Exact loop space sequences

by

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1. Introduction. Let (E, e_0) and (B, b_0) be topological spaces with base points and $p\colon (E, e_0) \to (B, b_0)$ a continuous map. The map p determines an infinite sequence

$$\ldots\ \Omega^n E \to \Omega^n B \to \Omega^{n-1}\Sigma p \to \Omega^{n-1}E \to \ldots$$

where $\Sigma p = \{(e, a) \in E \times B^I\colon p(e) = a(0) \text{ and } a(1) = b_0\}$ is the mapping track of p, ΩE is the space of loops in E with base point e_0 and $\Omega^n E = \Omega\Omega^{n-1}E$ for $n \geqslant 2$ with base point the degenerate loop $c^n(e_0)$. This sequence is exact up to homotopy and satisfies topological splitting theorems analogous to the algebraic theorems for the splitting of sequences of groups and homomorphisms.

If the fiber structure (E, p, B) has the weak covering homotopy property, the loop space sequence of p determines the homotopy sequence of the fibration. The homotopy sequence of a triplet is also obtained as a special case. The splitting theorems for loop space sequences give generalizations of the standard direct sum theorems [3, pp. 150–153] for triplets and weak Hurewicz fibrations.

2. Exact sequences.

DEFINITION. A pair (E, p, B) and (E', p', B) of fiber structures over the same base B have the *same homotopy type* or are *homotopy equivalent* means that there is a homotopy equivalence $f\colon E \to E'$ such that $p'f \sim p$ (homotopic).

DEFINITION. A sequence

$$A \xrightarrow{q} E \xrightarrow{p} B$$

of topological spaces with base points and continuous maps is *exact* means that

(1) the composition pq is null-homotopic (i.e. homotopic to the constant map whose only value is the base point of B); and

(2) for each space W and continuous map $h\colon W\to E$ such that ph is null-homotopic, there is a continuous map $h'\colon W\to A$ such that $qh'\sim h$.

DEFINITION. An *H-group* is a topological space X with a continuous multiplication $\cdot$ which is associative up to homotopy and has a homotopy unit and an inversion.

Note. The functions involved in this paper are not assumed to be base point preserving unless specifically stated. All function spaces are assigned the compact-open topology.

THEOREM 1. *Suppose that the sequence*

$$0\to A\xrightarrow{q}E\xrightarrow{p}B$$

is exact, p has a right homotopy inverse $\chi\colon B\to E$, A and E are H-groups with base points as homotopy units, and $q\colon A\to E$ is an H-homomorphism. If the map $g\colon E\to B$ defined by

$$g(e)=p\big(e\cdot j\chi p(e)\big)\qquad e\in E,$$

where j is the inversion on E, is null-homotopic and the map $m\colon B\times A\to B$ defined by

$$m(b,a)=p\big(q(a)\cdot\chi(b)\big)\qquad (b,a)\in B\times A$$

is homotopic to the projection π_1 on the first component, then (E,p,B) and $(B\times A,\pi_1,B)$ have the same homotopy type.

Proof. Define $f\colon E\to E$ by

$$f(e)=e\cdot j\chi p(e)\qquad e\in E\,.$$

Then $g=pf$ is null-homotopic so there is a map $f'\colon E\to A$ such that $qf'\sim f$. The exactness of the sequence implies that $f'q\sim \mathrm{id}_A$.

Define $G\colon E\to B\times A$ and $H\colon B\times A\to E$ by

$$G(e)=\big(p(e),f'(e)\big)\,,\qquad e\in E$$

$$H(b,a)=q(a)\cdot\chi(b)\,,\qquad (b,a)\in B\times A\,.$$

The function H has been used by Hilton and is known to be a weak homotopy equivalence [2, p. 104]. For the case being considered here, G and H are mutual homotopy inverses. To see this, observe that

$$HG=qf'\cdot\chi p\sim f\cdot\chi p\sim \mathrm{id}_E$$

and

$$\pi_1GH=pH\sim\pi_1\,.$$

If π_2 is the projection of $B\times A$ on A, then

$$\pi_2GH=f'H\sim f'qf'H\sim f'fH\sim f'(H\cdot j\chi m)\sim f'q\pi_2\sim\pi_2\,.$$

Since $\pi_1G=p$, it follows that (E,p,B) and $(B\times A,\pi_1,B)$ have the same homotopy type.

