $I_i$ are in opposed loops of $F_i$ and $F_i$ has Property $E$ with respect to $h_i h_j F(P_i \cup P_3)$ and $h_i h_j F(P_3 \cup P_i)$.

By a homeomorphism $h_3$ of $E^3$ onto $E^3$ which is the identity on the complement of a small neighborhood $W_i$ of $D_{N-1}$, each component of $h_i h_j F(P_i \cup P_3 \cup P_3 \cup P_i)$ may be pushed along the arc of $k$ until it intersects the complement of $D_{N-1}$ so that $h_i h_j h_3 F(P_i \cup P_3 \cup P_3 \cup P_i) = 0$ and $F_i$ has Property $E$ with respect to $h_i h_j h_3 F(P_i \cup P_3)$ and $h_i h_j h_3 F(P_3 \cup P_i)$.
The 4-od $h_1$ is contained in $D_{\mathcal{A}^{\mathcal{A}_{2}}_{\mathcal{A}^{\mathcal{A}_{2}}}1}$ and has endpoints only on Boundary($D_{\mathcal{A}^{\mathcal{A}_{2}}_{\mathcal{A}^{\mathcal{A}_{2}}}1}$). Let $W_i$ be a neighborhood of $D_{\mathcal{A}^{\mathcal{A}_{2}}_{\mathcal{A}^{\mathcal{A}_{2}}}1}$ contained in $W_j$. The cutting and sewing process of [1] may be applied which results in a homeomorphism $h_i$ of $P_i \cup P_j \cup P_k \cup P_l$ into $A_4$, such that $\mathcal{A}^{\mathcal{A}_{2}}_{\mathcal{A}^{\mathcal{A}_{2}}}1 \cap \mathcal{A}^{\mathcal{A}_{2}}_{\mathcal{A}^{\mathcal{A}_{2}}}1 \cap \mathcal{A}^{\mathcal{A}_{2}}_{\mathcal{A}^{\mathcal{A}_{2}}}1 = \emptyset$ and for each $i$, $h_i$ is the identity on Boundary($P_i$), $h_i$[Interior($P_i$)] $\subset$ Interior($A_4$), $h_i(P_i) = W_i \subset h_i h_i(P_i)$, and $\Phi_i$ has Property $E$ with respect to $h_i(P_i) \cup P_j$ and $h_i(P_j) \cup P_i$. An important point is that for each sequence $(N-1)\alpha, j \neq (N-1)\alpha, 1$ or $(N-1)\alpha, 2$, $h_i(P_i)$ intersects an arc $a_{i-1,2}$ in $\Gamma_{i-1,2}$ only if $P_i$ intersects $a_{i-1,2}$. Since $h_i(P_j) = W_j \subset h_i h_i(P_j)$ and $h_i h_i(P_j)$ is the identity on the complement of $A_{i-1,2} \cup A_{i-2,3}$, extend $h_i$ to a homeomorphism of Boundary($A_4$) $\cup \sum P_i$, $i = 1, \ldots, 4$, by defining $h_i$ as the identity on Boundary($A_4$).

