Small retractions of smooth dendroids onto trees

by

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1. Introduction. A continuum (compact connected metric space) which is arcwise connected and hereditarily unicoherent is a *dendroid*. A *dendrite* is a locally connected dendroid. A *tree* is a one-dimensional polyhedron (the carrier of a finite simplicial complex) containing no simple closed curve. Clearly, trees are dendrites having a finite set of end points. Dendroids are easily seen to be hereditarily decomposable, hence one-dimensional.

It is the purpose of this paper to show that a class of dendroids, called smooth, are very much like trees, in that they can be retracted onto trees by maps (continuous functions) which do not move points very far (Theorem 2). From this theorem, it follows that the product of any collection of smooth dendroids has the fixed point property (f.p.p.) and that the cone over a smooth dendroid has f.p.p. In [1] Borsuk showed that each dendroid has f.p.p.

A dendroid is *smooth* provided there is a point $p \in M$, called an *initial point*, such that if b is a sequence in M converging to b_0, then sequence of arcs $[p, b_1], [p, b_2], \ldots$ converges to the arc $[p, b_0]$. It is shown in ([2], Corollary 4) that each dendrite is a smooth dendroid. In the plane, the cone over the Cantor set (embedded in the usual way on $[0,1]$) is a smooth dendroid which is not a dendrite.

A collection $\mathcal{C}$ of subsets of a space is *coherent* provided that if $\mathcal{C}' \subset \mathcal{C}$, then some member of $\mathcal{C}-\mathcal{C}'$ intersects a member of $\mathcal{C}'$. We will denote the union of the members of $\mathcal{C}$ by $\mathcal{C}^*$. A *tree chain* $\mathcal{T}$ in a metric space X is a finite coherent collection of open sets, called *links*, such that no point of X is in more than two elements of $\mathcal{T}$ and $\mathcal{T}$ contains no circular chains. Thus a tree chain is a one-dimensional cover with nerve whose geometric realization is a tree. If each member of $\mathcal{T}$ has diameter $< \varepsilon$, then $\mathcal{T}$ is an *ε-tree chain*. If non-intersecting links are a positive distance apart, $\mathcal{T}$ is said to be *taut*. A continuum M is *tree-chainable* or *tree-like* provided that for each $\varepsilon > 0$, there is an ε-tree chain covering M. It follows from [6] that M is tree-like iff for each open cover $\mathcal{O}$ of M there is a tree N and a map $f\colon M\to N$ such that f is an $\mathcal{O}$-map (the inverse image of each point is contained in a member of $\mathcal{O}$).

2. Straignt trees and looping arcs. If $\mathcal{C}$ is a tree chain and D is a tree such that $D\subset\mathcal{C}^*$ then D is *straight* in $\mathcal{C}$ iff

(1) $\mathcal{C}$ is an essential cover of D (i.e. each link of $\mathcal{C}$ contains a point of D not in any other link of $\mathcal{C}$).

(2) If $T\in\mathcal{C}$ then $D\cap \operatorname{Bd} T$ contains exactly one point in each link of $\mathcal{C}-\{T\}$ intersecting T.

(In Figure 1, the tree D, which contains only the point r of the arc $[q, r]$, is straight in $\mathcal{C}$. The tree $D\cup[q,r]$ is not straight in $\mathcal{C}$.)

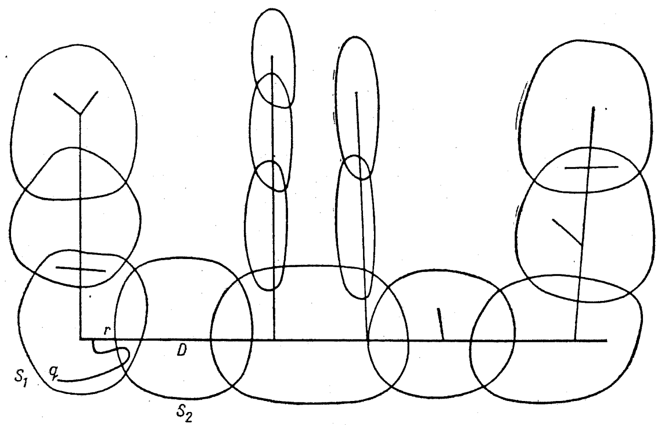

Fig. 1

PROPOSITION 1. *If D is straight in the tree chain $\mathcal{C}$, and $T\in\mathcal{C}$, then $\overline{T}\cap D$ is connected. Moreover, if T is an end link of $\mathcal{C}$, then $D-T$ is connected.*