COROLLARY. *Let*

$$0\to A\xrightarrow{q}E\xrightarrow{p}B$$

be an exact sequence such that A, E and B are H-groups and p and q are H-homomorphisms. If $\chi\colon B\to E$ is a right homotopy inverse for p, then (E,p,B) and $(B\times A,\pi_1,B)$ are homotopy equivalent.

THEOREM 2. *Under the hypotheses of Theorem 1 or its Corollary, it follows that Ω^nE and $\Omega^nB\times\Omega^nA$ are H-isomorphic for each positive integer n.*

Proof. Let $G\colon E\rightleftarrows B\times A\colon H$ denote the homotopy equivalence pair given in the proof of Theorem 1 and let a_0 denote the homotopy unit of A. The map H induces an H-isomorphism ΩH from $\Omega(B\times A)$ into the space $\Omega\big(E,H(b_0,a_0)\big)$ of loops in E with base point $H(b_0,a_0)$.

For $e\in E$, let $[e]$ denote the path component to which e belongs. Observe that

$$\begin{aligned}[e_0]&=[HG(e_0)]=[qf'(e_0)\cdot\chi p(e_0)]=[qf'q(a_0)\cdot\chi(b_0)]\\&=[q(a_0)\cdot\chi(b_0)]=[H(b_0,a_0)]\end{aligned}$$

so that e_0 and $H(b_0,a_0)$ belong to the same path component of E.

We then have the following sequence of H-isomorphisms:

$$\Omega B\times\Omega A\approx\Omega(B\times A)\approx\Omega\big(E,H(b_0,a_0)\big)\approx\Omega E\,.$$

An inductive application of this argument establishes the theorem.

DEFINITION. For the given map $p\colon(E,e_0)\to(B,b_0)$, define $q\colon\Omega B\to\to\Sigma p$ by

$$q(\beta)=(e_0,\beta)\,,\qquad \beta\in\Omega B\,.$$

The map $p\colon E\to B$ and the projection $\pi_1\colon\Sigma p\to E$ induce maps $\Omega p\colon\Omega E\to\Omega B$ and $\Omega\pi_1\colon\Omega\Sigma p\to\Omega E$ by composition. The resulting infinite sequence

$$\ldots\,\Omega^{n+1}B\xrightarrow{\Omega^nq}\Omega^n\Sigma p\xrightarrow{\Omega^n\pi_1}\Omega^nE\xrightarrow{\Omega^np}\Omega^nB\longrightarrow\ldots\,\Omega B\xrightarrow{q}\Sigma p\xrightarrow{\pi_1}E\xrightarrow{p}B$$

is called the *loop space sequence* of the map $p\colon(E,e_0)\to(B,b_0)$.

The essential ingredients for a proof of the following theorem are given in [2, pp. 95–99].

THEOREM 3. *The loop space sequence of a continuous map $p\colon(E,e_0)\to\to(B,b_0)$ is exact.*

3. Applications to weak Hurewicz fibrations. Throughout this section we assume that the fiber structure (E,p,B) has the weak covering homotopy property and basic fiber $F=p^{-1}(b_0)$. Since the natural map $U\colon F\to\Sigma p$ defined by

$$U(e)=\big(e,c(b_0)\big)\,,\qquad e\in F\,,\qquad c(b_0)(I)=b_0$$

is a homotopy equivalence, Σp may be replaced by F in the loop space sequence of p to obtain an exact sequence

$$\ldots \longrightarrow \Omega^{n+1}B \xrightarrow{d_n} \Omega^n F \xrightarrow{i} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \longrightarrow \ldots$$
$$\ldots \longrightarrow \Omega B \xrightarrow{d} F \xrightarrow{i} E \xrightarrow{p} B .$$

Here i denotes inclusion maps and $d\colon \Omega B \to F$ is defined by

$$d(\beta) = \lambda(e_0, \beta)(1), \quad \beta \in \Omega B$$

where λ is a weak lifting function. The sequence obtained by taking the induced maps on path components is the usual homotopy sequence of the fibration.