Let $h_i$ be a homeomorphism of $E^3$ onto $E^3$ which is the identity on the complement of Interior($A_{i-1,2} \cup A_{i-2,3} \cup D_{\mathcal{A}^{\mathcal{A}_{2}}}1$) and, as shown in Figure 10, expands Interior($D_{\mathcal{A}^{\mathcal{A}_{2}}}1$) so that $h_i$[Interior($D_{\mathcal{A}^{\mathcal{A}_{2}}}1$)] contains $(a_{i-1,2} \cup a_{i-2,3}) - U_{\mathcal{A}^{\mathcal{A}_{2}}}1$. The closure of $h_i[\Phi_i - D_{\mathcal{A}^{\mathcal{A}_{2}}}1]$ is composed of two arcs, $h_i[\Phi_i]$ and $h_i[\Phi_i]$. For each $i = 1, 2$, extend $h_i[\Phi_i]$ to a point in the interior of the component of $U_{\mathcal{A}^{\mathcal{A}_{2}}}1 - h_i[\Phi_i]$ it intersects and from this point construct an arc in $U_{\mathcal{A}^{\mathcal{A}_{2}}}1 - h_i[\Phi_i]$ to $\Gamma_{\mathcal{A}^{\mathcal{A}_{2}}}1$ - Boundary($U_{\mathcal{A}^{\mathcal{A}_{2}}}1$). Thus, a finite graph $\mathcal{A}_4$ composed of two simple closed curves $s_1$ and $s_2$ joined by a connecting arc has been constructed in $A_{i-1,2} \cup A_{i-2,3} \cup D_{\mathcal{A}^{\mathcal{A}_{2}}}1$. The simple closed curves $s_1$ and $s_2$ are linked and each links $A_{i-1,2}$ and $A_{i-2,3}$ in Interior($U_{\mathcal{A}^{\mathcal{A}_{2}}}1$). That part of $\mathcal{A}_4$ in the complement of $U_{\mathcal{A}^{\mathcal{A}_{2}}}1$, which is also part of the connecting arc in the component of $U_{\mathcal{A}^{\mathcal{A}_{2}}}1$ is $(a_{i-1,2} \cup a_{i-2,3}) - U_{\mathcal{A}^{\mathcal{A}_{2}}}1$. The connecting arc does not intersect $h_i[\Phi_i]$, hence $h_i[\Phi_i]$, $h_i[\Phi_i]$, and $h_i[\Phi_i]$, $h_i[\Phi_i]$ are each arcs in $\mathcal{A}_4$ with respect to $h_i[\Phi_i]$, $h_i[\Phi_i]$, and $h_i[\Phi_i]$, $h_i[\Phi_i]$, where $p$ and $q$ are points in, respectively, $s_1$ and $s_2$, there is an arc $pq$ in $\mathcal{A}_4$ such that $2$ is an Intersection Number of $pq$ with respect to $h_i[\Phi_i]$, $h_i[\Phi_i]$, and $h_i[\Phi_i]$, $h_i[\Phi_i]$. If $(N-1)\beta, j \neq (N-1)\alpha, 1$ or $(N-1)\alpha, 2$, $h_i[\Phi_i]$ intersects an arc $a_{i-1,2}$ in $\Gamma_{\mathcal{A}^{\mathcal{A}_{2}}}1$ only if $\mathcal{A}_4$ intersects $a_{i-1,2}$, hence $h_i$ is the identity on the complement of Interior($A_{i-1,2} \cup A_{i-2,3} \cup D_{\mathcal{A}^{\mathcal{A}_{2}}}1$).

Thus far, the definition of homeomorphisms and construction has been done relative to $A_{i-1,2}$, $A_{i-2,3}$, and $D_{\mathcal{A}^{\mathcal{A}_{2}}}1$. A similar definition of homeomorphisms and construction is to be done relative to $A_{i-1,2}$, $A_{i-2,3}$, and $D_{\mathcal{A}^{\mathcal{A}_{2}}}1$ resulting, as shown in Figure 11, in a homeomorphism $h_4$ of Boundary($A_4$) $\cup \sum P_i$, $i = 1, \ldots, 4$, into $A_4$, which is the identity on Boundary($A_4$), and a finite graph $\mathcal{A}_4$ in $A_{i-1,2} \cup A_{i-2,3} \cup D_{\mathcal{A}^{\mathcal{A}_{2}}}1$. In the complement of $A_{i-1,2} \cup A_{i-2,3} \cup D_{\mathcal{A}^{\mathcal{A}_{2}}}1$, for each $i$, $h_4(P_i)$ is contained in $h_4 h_4(P_i)$. Thus, for $(N-1)\beta, j \neq (N-1)\alpha, n, n = 1, \ldots, 4$, each arc $a_{i-1,2}$, $i$
sects $h_0(P_0)$ only if $F(P_0)$ intersects $d_{2N-(N-1)a}$. The finite graph $\Psi_m$ is composed of two simple closed curves $s_1$ and $s_2$ joined by a connecting arc. The connecting arc does not intersect $h_0(F_1 \cup F_2 \cup F_3 \cup F_4)$. If $p$ and $q$ are points in, respectively, $s_1$ and $s_2$ there is an arc $pq$ in $\Psi_m$ such that 2 is an Intersection Number of $pq$ with respect to $h_0(F_1 \cup F_2)$ and $h_0(F_2 \cup F_3)$.