Proof. Let $S_1, \ldots, S_n$ denote the links of $\mathcal{C}-\{T\}$ which intersect T. Now $\operatorname{Bd} T$ is the union of n mutually disjoint closed sets, $S_1\cap \operatorname{Bd} T$, $S_2\cap\operatorname{Bd} T, \ldots, S_n\cap \operatorname{Bd} T$. Suppose that the theorem fails and there are disjoint closed sets A and B such that $A\cup B = D\cap\overline{T}$. Since, for each i, $D\cap S_i\cap\operatorname{Bd} T$ is degenerate, $D\cap S_i\cap \operatorname{Bd} T$ intersects exactly one of A and B. Let $\mathcal{U}_i$ be the star-component (the maximal coherent collection) of $\mathcal{C}-\{T\}$ containing S_i, and let $L_i = (\mathcal{U}_i^* - T)\cap D$. Then $L_1, \ldots, L_n$ is a collection of mutually disjoint closed sets, and for each i, $D\cap S_i\cap \operatorname{Bd} T = L_i\cap\overline{T}$. It follows that for each i, L_i intersects exactly one of A and B.

Let

$$X = A\cup\left(\bigcup\{L_i\colon L_i\cap A\neq\emptyset\}\right) \quad\text{and}\quad Y = B\cup\left(\bigcup\{L_i\colon L_i\cap B\neq\emptyset\}\right).$$

Then X and Y are disjoint closed sets such that $X\cup Y = D$, and D is not connected. This is impossible, hence $D\cap\overline{T}$ is connected.

If T is an end link of $\mathcal{C}$, then $D\cap\operatorname{Bd}T$ is degenerate. Since each component of $D-T$ intersects $\operatorname{Bd} T$, $D-T$ is connected.

The principal reason we are interested in the notion of straightness is given by:

PROPOSITION 2. *If M is a compact metric space, $\mathcal{C}$ is a taut tree chain covering M and F is a tree contained in M and straight in $\mathcal{C}$, then there is a continuous retraction $r\colon M\to F$ such that r is a $\mathcal{C}$-map.*

Proof. Suppose T_1 is an end link of $\mathcal{C}$ and T_2 is the link of $\mathcal{C}-\{T_1\}$ which intersects T_1. Since no point of M belongs to three links of $\mathcal{C}$, $\operatorname{Bd} T_1 = T_2\cap \operatorname{Bd} T_1$. Moreover, since F is straight in $\mathcal{C}$, $F\cap\operatorname{Bd}T_1$ is a single point, d_2. Define a map

$$g_1\colon (F\cap\overline{T}_1)\cup\operatorname{Bd}T_1\to F\cap\overline{T}_1$$

by

$$g_1(x) = x, \quad \text{if } x\in F\cap\overline{T}_1,$$
$$g_1(x) = d_2, \quad \text{if } x\in \operatorname{Bd} T_1.$$

Clearly, g_1 is a continuous retraction onto $F\cap\overline{T}_1$. It follows from Proposition 1 that $F\cap\overline{T}_1$ is an AR, because it is connected. Thus there is a continuous extension f_1 of g_1, $f_1\colon \overline{T}_1\to F\cap\overline{T}_1$. Moreover, since $\operatorname{Bd}T_1\subset T_2$, f_1 is a $\mathcal{C}$-map.

Suppose $T_3, T_4, \ldots, T_n$ are the links of $\mathcal{C}-\{T_1, T_2\}$ which intersect T_2. Then $\operatorname{Bd}T_2$ is the union of the mutually disjoint closed sets $T_1\cap\operatorname{Bd}T_2$, $T_3\cap\operatorname{Bd}T_2, \ldots, T_n\cap\operatorname{Bd}T_2$. Since F is straight in $\mathcal{C}$, there are point $d_1, d_3, \ldots, d_n$ of F such that for each i, $T_i\cap\operatorname{Bd}T_2 = \{d_i\}$. Define

$$g_2\colon \overline{T}_1\cup(F\cap\overline{T}_2)\cup\operatorname{Bd}T_2\to F\cap(\overline{T}_1\cup\overline{T}_2)$$

by

$$g_2(x) = f_1(x), \quad \text{if } x\in\overline{T}_1,$$
$$g_2(x) = x, \quad \text{if } x\in F\cap\overline{T}_2,$$
$$g_2(x) = d_i, \quad \text{if } x\in T_i\cap\operatorname{Bd}T_2,\ i\geqslant 3.$$