THEOREM 4. *Let (E, p, B) be a weak Hurewicz fibration such that E and B are H-groups, p is an H-homomorphism and the fiber F is an H-subgroup of E. Then (E, p, B) is fiber homotopy equivalent to $(B\times F, \pi_1, B)$ provided that (E, p, B) has a base point preserving cross section.*

Proof. Let $\chi\colon B\to E$ be a cross section such that $e_0 = \chi(b_0)$. Since the sequence

$$\Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{d} F \xrightarrow{i} E \xrightarrow{p} B$$

is exact and $\Omega p\,\Omega\chi$ is the identity on ΩB, it follows that d is null-homotopic. Hence (E, p, B) and $(B\times F, \pi_1, B)$ are homotopy equivalent. The desired fiber homotopy equivalence follows from [1, Theorem 6.1].

THEOREM 5. *If F is a retract of E, then*

(a) *$(\Omega E, \Omega p, \Omega B)$ and $(\Omega B\times \Omega F, \pi_1, \Omega B)$ are homotopy equivalent, and*

(b) *$\Omega^n E \approx \Omega^n B \times \Omega^n F$ for $n \geqslant 2$.*

Proof. Since F is a retract of E, the inclusion maps $i\colon F\to E$ and $i\colon \Omega F\to\Omega E$ have left homotopy inverses. Hence d and d_2 are null-homotopic. Since d is null-homotopic, there is a map $\chi\colon \Omega B\to\Omega E$ such that $\Omega p\chi \sim \mathrm{id}_{\Omega B}$. The conclusions follow from the Corollary and Theorem 2 respectively.

THEOREM 6. (a) *The fiber structures $(\Omega F, i, \Omega E)$ and $(\Omega E\times\Omega^2 B, \pi_1, \Omega E)$ are homotopy equivalent if and only if ΩE is deformable into ΩF.*

(b) *If E is deformable into F, then $\Omega^n F$ and $\Omega^n E\times\Omega^{n+1}B$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n\geqslant 2$.*

Proof. (a) If ΩE is deformable into ΩF, the maps $\Omega p\colon \Omega E\to\Omega B$ and $\Omega^2 p\colon \Omega^2 E\to\Omega^2 B$ are null-homotopic. Hence the H-group sequence

$$0\to\Omega^2 B\to\Omega F\to\Omega E\to 0$$

is exact and splits up to homotopy type.

Conversely, suppose that $K\colon \Omega F \rightleftarrows \Omega E\times\Omega^2 B\colon L$ is a homotopy equivalence pair such that

$$\pi_1 K \sim i, \quad iL\sim \pi_1 .$$

Define $G\colon \Omega E\to\Omega F$ by

$$G(a) = L\big(a, c^2(b_0)\big), \quad a\in L .$$

Then

$$iG\sim\pi_1 KG\sim \mathrm{id}_{\Omega E}$$

so that ΩE is deformable into ΩF.

(b) If E is deformable into F, then p, Ωp and $\Omega^2 p$ are null-homotopic and the conclusion follows from Theorem 2.

THEOREM 7. (a) *The fiber structures $(\Omega B, d, F)$ and $(F\times\Omega E, \pi_1, F)$ are homotopy equivalent if and only if F is contractible in E.*

(b) *If F is contractible in E, then $\Omega^{n+1}B\approx\Omega^n F\times\Omega^{n+1}E$ for $n\geqslant 1$.*

Proof. If F is contractible in E, then $i\colon F\to E$ and $i\colon\Omega F\to\Omega E$ are null-homotopic. Hence the sequence

$$0\longrightarrow\Omega E\xrightarrow{\Omega p}\Omega B\xrightarrow{d} F\longrightarrow 0$$

is exact. Although F may not be an H-group, the hypotheses of Theorem 1 are satisfied. This establishes (b) and half of (a).

Conversely, suppose that the indicated fiber structures have the same homotopy type. Then d has a right homotopy inverse and hence $i\colon F\to E$ is null-homotopic.