All four simple closed curves in $\Psi_1$ and $\Psi_2$ are linked in Interior ($U_{(N-a)}$). The sum of $\Psi_1$ and $\Psi_2$ in the complement of $U_{(N-a)}$, which is also the sum of the connecting arcs in the complement of $U_{(N-b)}$, is $I_{(N-a)} - U_{(N-a)}$.

Since the four simple closed curves in $\Psi_1$ and $\Psi_2$ are linked in $U_{(N-a)}$, by Theorems 3 and 5 of [3], there is a graph $\Psi$ of $U_{(N-a)}$ such that $\Psi$ is the unique simple closed curve in $U_{(N-a)}$ such that 2 is an Intersection Number of $pq$ with respect to $h_0(F_1 \cup F_2)$ and $h_0(F_2 \cup F_3)$. We select a point $p_i$ in $U_{(N-a)}$ such that $p_i$ is an intersection point of $U_{(N-a)}$ and $U_{(N-a)}$. The sum of the curves $\pm p_i F_{1,2,3,4}$, $i = 1, \ldots, 4$, is a figure eight, $\Phi$. By a homeomorphism $h_0$ of $F^0$ onto $F^0$, we can find an identity on the complement of $U_{(N-a)}$ such that there is a homeomorphism $h_0$ of $F^0$ onto $F^0$ which satisfies

(i) $\Phi$ is isotopic to the identity by an isotopy which is fixed on the complement of $U_{(N-a)}$, and

(ii) each $g(\alpha_{a,b})$ in each $\alpha_i$ intersects at most one of $P_i \cup P_i$ and $P_i \cup P_i$.

**Definition.** If $F^0 \mid G$ is topologically $E^0$, the shrinking number $L$ of $F^0 \mid G$ is the least integer such that there is a homeomorphism $h_0$ of $F^0$ onto $F^0$ which satisfies

(i) $\Phi$ is isotopic to the identity on $U_{(N-a)}$, and

(ii) each $g(\alpha_{a,b})$ in each $\alpha_i$ intersects at most one of $P_i \cup P_i$ and $P_i \cup P_i$.

**Definition.** If $F^0 \mid G$ is topologically $E^0$, the shrinking number $L$ of $F^0 \mid G$ is the least integer such that there is a homeomorphism $h_0$ of $U_{(N-a)}$ such that there is a homeomorphism $h_0$ of $F^0$ onto $F^0$ which is isotopic to the
Some characterizations of para-compactness in $k$-spaces

by

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1. Introduction. Paracompact spaces and $k$-spaces both have the distinction of being simultaneous generalizations of metric and compact spaces. The purpose of this paper is to present some of the interactions between these seemingly unrelated notions. Throughout this paper the underlying topological structures will be the $k$-spaces (e.g. first countable, Fréchet, sequential, locally compact, $k'$-space, and $k$-space). Specifically, for spaces within the class of $k$-spaces, those with the paracompact property are characterized. For this purpose, four generalizations of paracompactness are introduced. These generalizations are defined in terms of refinements which have some finiteness condition on the elements of a given collection of subsets. With the additional structure of the $k$-spaces these refinements have the properties required for the characterizations. These characterizations are given in § 3 and are summarized in the implication diagram which appears in Figure 3.2.

The fundamental notions used in this study are developed in § 2. Applications of these concepts to metrizability of spaces are given in § 4. Some examples are presented in § 5. The term "space" will mean a Hausdorff topological space and the term "family" will mean a family of subsets.

2. Preliminaries. The fundamental notions involved in this work will be developed in the general setting of $F$-hereditary collections and weak topology in the sense of Whitehead. A family $\mathcal{K} = \{K_a : a \in A\}$ in a space $X$ is said to be an $F$-hereditary collection provided: (i) $\mathcal{K}$ is a covering of $X$ and (ii) for each closed set $F \subseteq X$, $F \cap K_a$ is $\mathcal{K}$ for each $a \in A$. Some mapping properties of collections with property (ii) were investigated by Remmert [11]. For all $F$-hereditary collections of interest, the singletons are in $\mathcal{K}$, and (i) is satisfied. For instance, the collection of all compact

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