Since f_1 is a retraction onto $F\cap\overline{T}_1$, g_2 is well-defined. Clearly g_2 is continuous and a $\mathcal{C}$-map. Applying Proposition 1, we see that $F\cap\overline{T}_1$ and $F\cap\overline{T}_2$ are connected, thus $F\cap(\overline{T}_1\cup\overline{T}_2)$ is connected and an AR. We may continuously extend g_2 by $f_2\colon \overline{T}_1\cup\overline{T}_2\to F\cap(\overline{T}_1\cup\overline{T}_2)$. After a finite number of repetitions of this process, we obtain r.

We should note here that if F is a tree which is straight in $\mathcal{C}$, F need not be a geometric realization of (nerve $\mathcal{C}$); for instance, see Figure 1.

Even if F is a geometric realization of (nerve $\mathcal{C}$), the mapping r is not the usual "barycentric $\mathcal{C}$-map" of [6].

If $\mathcal{C}$ is a tree chain covering M, and $[a, b]$ is an arc in M, then $[a, b]$ *loops* in $\mathcal{C}$ provided there are distinct links T_1 and T_2 in $\mathcal{C}$ and points x, y, z of $[a, b]$ such that $a \leqslant x < y < z \leqslant b$, $x, z \in \mathrm{Bd}\, T_1$ and $y \in \mathrm{Bd}\, T_2$. (Here $<$ denotes the usual order from a to b on $[a, b]$.) One can see that if D is a tree which is straight in $\mathcal{C}$, then no arc in D loops in $\mathcal{C}$. The converse of this theorem is false, as can be seen by Figure 1. No arc in $D \cup [q, r]$ loops in $\mathcal{C}$, but $D \cup [q, r]$ is not straight in $\mathcal{C}$.

PROPOSITION 3. *Suppose M is compact, $\mathcal{S}$ is a taut tree chain covering M, D is a tree contained in M, and straight in $\mathcal{S}$. Suppose $q \in M - D$ and no arc from q to D loops in $\mathcal{S}$. If $[q, r]$ is the irreducible arc from q to D then $[q, r]$ is contained in a single link of $\mathcal{S}$ and $[q, r]$ intersects the closure of at most one other link of $\mathcal{S}$.*

Proof. Suppose the proposition fails, and $q \in S_1 \in \mathcal{S}$. Since $[q, r] \not\subset S_1$, there is a link $S_2 \in \mathcal{S}$, and a point x such that $x \in [q, r] \cap S_2 \cap \mathrm{Bd}\, S_1$. Similarly, there is a point $y \in [q, r] \cap \mathrm{Bd}\, S_2$. Without loss of generality, we may assume $q \leqslant x < y \leqslant r$. Since $\mathcal{S}$ is an essential cover of D, there is a point $d \in D \cap S_2 \cap \mathrm{Bd}\, S_1$. Inasmuch as $[q, r]$ is irreducible from q to D, $[q, r] \subset [q, d]$. Thus $q \leqslant x < y < d$, and $[q, d]$ loops in $\mathcal{S}$. This is impossible, hence $[q, r]$ is contained in a single link S_1 of $\mathcal{S}$.

Suppose S_2 and S_3 are distinct links of $\mathcal{S} - \{S_1\}$ such that $[q, r] \cap \overline{S}_2 \neq \emptyset$ and $[q, r] \cap \overline{S}_3 \neq \emptyset$. We may assume that $[q, r] \not\subset S_2 \cup S_3$. Thus there are points w and z such that $w \in [q, r] \cap S_1 \cap \mathrm{Bd}\, S_2$ and $z \in [q, r] \cap S_1 \cap \mathrm{Bd}\, S_3$. Since $\mathcal{S}$ is taut, $w \neq z$. At most one of w and z is r, assume $w \neq r$. Since D is straight in $\mathcal{S}$, there is a point $e \in D \cap S_1 \cap \mathrm{Bd}\, S_2$. Then $w \neq e \neq z$ and the arc $[q, z]$ loops in $\mathcal{S}$. This contradiction establishes the proposition.