4. Applications to triplets. Let (X, A, x_0) be a triplet and $i\colon A\to X$ the inclusion. Then Σi is the space $\Omega(X, A)$ of paths in X with initial point in A and terminal point x_0. The loop space sequence of the map $i\colon (A, x_0)\to(X, x_0)$ is the sequence

$$\ldots\,\Omega^{n+1}X\xrightarrow{i}\Omega^{n+1}(X, A)\longrightarrow\Omega^n A\xrightarrow{i}\Omega^n X\longrightarrow\ldots$$
$$\ldots\,\Omega X\xrightarrow{i}\Omega(X, A)\xrightarrow{\partial} A\xrightarrow{i} X$$

where i denotes inclusion maps and $\partial(a) = a(0)$ for each $a\in\Omega(X, A)$. The sequence obtained by taking path components and induced maps is the usual homotopy sequence of the triplet (X, A, x_0).

The splitting of the above loop space sequence is summarized in the following theorems. The proofs are similar to those of the preceding section.

THEOREM 8. (a) *The fiber structures $(\Omega A, i, \Omega X)$ and $\big(\Omega X\times\Omega^2(X, A), \pi_1, \Omega X\big)$ are homotopy equivalent if and only if ΩX is contractible in $\Omega(X, A)$.*

(b) *If X is deformable into A relative to x_0, then $\Omega^n A$ and $\Omega^n X \times \Omega^{n+1}(X, A)$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

THEOREM 9. (a) *The fiber structures $(\Omega^2(X, A), \partial, \Omega A)$ and $(\Omega A \times \Omega^2 X, \pi_1, \Omega A)$ are homotopy equivalent if and only if ΩA is contractible in ΩX.*

(b) *If A is contractible in X relative to x_0, then $\Omega^{n+1}(X, A)$ and $\Omega^n A \times \Omega^{n+1} X$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

THEOREM 10. (a) *The fiber structures $(\Omega X, i, \Omega(X, A))$ and $(\Omega(X, A) \times \Omega A, \pi_1, \Omega(X, A))$ are homotopy equivalent if and only if $\partial\colon \Omega(X, A) \to A$ is null-homotopic.*

(b) *If A is a retract of X, then $\Omega^n X$ and $\Omega^n(X, A) \times \Omega^n A$ are homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

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Topologies for probabilistic metric spaces

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1. Introduction. The purpose of the present paper is to generalize some of the topological notions for probabilistic metric spaces introduced by Schweizer and Sklar [4], and by Thorp [5]. The fundamental tool for this task is the "profile function" a monotone non-decreasing function defined on the non-negative half of the real line and having its values in the closed unit interval. It will be shown that such a function gives rise to a generalized topology [3] on any PM-space $(S, \mathcal{F})$.

A condition sufficient to strengthen these g-topologies to topologies is established but its non-necessity is shown by several examples which are of some interest in their own right, with a view, perhaps, towards possible applications.

Finally, this approach to generalized topologies for PM-spaces is compared with that of Thorp [5] and a rather mild condition for the equivalence of these two is demonstrated.

The only concepts required for an understanding of these results are those of PM-space, triangular norm (t-norm) and Menger space. These may be found in Schweizer and Sklar [4], among others.

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2.

DEFINITION 2.1. A function φ is a *profile function* if $\mathrm{Dom}\,\varphi = [0, \infty)$, φ is non-decreasing and $0 \leqslant \varphi(x) \leqslant 1$ for all x in $[0, \infty)$.

DEFINITION 2.2. Let $(S, \mathcal{F})$ be a PM-space, let φ be a profile function, let $p \in S$, let $A \subseteq S$, and let $\varepsilon, \lambda > 0$ be given. Then:

a. The set $N_p(\varphi;\ \varepsilon, \lambda) = \{q \in S\colon\ F_{pq}(\varepsilon) > \varphi(\varepsilon) - \lambda\}$ is called the *$(\varphi;\ \varepsilon, \lambda)$-neighborhood of p*;

b. p is a *φ-accumulation point of A* if $(N_p(\varphi;\ \varepsilon, \lambda) - \{p\}) \cap A \neq \emptyset$ for every $\varepsilon, \lambda > 0$;

c. A is *φ-closed* if $\varphi(A) \subseteq A$, where $\varphi(A)$ is the set of φ-accumulation points of A.