3. Principal results.

THEOREM 1. *Suppose M is a smooth dendroid with metric ϱ and $\varepsilon > 0$. Then there is a tree $F \subset M$ and an ε-tree chain $\mathcal{W}$ such that*

(1) *$\mathcal{W}$ is a taut tree chain covering M,*

(2) *F is straight in $\mathcal{W}$.*

Proof. Suppose that the theorem fails and let p be an initial point of M. It follows from [3] that M is tree-like; let $\mathcal{R}$ be a taut ε-tree chain covering M. If N is a subcontinuum of M, then N has *property Q* provided there does not exist a refinement $\mathcal{S}$ of $\mathcal{R}$ and a tree $D \subset N$ such that

(i) $\mathcal{S}$ is a taut tree-chain covering N,

(ii) D is straight in $\mathcal{S}$,

(iii) If $n \in N$ then the arc $[p, n] \cap N$ does not loop in $\mathcal{S}$.

We will show that property Q is inductive. Suppose L is a decreasing sequence of subcontinua of M, each having property Q. Clearly, $\bigcap L_i$ is a continuum. Suppose $\mathcal{S}$ refines $\mathcal{R}$ and D is a tree such that $\mathcal{S}$ and D satisfy (i) and (ii) for $N = \bigcap L_i$. We will show that $\bigcap L_i$ has property Q by showing that (iii) does not hold. Since $\mathcal{S}^*$ is an open set containing $\bigcap L_i$, $\mathcal{S}^*$ contains all but finitely many L_i. We will assume that each $L_i \subset \mathcal{S}^*$. Since, for each i, L_i, $\mathcal{S}$ and D satisfy (i) and (ii), L_i must fail to satisfy (iii). Thus there is a point $n_i \in L_i$ such that $L_i \cap [p, n_i]$ loops in $\mathcal{S}$. There are distinct links S_{j_i} and S_{k_i} of $\mathcal{S}$ and points x_i, y_i, z_i such that $p \leqslant x_i < y_i < z_i \leqslant n_i$, $x_i, z_i \in L_i \cap \mathrm{Bd}\, S_{k_i}$ and $y_i \in L_i \cap \mathrm{Bd}\, S_{j_i}$. Since $\mathcal{S}$ is a finite collection, some pair of links is chosen infinitely often. By passing to convergent subsequences and relabeling, we may assume that there are links S_j and S_k of $\mathcal{S}$ such that for each i, $S_j = S_{j_i}$ and $S_k = S_{k_i}$, and that the sequences x, y, z, n converge to x_0, y_0, z_0 and n_0, respectively. Because L is a decreasing sequence, $x_0, z_0 \in (\bigcap L_i) \cap \mathrm{Bd}\, S_k$, $y_0 \in (\bigcap L_i) \cap \mathrm{Bd}\, S_j$ and $n_0 \in \bigcap L_i$. Clearly, $\lim[p, x_i] \subset \lim[p, y_i] \subset \lim[p, z_i] \subset \lim[p, n_i]$. Since M is smooth, these limits are the arcs $[p, x_0]$, $[p, y_0]$, $[p, z_0]$ and $[p, n_0]$, respectively. Thus $p \leqslant x_0 \leqslant y_0 \leqslant z_0 \leqslant n_0$. Since $S_j \neq S_k$ and $\mathcal{S}$ is taut, $(\bigcap L_i) \cap \mathrm{Bd}\, S_j \cap \mathrm{Bd}\, S_k = \emptyset$. It follows that $x_0 \neq y_0 \neq z_0$ and so $p \leqslant x_0 < y_0 < z_0 \leqslant n_0$; thus the arc $[p, n_0] \cap (\bigcap L_i)$ loops in $\mathcal{S}$. Since (iii) does not hold, $\bigcap L_i$ has property Q and property Q is inductive.

Clearly M has property Q, so there is a subcontinuum M' of M which is irreducible with respect to property Q. If $p \notin M'$, then there is a point $p' \in M'$ such that $[p, p'] \cap M' = \{p'\}$. It follows from ([2], Corollary 6) that M' is a smooth dendroid with initial point p'. Suppose N is a subcontinuum of M' and $n \in N$. Then $[p, n] \cap N$ loops in $\mathcal{S}$ iff $[p', n] \cap N$ loops in $\mathcal{S}$. (In fact, $[p', n] \cap N = [p, n] \cap N$.) Rather than use M' and p', we will simply assume that $p \in M'$ and $M' = M$.

It is shown in ([2], Theorem 10) that there is a metric d for M which is radially convex with respect to p (i.e. if $x \in [p, y]$ and $x \neq y$ then $d(p, x) < d(p, y)$). Let W be an open sphere about p in this metric d. It is easy to see that W is arcwise connected and, since M is hereditarily unicoherent, $\mathrm{Bd}\, W$ contains exactly one point from each component of $M - W$. Since the metrics ϱ and d are equivalent, we may assume that W was chosen so that if $R_i \cap W \neq \emptyset$, then $\mathrm{diam}(R_i \cup W) < \varepsilon$ in our original metric ϱ. Let K denote a component of $M - W$ and let $\{q\} = K \cap \mathrm{Bd}\, W$. Since K is a proper subcontinuum of M, K does not have property Q. Thus there is a refinement $\mathcal{S}$ of $\mathcal{R}$ and a tree D such that $\mathcal{S}$ and D satisfy (i), (ii) and (iii) for $N = K$.

We would like to be able to conclude that $q \in D$ and that only one link of $\mathcal{S}$ intersects W. If $q \notin D$, there is a point $r \in D$ such that $[q, r] \cap D = \{r\}$. Let $E = [q, r] \cup D$. Note that E is a tree, although we do not know that E is straight in $\mathcal{S}$. We will refine $\mathcal{S}$ to obtain a tree chain $\mathcal{T}$ so that $\mathcal{T}$ and E satisfy (i) and (ii) for $N = K$, and only one link of $\mathcal{T}$ intersects W. Since $[q, r] = [p, r] \cap K$, $[q, r]$ does not loop in $\mathcal{S}$. Applying Proposition 3, there are distinct, intersecting links S_1 and S_2 such that $[q, r] \subset S_1$ and $[q, r] \cap \bar{S}_j = \emptyset$, if $1 \neq j \neq 2$. See Figure 1. Using ([5], Lemma 1), we obtain a set V, open in M, such that $r \in V \subset \bar{V} \subset S_1$, $V \cap S_j = \emptyset$ if $1 \neq j \neq 2$ and $\mathrm{Bd}\,V$ contains exactly one point from each component of $E - \{r\}$.

We show that $\overline{S_2 - S_1} \cap (E - V)$ is connected. Since $E = D \cup [q, r]$ and $[q, r] \subset S_1$, $\overline{S_2 - S_1} \cap (E - V) = \overline{S_2 - S_1} \cap (D - V)$. Since $V \subset S_1$, $\overline{S_2 - S_1} \cap (D - V) = \overline{S_2 - S_1} \cap D$. Now $\bar{S}_2 - S_1 = \overline{S_2 - S_1}$, so $\overline{S_2 - S_1} \cap D = (\bar{S}_2 - S_1) \cap D = (\bar{S}_2 \cap D) - S_1$. According to Proposition 1, $\bar{S}_2 \cap D$ is a tree. Let $\mathcal{B}$ denote the links of $\mathcal{S}$ intersecting S_2. Then $\mathcal{B}$ is a tree chain covering $\bar{S}_2 \cap D$ and $\bar{S}_2 \cap D$ is straight in $\mathcal{B}$. Since S_1 is an end link of $\mathcal{B}$, we apply Proposition 1 again and conclude that $(\bar{S}_2 \cap D) - S_1 = \overline{S_2 - S_1} \cap (E - V)$ is connected.

Let L denote the component of $E - V$ containing $\overline{S_2 - S_1} \cap (E - V)$. Since $E - V$ has only finitely many components, L is both open and closed in $E - V$. Thus L and $(E - V) - L = E - (V \cup L)$ are disjoint and closed in M. It follows that $L \cup (\overline{S_2 - S_1})$ and $E - (V \cup L)$ are disjoint. Using normality, we obtain an open set Y such that $E - (V \cup L) \subset Y$ and $\bar{Y} \cap (L \cup (\overline{S_2 - S_1})) = \emptyset$. Let $\mathcal{T}$ be defined by $T_i = S_i$, if $i \neq 2$; $T_2 = S_2 - (\overline{Y \cup V})$. Since $S_2 \cap (\overline{Y \cup V}) \subset S_1$, $\mathcal{T}$ is a taut ε-tree chain covering K. Moreover, $E \cap T_1 \cap \mathrm{Bd}\,T_2 = (E - V) \cap S_1 \cap \mathrm{Bd}\,T_2$, which is contained in exactly one of the degenerate sets $L \cap \mathrm{Bd}\,V$ and $L \cap \mathrm{Bd}\,S_2$. Therefore E is straight in $\mathcal{T}$ and so $\mathcal{T}$ and E satisfy (i) and (ii) and $q \in E$. We may assume that only one link T_1 of $\mathcal{T}$ intersects W. (If necessary, remove $\bar{W}$ from each of $\mathcal{T} - \{T_1\}$. This new tree chain will still cover K, because $(\bar{W} - W) \cap K = \{q\} \subset T_1$.)

One further refinement $\mathcal{U}$ of $\mathcal{T}$ is necessary. Since $\mathcal{T}$ covers K, no continuum in $M - W$ intersects both K and $(M - W) - \mathcal{T}^*$. Thus there is an open set X such that $K \subset X \subset \mathcal{T}^*$ and $\mathrm{Bd}\,X \subset W$. Define $\mathcal{U}$ by $U_i = T_i \cap X$. Thus for each component K of $M - W$, we obtain a refinement $\mathcal{U}$ of $\mathcal{R}$ and a tree E such that (i) $\mathcal{U}$ is a taut tree-chain covering K, (ii) E is straight in $\mathcal{U}$, (iv) $K \cap \mathrm{Bd}\,W = \{q\} \subset E$, (v) U_1 is the only link of $\mathcal{U}$ intersecting W, and (vi) $\mathrm{Bd}\,\mathcal{U}^* \subset W$.

There is a finite collection $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ of these tree-chains whose union covers $M - W$. Denote the corresponding components of $M - W$ by K_j, the trees by E_j and $K_j \cap \mathrm{Bd}\,W = \{q_j\} \subset E_j$. Now $K_1 \cup \big((M - W) - \bigcap_{j \neq 1} \mathcal{U}_j^*\big)$ and $\big((M - W) - \mathcal{U}_1^*\big) \cup \big(\bigcup_{j \neq 1} K_j\big)$ are disjoint closed subsets of $M - W$. No continuum in $M - W$ intersects both these closed sets, because, for each j, $\mathrm{Bd}\,\mathcal{U}_j^* \subset W$. Thus there is an open set Z_1 such that $K_1 \cup \big((M - W) - \bigcup_{j \neq 1} \mathcal{U}_j^*\big) \subset Z_1$ and $Z_1 \cap \Big(\big((M - W) - \mathcal{U}_1^*\big) \cup \big(\bigcup_{j \neq 1} K_j\big)\Big) = \emptyset$. It follows that $\bar{Z}_1 \subset \mathcal{U}_1^* \cup W$. Define a refinement $\mathcal{V}_1$ of $\mathcal{U}_1$ by $V_{1k} = U_{1k} \cap Z_1$. Then $\mathcal{V}_1$ has properties (i), (ii), (iv), (v), (vi) and in addition $\mathcal{V}_1^* \cap \mathcal{U}_j^* \subset W$, if $j \neq 1$. Repeating this process $m-1$ times yields tree chains $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ which have properties (i), (ii), (iv), (v), (vi) and $\mathcal{V}_i^* \cap \mathcal{V}_j^* \subset W$, if $i \neq j$. For each j, V_{j1} is the only link of $\mathcal{V}_j$ which intersects W. Since each $\mathcal{V}_j$ refines $\mathcal{R}$, $\mathrm{diam}(W \cup V_{j1}) < \varepsilon$. Thus $\mathcal{W}_j = \{V_{j1} \cup W, V_{j2}, \ldots, V_{jk_j}\}$ is an ε-tree chain covering the tree $[p, q_j] \cup E_j$. Moreover, $[p, q_j] \cup E_j$ is straight in $\mathcal{W}_j$ and $\mathcal{W}_i^* \cap \mathcal{W}_j^* = W = W_{i1} \cap W_{j1}$, if $i \neq j$. Let $F = \bigcup_{j=1}^{m} ([p, q_j] \cup E_j)$. Then F is a tree which is straight in the tree chain $\bigcup_{j=1}^{m} \mathcal{W}_j$, and this tree chain covers M. This concludes the proof.

We note that we could have proved that smooth dendroids are tree-like using an argument similar to, and somewhat simpler than, that given in the proof of Theorem 1. However, Cook's theorem makes this unnecessary.

Our main result now follows immediately from Theorem 1 and Proposition 3.

THEOREM 2. *Suppose M is a smooth dendroid with metric ϱ and $\varepsilon > 0$. Then there is a tree $F \subset M$ and a retraction $r\colon M \to F$ such that r moves no point of M as much as ε.*

One would like to be able remove "smooth" from the hypothesis of Theorem 2. In [5] it is shown that we may to this if M is assumed to be a fan (a dendroid with one ramification point).

For completeness, we state two theorems on fixed points, which are the joint work of C. A. Eberhart and the writer. For other theorems along these lines, and detailed arguments, see [4].

THEOREM 3. *The product of any collection of smooth dendroids has* f.p.p.

Proof. It suffices to show that the product $\prod X_i$ of any finite collection $X_1, \ldots, X_n$ of smooth dendroids has f.p.p. If $g\colon \prod X_i \to \prod X_i$ is a fixed-point-free map, then there is a $\delta > 0$ such that each point of $\prod X_i$ is moved at least δ by g. Repeatedly applying Theorem 2, there is a collection $F_1, \ldots, F_n$ of trees and retractions $r_i\colon X_i \to F_i$, each of which moves no point of X_i as much as $\delta/2^n$. Then $\prod r_i$ is a retraction of $\prod X_i$ onto $\prod F_i$ moving no point as much as $\delta/2^{n-1}$. The composition of $\prod r_i$ restricted to $\prod F_i$, followed by g is a map of $\prod F_i$ into itself which moves each point of $\prod F_i$ at least $\delta/2$; thus this map is fixed-point-free. However, $\prod F_i$ is an AR, hence has the fixed point property. This contradiction establishes the theorem.

THEOREM 4. *The cone $C(M)$ over a smooth dendroid M has* f.p.p.

Proof. It follows from ([2], Corollary 12) that M is contradictible. According to [7], $C(M)$ has f.p.p. iff $M\times I$ has f.p.p. Since $M\times I$ is a product of smooth dendroids, we apply Theorem 3 and conclude that $C(M)$ has f.p.p.

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Some relations between k-analytic sets and generalized Borel sets

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§ 1. Introduction. A. H. Stone introduced the family of k-analytic sets, where k is an arbitrary cardinal, and established some of their properties ([7], hereafter referred to simply as Stone). He conjectured that the relationships between classical analytic and Borel sets, such as Souslin's theorem, that if a set and its complement are both analytic, they are both Borel, would generalize to relationships between k-analytic sets and k-hyperborel sets, defined as the smallest family of sets containing all closed sets and closed under intersections of $\aleph_0$ and unions of k of them. While this seems to be the correct Borel family there are difficulties in working with different cardinal numbers.

In this paper we establish some relations between Souslin$(\alpha)\mathfrak{F}$ sets (k-analytic sets if $k=\aleph_\alpha$) and generalized Borel families of sets which contain the k-hyperborel sets, but admit intersections of k elements. We remark that Maximoff [2] was led to a similar Borel family while studying a relation between Borel sets and sets analogous to k-analytic sets. The method used is a generalization of Lusin's theory of sieves. Most of the proofs in this paper are direct generalizations of proofs of Lusin [1].

Specifically, in § 3 is developed the basic sieve theory; in corollary 4, Souslin$(\alpha)\mathfrak{F}$ sets are characterized as the sifted sets of a certain class of sieves. In § 4 we establish the decomposition of sifted sets and their complements into disjoint Borel$(\alpha)\mathfrak{F}$ sets (see § 2 for definitions) and apply this result to express a Souslin$(\alpha)\mathfrak{F}$ set as a union and intersection of $\aleph_{\alpha+1}$ Borel$(\alpha)\mathfrak{F}$ sets (theorem 6). Finally in § 5 we show (theorem 8) that disjoint Souslin$(\alpha)\mathfrak{F}$ sets can be separated by disjoint Borel$(\alpha)\mathfrak{F}$ sets and use this result to prove (corollary 9) that if a set and its complement are Souslin$(\alpha)\mathfrak{F}$, then they are Borel$(\alpha)\mathfrak{F}$, and that (theorem 11) a continuous, one-to-one image of $I(\alpha)$ (a generalization of the irrationals, see § 2) is a Borel$(\alpha)\mathfrak{F}$ set.

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